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Preprint series, Vol. 43 (2005), 982

TREE AMALGAMATION OF GRAPHS AND TESSELLATIONS OF THE CANTOR SPHERE

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ISSN 1318-4865

June 2, 2005

Ljubljana, June 2, 2005

Tree amalgamation of graphs and tessellations of the Cantor sphere

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Abstract

A general method is described which gives rise to highly symmetric tessellations of the Cantor sphere, i.e., the 2-sphere with the Cantor set removed and endowed with the hyperbolic geometry with constant negative curvature. These tessellations correspond to almost vertex-transitive planar graphs with infinitely many ends. Their isometry groups have infinitely many ends and are free products with amalgamation of other planar groups, possibly one or two-ended or finite. It is conjectured that all vertex-transitive tessellations of the Cantor sphere can be obtained in this way.

Although our amalgamation construction is rather simple, it gives rise to some extraordinary examples with properties that are far beyond expected. For example, for every integer k, there exists a k-connected vertex-transitive planar graph such that each vertex of this graph lies on at least k infinite faces. These examples disprove a conjecture of Bonnington and Watkins [2] that there are no 5-connected vertex-transitive planar graphs with infinite faces. This also disproves another conjecture that in a 4-connected vertex-transitive planar graph each vertex lies on the boundary of at most one infinite face. Further examples give rise to counterexamples of some other conjectures of similar flavor.

*Supported in part by the Ministry of Science and Technology of Slovenia, Research Program P1–0297 and Research Project J1–6150.

1 Introduction

In the 1980's, Mark Watkins [20] asked if there exists a 3-connected vertex-transitive planar graph with infinitely many ends. The author found an example [15] which gave rise to a general construction yielding a variety of such graphs of arbitrarily large connectivity. This construction, which we shall call the *tree amalgamation* of graphs, is closely related to the free product with amalgamation known from the theory of groups, cf., e.g., [12, 13]. In fact, in the most interesting examples, the automorphism group of the constructed graphs would be isomorphic to the free product with amalgamation of automorphism groups of the graphs used in the construction.

Stallings [19] proved that every finitely presented group with infinitely many ends is either a free product with amalgamation or an HNN-product of "smaller" groups. Later, Dunwoody [7] (see also [6]) proved that every finitely presented group can be obtained from a finite number of at most one-ended groups by means of these two operations.

In 1988, the author expected that the tree amalgamation operation would be powerful enough to yield a classification of (3-connected) vertex-transitive planar graphs, or at least planar Cayley graphs, with infinitely many ends in terms of finite and one-ended infinite planar vertex-transitive graphs which are well understood. However, more than 16 years after first thoughts, such a classification has not been made, although many people have been aware of this question. Therefore, we have decided to present our tree amalgamation construction and to show some extraordinary examples of tessellations obtained in this way.

Bonnington and Watkins [2] investigated planar vertex-transitive graphs with infinite faces. In the first version of their paper, they conjectured that such graphs cannot be 5-connected. They also conjectured that in a 4-connected vertex-transitive planar graph each vertex lies on the boundary of at most one infinite face. We disprove both conjectures by constructing, for every positive integer k, a k-connected vertex-transitive planar graph such that each vertex of this graph lies on at least k infinite faces. In the printed version of [2] it is also conjectured that a precisely 3-connected vertex-transitive planar graph cannot have infinite faces. We also disprove this conjecture by proving that such graphs can have arbitrarily many infinite faces incident with each vertex. See Theorems 5.1 and 5.2.

We also give examples of 2-connected arc-transitive plane graphs in which all faces are infinite (Theorem 5.5).

The underlying surface of tessellations corresponding to tree amalgamations is homeomorphic and isometric to the 2-sphere with the Cantor set removed and endowed with the hyperbolic geometry so that it has constant negative curvature. We shall use the term *Cantor sphere* to refer to this space. It should be mentioned that all Cantor spheres are homeomorphic to each other, but unlike the simply connected planar surfaces, there are infinitely many nonisometric realizations of the Cantor sphere, even when the curvature is -1 everywhere. Tessellations of Cantor spheres can be made particularly nice by using circle packing theorems [3, 4, 17, 1].

Since the fundamental work of Gromov [9], there has been an increased interest in hyperbolic groups. Free products with amalgamation are, except in some trivial cases, obviously hyperbolic in nature. Our tessellation representation of tree amalgamations of planar graphs (and their isometry groups) gives yet another view of hyperbolic groups.

All graphs in this paper are locally finite. They may be finite or infinite. We shall use standard graph theory terminology and established notation.

The group of all automorphisms of a graph G is the *automorphism group* of G and is denoted by $\mathrm{Aut}(G)$. A graph G is *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$. It is *edge-transitive* if $\mathrm{Aut}(G)$ acts transitively on $E(G)$, and it is *arc-transitive* if $\mathrm{Aut}(G)$ acts transitively on pairs $(v, e) \in V(G) \times E(G)$ where v and e are incident. If G is embedded in some surface and $F(G)$ is the set of faces, then G is said to be *flag-transitive* if $\mathrm{Aut}(G)$ acts transitively on the set of all *flags*, i.e., the triples $(v, e, f) \in V(G) \times E(G) \times F(G)$ such that v is incident with e, and e is incident with f (where double incidences give rise to different flags).

2 Tree amalgamation of graphs

Let $p_1, p_2 \in \{1, 2, 3, \dots\} \cup \{\infty\}$, and let T be the (p_1, p_2)-semiregular tree, i.e., if $V(T) = V_1 \cup V_2$ is the bipartition of T, then every vertex in V_i has degree p_i, $i = 1, 2$. If $p_i = \infty$, the degree is countably infinite. In particular, T is infinite if $p_1 \geq 2$ and $p_2 \geq 2$.

Suppose that there is a mapping c which assigns to each edge of T a pair (k, l), $0 \leq k < p_1$, $0 \leq l < p_2$, such that for every vertex $v \in V_1$, all first coordinates of the pairs in $\{c(e) \mid v \text{ is incident with } e\}$ are distinct and take all values in the set $\{k \mid 0 \leq k < p_1\}$, and for every vertex in V_2, all second coordinates are distinct and exhaust all values in the set $\{l \mid 0 \leq l < p_2\}$.

Let G_1 and G_2 be graphs. Suppose that $\{S_k \mid 0 \leq k < p_1\}$ is a family of subsets of $V(G_1)$, and $\{T_l \mid 0 \leq l < p_2\}$ is a family of subsets of $V(G_2)$. We shall assume that all sets S_k and T_l have the same cardinality, and we let $\varphi_{kl} : S_k \to T_l$ be a bijection. The maps φ_{kl} are called *identifying maps*.

For each vertex $v \in V_i$, take a copy G_i^v of the graph G_i, $i = 1, 2$. Denote by S_k^v (if $i = 1$) and T_l^v (if $i = 2$) the corresponding copies of S_k or T_l in $V(G_i^v)$. Let us take the disjoint union of graphs G_i^v, $v \in V_i$, $i = 1, 2$. For every edge $st \in E(T)$ ($s \in V_1, t \in V_2$) with $c(st) = (k, l)$, we identify each vertex $x \in S_k^s$ with the vertex $y = \varphi_{kl}(x)$ in T_l^t. The resulting graph Y is called the *tree amalgamation* of graphs G_1 and G_2 over the *connecting tree* T.

For a vertex $v \in V(G_i)$, we define the *degree of identification*, denoted by $\mu(v)$, as the number of sets S_k (if $i = 1$) or T_l (if $i = 2$) that contain v. To prevent identifications of vertices in graphs G_i^s and G_j^t, where $s \in V_i$ and $t \in V_j$ are far apart in T, we shall impose the following requirement:

(A1) For every $st \in E(T)$ ($s \in V_1, t \in V_2$) with $c(st) = (k, l)$, and for every $x \in S_k$, either $\mu(x) = 1$ or $\mu(\varphi_{kl}(x))) = 1$.

Having (A1), every vertex $x \in S_k^s \subseteq V(G_1^s)$ with $\mu(x) > 1$ is identified with precisely $\mu(x)$ other vertices which belong to distinct neighboring graphs G_2^t. If $\mu(x) = 1$ and $st \in E(T)$ is the edge with $c(st) = (k, l)$, then x is identified with precisely $\mu(\varphi_{kl}(x))$ other vertices. Apart from $\varphi_{kl}(x) \in V(G_2^t)$, they belong to distinct neighboring graphs G_1^r of G_2^t. Similar conclusion holds for vertices in G_2^t.

Proposition 2.1 *Suppose that Y is a tree amalgamation of G_1 and G_2 with respect to identifying families $\mathcal{C}^1 = \{S_k \mid 0 \le k < p_1\}$ and $\mathcal{C}^2 = \{T_l \mid 0 \le l < p_2\}$, such that every vertex of G_2 is contained in precisely one element of $\mathcal{C}^2$. Suppose that G_1 is k-connected and that for every $C, C' \in \mathcal{C}^2$, we have $|C| \ge k$ and for every k-set X of vertices in C, there are k disjoint paths from X to C'. Then Y is k-connected.*

Proof. Choose an infinite path $v_1v_2v_3\ldots$ in T. Let x be a vertex in Y that belongs to $V(G_1^v)$. Let $u_1u_2u_3\ldots$ be a path in T such that $u_1 = v$ and there are integers p, q such that $u_{p+i} = v_{q+i}$ for every $i > \min\{p, q\}$. By using the assumptions on k-connectivity of G_1 and the linkage property of G_2, it is easy to see that there exists a collection of k internally disjoint rays (one-way infinite paths) starting at x and passing through $G_1^{u_1}, G_2^{u_2}, G_1^{u_3}, G_2^{u_4}, \ldots$.

Suppose that S is a vertex set of cardinality at most $k-1$ that separates vertices x and y in Y. Consider k internally disjoint rays starting at x. At least one of them, call it R_x, does not intersect S. Similarly, there is a ray R_y starting at y that is disjoint from S and passes through the same sequence of graphs $G_1^{v_i}$ and $G_2^{v_j}$ as R_x. In particular, R_x and R_y belong to the same end in Y. Consequently, there are k disjoint paths from R_x to R_y. At least one of them is disjoint from S. This contradicts the assumption that S separates x and y. □

If γ is a simple closed curve in the 2-sphere, then γ separates the sphere into two discs. If γ is oriented, then we call the disc which is on the right hand side of γ the *interior* of γ. The other disc is called the *exterior* of γ.

Suppose that G_1 is a plane graph, i.e., G_1 is considered together with some fixed embedding in the 2-sphere. Let $\mathcal{C}^1 = \{S_k \mid 0 \le k < p_1\}$ and $\mathcal{C}^2 = \{T_l \mid 0 \le l < p_2\}$ be the families of all identifying sets in G_1 and G_2 (viewed as multisets). We say that $\mathcal{C}^1$ is *facial* if for every $S_k \in \mathcal{C}^1$, there is a simple closed curve $\gamma(S_k)$ in the sphere such that $\gamma(S_k) \cap G_1 = S_k$, the interior of $\gamma(S_k)$ contains neither vertices nor edges of G_1, and for any distinct members S_k, S_l, the interiors of $\gamma(S_k)$ and $\gamma(S_l)$ are disjoint. The same definition applies to $\mathcal{C}^2$. The identifying map φ_{kl} is *facial* if it maps the set S_k onto T_l in such a way that the cyclic order of vertices of S_k on $\gamma(S_k)$ corresponds to the cyclic order (in either direction) of the image on $\gamma(T_l)$.

The following proposition is not a surprise, see [15].

Proposition 2.2 *Suppose that G_1 and G_2 are plane graphs and that $\mathcal{C}^1, \mathcal{C}^2$ are facial. If all identifying maps are facial, then the tree amalgamation Y has an embedding in the plane such that the induced embedding of each copy of G_1 or G_2 is homeomorphic to its given plane embedding (possibly with reverse orientation).*

Proof. Let $t_0, t_1, t_2, \ldots$ be an enumeration of vertices of T such that for every $i > 1$, t_i has a neighbor $t_i' \in \{t_0, \ldots, t_{i-1}\}$.

For $i = 1, 2, \ldots$, we shall define a planar map M_i whose graph is obtained from the disjoint union of copies of graphs G_1 and G_2 corresponding to vertices $t_0, t_1, \ldots, t_i$ and making all identifications used in the amalgamation corresponding to the edges among these vertices in T.

We may assume that $t_0 \in V_1$. Then M_0 is G_1 embedded in the 2-sphere. Assuming that we have the map M_{i-1}, we define M_i as follows. The map M_{i-1} has disks with pairwise disjoint interiors and bounded by curves $\gamma(S_k^s)$ and $\gamma(T_l^t)$, where $s \in \{t_0, \ldots, t_{i-1}\} \cap V_1$, $t \in \{t_0, \ldots, t_{i-1}\} \cap V_2$, and where k (and l) are such that there exists a neighbor $u \notin \{t_0, \ldots, t_{i-1}\}$ of s (of t) such that $c(su) = (k, l')$ (or $c(tu) = (k', l)$). (This is easily seen by induction.) To simplify notation, we shall assume that $t_i \in V_2$. Let us consider the embedding of the graph $G = G_2^{t_i}$ corresponding to the vertex t_i of T and suppose that $c(t_i' t_i) = (k, l)$. Delete the interior of $\gamma(T_l^{t_i})$ to get a disk D. Now replace the disk bounded by $\gamma(S_k^{t_i'})$ in M_{i-1} by D in such a way that the vertices of $S_k^{t_i'}$ are identified with $T_l^{t_i}$ as required by the identifying map φ_{kl}. Since the identification maps are facial, this is possible (maybe after reversing the orientation of D), and we get the map M_i in the 2-sphere.

Since the map M_i extends the embedding of M_{i-1}, the limiting map for $i \to \infty$ exists. It is obvious that this map is an embedding of the amalgamation Y in the 2-sphere and that it has the property stated in the proposition. $\square$

It is worth remarking that (A1) is not needed for the conclusion of Proposition 2.2.

We say that a family $\mathcal{C}$ of vertex sets is a *cover* of G if $\cup\mathcal{C} = V(G)$. Let G be a plane graph and let $\mathcal{C}$ be a facial cover of G. Denote by $\mathrm{Aut}(G, \mathcal{C})$ the group of all automorphisms of G which preserve $\mathcal{C}$ and its facial structure, i.e., every such automorphism ϕ induces a permutation of $\mathcal{C}$, and for every $C \in \mathcal{C}$, the cyclic order of C on $\gamma(C)$ induces the cyclic order of $\phi(C)$ which is the same or opposite to the cyclic order of $\phi(C)$ on $\gamma(\phi(C))$. The pairs $(v, C) \in V(G) \times \mathcal{C}$ for which $v \in C$ are called *$\mathcal{C}$-flags*. Clearly, $\mathrm{Aut}(G, \mathcal{C})$ acts on $\mathcal{C}$-flags; if this action is transitive, then we say that $\mathcal{C}$ is a *strongly transitive* cover in G.

Proposition 2.3 *Suppose that G_1 and G_2 are plane graphs with strongly transitive facial covers $\mathcal{C}^1$ and $\mathcal{C}^2$, respectively. Let Y be a tree amalgamation of G_1 and G_2 with respect to $\mathcal{C}^1$ and $\mathcal{C}^2$. If all identifying maps are facial, then Y is a vertex-transitive graph.*

If $\mathrm{Aut}(G, \mathcal{C})$ has two orbits on $\mathcal{C}$-flags, then we say that $\mathcal{C}$ is *semitransitive*. Suppose that $G_1 = G_2 = G$ and $\mathcal{C}^1 = \mathcal{C}^2 = \mathcal{C}$. An identifying map φ_{kl} is *symmetry increasing* if for every $x \in S_k$, its image $\varphi_{kl}(x)$ and x belong to distinct orbits of the action of $\mathrm{Aut}(G, \mathcal{C})$ on $\mathcal{C}$-flags.

Proposition 2.4 *Suppose that G is a plane graph with a semitransitive facial cover $\mathcal{C}$. Let Y be a tree amalgamation of G with itself with respect to $\mathcal{C}$. If all identifying maps are facial and symmetry increasing, then Y is a vertex-transitive graph.*

We will also make use of another, less obvious criterion of vertex-transitivity.

Proposition 2.5 *Suppose that G_1 and G_2 are plane graphs with facial covers $\mathcal{C}^1$ and $\mathcal{C}^2$, respectively. Suppose that the action of* $\mathrm{Aut}(G_1, \mathcal{C}^1)$ *on the $\mathcal{C}^1$-flags has r orbits $\Omega_1^1, \ldots, \Omega_r^1$ and that the action of* $\mathrm{Aut}(G_2, \mathcal{C}^2)$ *on the $\mathcal{C}^2$-flags has s orbits $\Omega_1^2, \ldots, \Omega_s^2$. Let Y be a tree amalgamation of G_1 and G_2 with respect to $\mathcal{C}^1$ and $\mathcal{C}^2$. For a vertex $v \in V(Y)$, consider the multiset Π_v of all pairs (i, j), for which there exists a $\mathcal{C}^1$-flag in Ω_i^1 and a $\mathcal{C}^2$-flag in Ω_j^2 that are identified into v with some identification map. If the identifying maps are such that Π_v is independent of v, then Y is a vertex-transitive graph.*

There is also a "bipartite version" of Proposition 2.5 that is similar to that of Proposition 2.4.

Proofs of Propositions 2.3–2.5 are left to the reader. Vertex-transitivity of tree amalgamations is not surprising, but explicit description of automorphisms is rather tedious. One can show even more than stated. It follows that the tree amalgamation Y is uniquely determined up to graph isomorphism and is independent of the choice of identifying maps (as long as they satisfy the necessary properties stated in the propositions).

Propositions 2.3–2.5 can be easily generalized. For example, the condition on planarity is not really needed. However, we shall only need the planar case. Because of planarity, it is easier to formulate the sufficient condition on how the identifying maps and the action of $\mathrm{Aut}(G, \mathcal{C})$ are related in order to achieve transitivity (it suffices that identifying maps are facial).

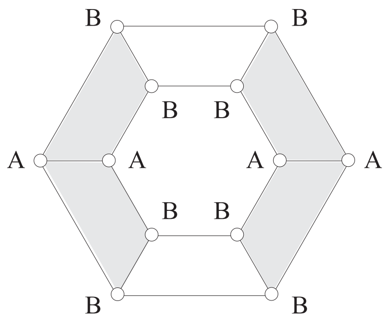

Figure 1: Vertex-transitive amalgamation with identifications of degree 2

The action of $\mathrm{Aut}(G, \mathcal{C})$ on $V(G)$ preserves identification degrees $\mu(x)$ of vertices with respect to the cover $\mathcal{C}$. An example with identifications of degree 1 and 2 yielding a vertex-transitive tree amalgamation is shown in Figure 1. Here, Proposition 2.4 is applied to the graph G shown in the figure and its cover consisting of four shaded quadrangles. The vertices labeled A have identification degree 2. Identifying maps satisfying (A1) interchange vertices labeled A and B, so they are symmetry increasing.

3 Planar Cayley maps

In this section we give some examples of planar Cayley graphs and relate the tree amalgamation with the amalgamated free product of groups.

Let Γ_1 and Γ_2 be groups. Suppose that C is a subgroup in both of them. The *free product with amalgamation* over C is the group denoted by $\Gamma_1 *_C \Gamma_2$, which is obtained from the free product $\Gamma_1 * \Gamma_2$ of Γ_1 and Γ_2 by identifying (amalgamating) subsets which correspond to cosets of C in Γ_1 and Γ_2. The free product with amalgamation occurs, for example, in the Seifert and van Kampen theorem (see, e.g., [14]), and is treated in many text books on group theory, e.g. [13, Chapter 4].

Suppose that for $i = 1, 2$, G_i is a Cayley graph of Γ_i (with respect to some finite generating set) and that $\mathcal{C}^i$ is the cover of G_i consisting of all cosets of C in Γ_i. For each coset we fix a representative g such that the coset is equal to gC. Let $p_i = [\Gamma_i : C]$, and let T be the (p_1, p_2)-semiregular tree. Then we define the identifying maps such that the coset $gC \subseteq V(G_1)$ is identified with a coset $hC \subseteq V(G_2)$ pointwise such that $gc \mapsto hc$ for every $c \in C$. In this case, the tree amalgamation of Cayley graphs G_1 and G_2 is isomorphic to the Cayley graph of the free product of Γ_1 and Γ_2 with amalgamation over C.

Let us consider, for example, the n-prism graph as the Cayley graph of the direct product $\Gamma_1 = \mathbb{Z}_n \times \mathbb{Z}_2$ corresponding to the presentation

$$\Gamma_1 = \langle a, b \mid a^2 = b^n = 1, ab = ba \rangle.$$

Let Γ_2 be the group isomorphic to Γ_1 with presentation

$$\Gamma_2 = \langle c, b \mid c^2 = b^n = 1, cb = bc \rangle.$$

The cosets of the common subgroup $C = \mathbb{Z}_n \times 0 = \langle b \mid b^n = 1 \rangle$ determine a facial cover consisting of two n-cycles. The tree amalgamation is the Cayley graph of the group

$$\Gamma_1 *_C \Gamma_2 = \langle a, b, c \mid a^2 = b^n = c^2 = 1, ab = ba, cb = bc \rangle \cong \mathbb{Z}_n \times (\mathbb{Z}_2 * \mathbb{Z}_2)$$

and is shown in Figure 2 for the case $n = 6$. This group has two ends.

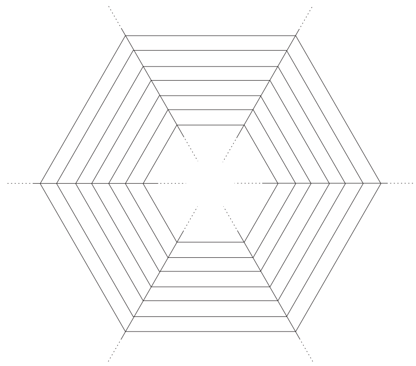

Figure 2: The amalgamation of two 6-prisms

An example with infinitely many ends is obtained, for instance, by taking the octahedron O and the icosahedron I, which are Cayley graphs of the following groups:

$$\Gamma_1 = \langle a, b, c \mid a^2 = b^3 = c^2 = abc = 1 \rangle \cong S_3$$

and

$$\Gamma_2 = \langle a', b, c' \mid a'^2 = b^3 = c'^3 = abc = 1 \rangle \cong A_4.$$

The generator b determines a subgroup of order 3, and its cosets form facial covers in I and O, respectively.

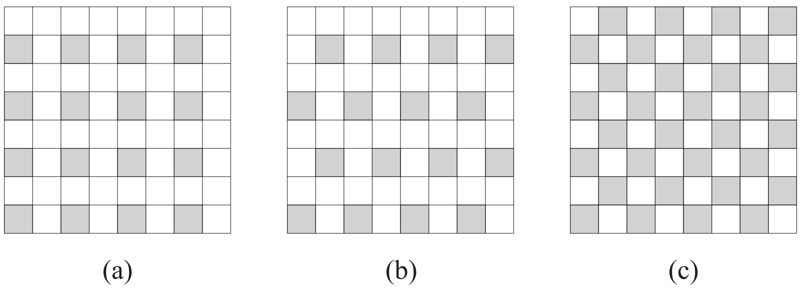

Figure 3: Strongly transitive facial covers of the integer lattice

An even more interesting example is obtained by starting with the group $\Gamma_2 = (\mathbb{Z}_2 * \mathbb{Z}_2)^2$ with the following presentation

$$\Gamma = \langle a_1, a_2, b_1, b_2 \mid a_i^2, b_j^2, (a_i b_j)^2, i, j = 1, 2 \rangle.$$

Its Cayley graph G with respect to the above presentation is isomorphic to the integer lattice graph. Then $C = \{0, a_1, b_1, a_1 b_1\}$ is a subgroup of Γ of infinite index whose cosets form a facial cover of G. It is represented in Figure 3(a) by shaded faces. The amalgamation of G with itself is the Cayley graph for the free product with amalgamation over C of two copies of Γ. It is represented in Figure 4. This graph is an example of a vertex-transitive planar graph with infinitely many thick ends. (Recall that an end is *thick* if it contains infinitely many pairwise disjoint rays.) This is also an example of the amalgamation construction where the degrees p_1, p_2 of the underlying tree T are infinite.

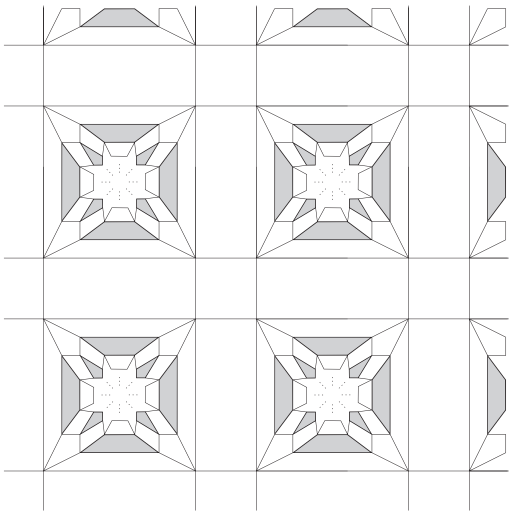

Figure 4: The amalgamation $\Gamma *_C \Gamma$

The integer lattice graph admits other strongly transitive facial covers. Besides the three strongly transitive covers shown in Figure 3, there are two others (up to symmetries of the plane): the first one contains all facial 4-cycles, while the second one consists of horizontal strips in every second row. There are several semitransitive facial covers which are not strongly transitive. Some of them are shown in Figure 5. These examples can be used to get further examples of infinite vertex-transitive planar graphs with thick ends.

Stallings [19] proved that every finitely generated group with infinitely many ends is either an amalgamation or an HNN-extension. Cayley graphs of HNN-extensions can also be expressed as tree amalgamations (with the

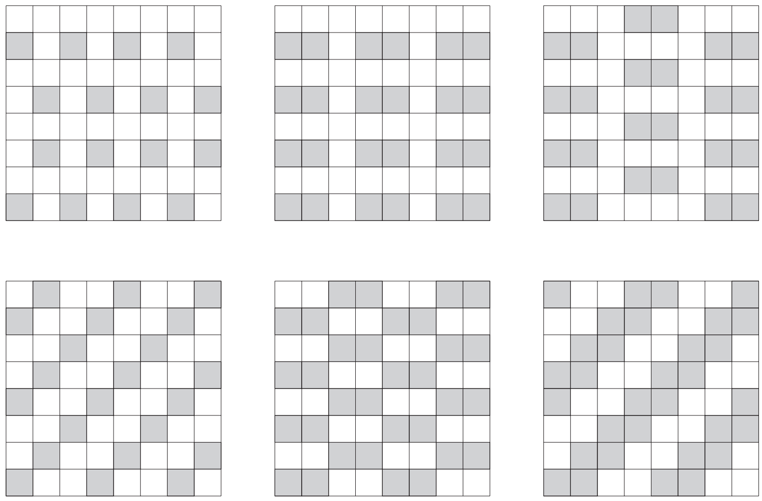

Figure 5: Some semitransitive facial covers of the integer lattice

degree of identification equal to 2). One example that is very close to HNN-extension structure is given in Figure 6. On the other hand, expressing the group as an amalgamation of two other groups does not guarantee that these groups are "simpler". However, Dunwoody [7, 6] proved that repeated application of Stallings' result leads to groups with at most one end in a finite number of steps if the group is finitely presented. This leads us to

Conjecture 3.1 *Let G be a planar Cayley graph. Then G can be obtained as the amalgamation of (one or more) Cayley planar graphs, each of which is either finite or infinite with one end only.*

In a conversation with Tomaž Pisanski, Tom Tucker, and Mark Watkins in 1988, we developed some arguments in support of this conjecture.

If G is a planar graph with at most one end, then its group of automorphisms acts either on the sphere (in which case G is finite), the Euclidean plane, or the hyperbolic plane. Groups acting on the Euclidean plane are known as *crystallographic groups*. They are easy to classify and well understood. Also, the groups acting on the hyperbolic plane are well understood. They are known as the *triangular groups* and have presentations of the form

$$\begin{aligned} T(r,s,t) &= \langle x,y,z \mid x^r = y^s = z^t = xyz = 1\rangle \\ &\cong \langle x,y \mid x^r = y^s = (xy)^t = 1\rangle. \end{aligned}$$

See, e.g., [5]. Now, Conjecture 3.1 is related to the following

Conjecture 3.2 *Let* Γ *be a (finitely generated) group of isometries of a surface that is homeomorphic to a subset of the 2-sphere. Then* Γ *is isomorphic to a free product with amalgamation and/or HNN-extension of finitely many groups, each of which is either finite or a subgroup of some crystallographic or some triangular group with one end.*

4 Tessellations of the Cantor sphere

By a *Cantor sphere* we mean any surface S that is homeomorphic to the 2-sphere with a copy of the Cantor set removed, endowed with the hyperbolic geometry with constant negative curvature, and such that the group of isometries $\mathrm{Aut}(S)$ is cocompact in S, i.e., $S/\mathrm{Aut}(S)$ is compact. All such surfaces are homeomorphic [18], but not always isometric to each other.

The set of ends of a Cantor sphere S is homeomorphic to the Cantor set. If G is any graph that is 2-cell embedded in S with finite faces, then its set of ends is also homeomorphic to the Cantor set [16], and if a group of automorphisms $\Gamma \leq \mathrm{Aut}(G)$ of G acts regularly on $V(G)$ and has only finitely many orbits, then Γ has the same set of ends. Hence, the isometry groups of Cantor spheres are infinitely ended.

Suppose that for $i = 1, 2$, G_i is a map realized as a metric space and let Γ_i be the group of isometries of the corresponding surface. Suppose that $\mathrm{Aut}(G_i, \mathcal{C}^i) \cap \Gamma_i$ acts transitively on the $\mathcal{C}^i$-angles. Suppose, moreover, that $\mathcal{C}^1$ and $\mathcal{C}^2$ are facial covers in G_1 and G_2, respectively, and that the corresponding curves $\gamma(S)$ $(S \in \mathcal{C}^1)$ and $\gamma(T)$ $(T \in \mathcal{C}^2)$ are all pairwise isometric, so that identifications can be made without changing the local metric. Then the tree amalgamation Y of G_1 and G_2 can be realized as a map whose group of isometries acts transitively on $V(Y)$. Most of the examples presented in this paper can be realized in this way.

Particularly nice examples are obtained if the maps G_1 and G_2 have constant Gaussian curvature. Such representations of maps can be obtained, for example, by using circle packing representations; see, e.g., [1, 3, 4, 17].

5 Tessellations with infinite faces

3-connected planar graphs have essentially unique embeddings in the plane in the sense that facial walks are uniquely determined. This was proved for finite graphs by Whitney [21] and extended to infinite locally finite graphs by Hotz [10]; see also Imrich [11] whose proof does not need local finiteness. In particular, in a 3-connected vertex-transitive graph G embedded in the plane, the set of the lengths of faces that are incident with a particular vertex are the same for all vertices. If G has infinite faces, then every vertex is incident with an infinite face.

Bonnington and Watkins [2] presented a 4-connected vertex-transitive planar graph with infinite faces. This example whose discovery is attributed to Grünbaum, can be obtained as the tree amalgamation of a cycle C_{4n} of length $4n$ (where $n \geq 2$) with the (4×2)-grid graph and with facial covers and identification maps as shown in Figure 6. The reason that infinite faces arise is in the fact that at least two identifications arise in each facial walk of both graphs used in the amalgamation.

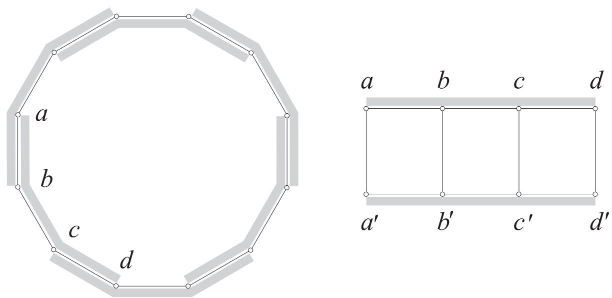

Figure 6: An amalgamation with infinite faces

In this section we exhibit additional vertex-transitive graphs with infinite faces.

Theorem 5.1 *For every integer k there exists a k-connected vertex-transitive planar graph such that every vertex is incident with at least k infinite faces.*

In the preprint version of [2], it was conjectured that a vertex-transitive planar graph with infinite faces cannot be 5-connected. It was also conjectured that in a 4-connected vertex-transitive planar graph each vertex lies on the boundary of at most one infinite face. Graphs of Theorem 5.1 disprove both of these conjectures. Another conjecture claiming that a precisely 3-connected vertex-transitive planar graph cannot have infinite faces appeared in [2]. We also disprove this conjecture by showing:

Theorem 5.2 *For every integer k there exists a vertex-transitive planar graph that is precisely 3-connected and such that every vertex is incident with at least k infinite faces.*

Proofs of Theorems 5.1 and 5.2 are deferred to the end of this section.

A tessellation of the plane is said to by of *type* $\{q_1 q_2 \ldots q_t\}$ if every vertex is of degree t and face lengths around every vertex in the clockwise order are equal to $q_1, \ldots, q_t$.

Lemma 5.3 *Let $k \geq 3$, $q_1 \geq 3$, and $q_2 \geq 3$ be integers. Then there exists a tessellation of Euclidean or hyperbolic plane of type $\{(q_1 q_2)^k\}$. Its graph $G_k(q_1, q_2)$ is $2k$-connected. The automorphism group of $G_k(q_1, q_2)$ acts transitively on flags corresponding to the faces of length q_1 and acts transitively on flags corresponding to the faces of length q_2. In particular, $G_k(q_1, q_2)$ is arc-transitive.*

Proof. Existence, uniqueness and transitivity properties of $G = G_k(q_1, q_2)$ are well-known. To prove the claim about connectivity, suppose that S is a vertex set of cardinality at most $2k - 1$ that separates two vertices x and y of G. Consider the $2k$ straight-ahead walks starting at x. They are pairwise disjoint and each of them gives rise to a ray (one-way infinite path) starting at x. So, at least one of them, call it R_x, does not intersect S. Similarly, there is a ray R_y starting at y that is disjoint from S. Since G has only one end, there are $2k$ disjoint paths from R_x to R_y. At least one of them is disjoint from S. However, this contradicts the assumption that S separates x and y.
□

Graphs $G_k(q_1, q_2)$ usually tesselate hyperbolic plane. The only one that is Euclidean is $G_3(3, 3)$, the tessellation of the plane with equilateral triangles. Two further Euclidean examples are obtained for $k = 2$, namely $G_2(3, 6)$ and $G_2(4, 4)$. Other examples exist for $k = 2$ but they are finite; $G_2(3, 3)$ is the octahedron, $G_2(3, 4)$ is the line graph of the 3-cube, and $G_2(3, 5)$ is the line graph of the dodecahedron.

The dual map of $G_k = G_k(q_1, q_2)$ is bipartite; the corresponding bipartition $\mathcal{F}_1, \mathcal{F}_2$ of the faces has all faces of length q_i in $\mathcal{F}_i$ ($i = 1, 2$). If v is a vertex of G_k and $F \in \mathcal{F}_i$ is a face incident with v, then the pair (v, F) is called an *$\mathcal{F}_i$-angle*. The collection of all $\mathcal{F}_i$-angles will be denoted by $\mathcal{A}_i$.

Lemma 5.4 *Suppose that k and q_1 are both multiples of an integer $s \geq 1$, $q_1 = r_1 s$, $k = rs$. Then there exists a mapping $\varphi : \mathcal{A}_1 \to \{1, \ldots, s\}$ such that the following holds:*

(a) *If $F \in \mathcal{F}_1$ and the facial walk of F in the clockwise direction is $v_1 v_2 \ldots v_{q_1} v_1$, then the cyclic order of $\varphi(v_1, F), \varphi(v_2, F), \ldots, \varphi(v_{q_1}, F)$ is equal to $(12 \ldots s)^{r_1}$.*

(b) *If v is a vertex and the faces incident to v are $F_1, \dots, F_{2k}$ in the clockwise cyclic order around v, where $F_1 \in \mathcal{F}_1$, then the cyclic order of $\varphi(v, F_1), \varphi(v, F_3), \dots, \varphi(v, F_{2k-1})$ is equal to $(s(s-1)\dots 1)^r$.*

Proof. The mapping φ can be constructed as follows. Let $a_0 = (v_0, F_0)$ be an angle in $\mathcal{A}_1$ and set $\varphi(a_0) = 1$. Let W be a walk in G_k starting at v_0. If W is a path, then by following W and applying conditions (a)–(b), we see that φ can be extended to all $\mathcal{F}_1$-angles incident with vertices on W. In this way we can extend φ to $\mathcal{A}_1$ in a unique way. However, an extension exists if and only if it is independent of the path chosen. This is equivalent to asking that for any closed walk W starting at v_0, after we return back to v_0, the forced values at the angles at v_0 match their initial values.

To prove this, let $W = v_0 v_1 \dots v_n v_0$ be a closed walk. Suppose that after following W and returning back to v_0, application of rules (a)–(b) along W yields the value t at the angle (v_0, F_0). Then we write $\psi(W) = t$.

If $v_{i+2} = v_i$ for some $i \in \{0, \dots, n-1\}$, let $W' = v_0 \dots v_{i-1} v_{i+2} \dots v_n v_0$. Then it is clear that $\psi(W') = \psi(W)$. Suppose now that the edge $v_i v_{i+1}$ belongs to a facial walk $F = v_i v_{i+1} u_1 \dots u_m v_i$. Let $W'' = v_0 \dots v_i u_m u_{m-1} \dots u_1 v_{i+1} \dots v_n v_0$. If $F \in \mathcal{F}_1$, then (a) implies that $\psi(W'') = \psi(W)$. On the other hand, if $F \in \mathcal{F}_2$, then following W'' around F yields only two values at angles in $\mathcal{A}_1$ that are adjacent to F. This implies that $\psi(W'') = \psi(W)$ in this case, too.

The plane is simply connected. Therefore, every closed walk W can be reduced to a trivial walk $W_0 = v_0$ by using the two types of changes ($W \mapsto W'$, $W \mapsto W''$, and their inverses) that we have described above. Consequently, $\psi(W) = \psi(W_0) = 1$, which we were to prove. □

Proof of Theorems 5.1 and 5.2. The graphs satisfying conclusions of both theorems are amalgamations. For the first amalgamation factor we take $G_1 = G_k(q_1, q_2)$, where $k = rs$ is the value from the theorems, $q_1 = r_1 s$ and $q_2 \geq 3$. We also assume that $r \geq k/2$, $r_1 \geq 2$, and $s \geq 3$. Let φ be a mapping from Lemma 5.4. The facial cover $\mathcal{C}^1$ of G_1 consists of segments of faces in $\mathcal{F}_1$ with consecutive angles $A_1 A_2 \dots A_s$ such that $\varphi(A_j) = j$ for all $j \in \{1, \dots, s\}$. Let us observe that $\Gamma_1 = \mathrm{Aut}(G_1, \mathcal{C}^1)$ has s orbits on $\mathcal{C}^1$-flags (v, C) ($v \in V(G_1)$, $C \in \mathcal{C}^1$).

As the second factor G_2 we take the $(s \times 2)$-grid graph $P_s \Box K_2$ with facial cover $\mathcal{C}^2$ consisting of two sets, each containing the vertices of a copy of P_s in the grid.

By Proposition 2.5, the amalgamation Y of $(G_1, \mathcal{C}^1)$ with $(G_2, \mathcal{C}^2)$, using the obvious facial identification maps, is a vertex-transitive graph. By

Proposition 2.2, Y is planar, and by Proposition 2.1 and Lemma 5.3 it is l-connected, where $l = \min\{2k, s\}$.

It can also be seen that every vertex of Y is incident with precisely $2r \geq k$ infinite faces. We leave the details to the reader.

The above construction gives an example satisfying the conclusion of Theorem 5.1 if $s \geq k$. If we take $s = 3$, then we obtain examples for Theorem 5.2. $\square$

Edge-transitive planar graphs are treated in [8]. Here we present interesting examples of some arc-transitive planar graphs with infinite faces.

Let T_r ($r \geq 3$) be the infinite r-regular tree. The Cartesian product $N_r = T_r \square K_2$ is planar and 2-connected. It can be obtained as a tree amalgamation of the 4-cycle, $G_1 = C_4$ (with consecutive vertices 0, 1, 2, 3), and the 2-path, $G_2 = K_2$ (with vertices 0 and 1), by taking the covers $\mathcal{C}^1 = \{\{0,1\},\{2,3\}\}$ and $\mathcal{C}^2 = \{\{0,1\},\dots,\{0,1\}\}$ (the multiset consisting of r copies of the pair $\{0,1\}$). The graph N_r is vertex-transitive and its automorphism group has two orbits on the edges. The first orbit E_1 contains all edges that correspond to the factor K_2 in the Cartesian product. The second orbit E_2 corresponds to all edges in both copies of T_r in the Cartesian product.

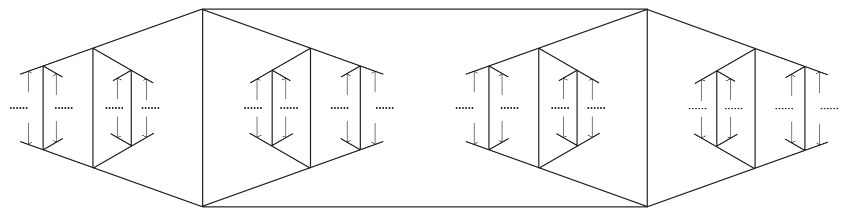

Figure 7: $T_3 \square K_2$ with infinite faces

For $i = 1, 2$, let $\mathcal{D}^i$ be the facial cover of N_r consisting of all pairs of vertices corresponding to edges in E_i. Finally, let M_r be the tree amalgamation of N_r (with cover $\mathcal{D}^1$) with itself (but this time taken with the cover $\mathcal{D}^2$). Let us observe that the degrees of identification are equal to 1 and r, respectively, so that (A1) is satisfied.

Theorem 5.5 *The graph M_r is a 2-connected $(r^2 - r)$-regular arc-transitive planar graph that admits an embedding in the plane under which all faces are infinite.*

Proof. It is easy to see that M_r is 2-connected, $(r^2 - r)$-regular, arc-transitive and planar. Bonnington and Watkins [2] studied embeddings of N_r in the plane. They proved that N_r admits an embedding in which all faces are infinite, see also Figure 7. By taking such embeddings when making identifications in the process of constructing the amalgamation and its planar embedding as presented in the proof of Proposition 2.2, all faces of the resulting limiting map M_r are clearly infinite. □

6 Conclusion

The purpose of this paper is two-fold. First, exhibiting various examples of "unusual" amalgamations, we obtain a variety of planar vertex-transitive graphs with infinitely many ends and with rather odd properties. Secondly, we show that amalgamations of plane graphs lead to nice tessellations of Cantor spheres – objects that would deserve to receive further attention because of their natural appearance in differential geometry, combinatorial group theory, graph theory and several other related fields.

Acknowledgements. The author is grateful to Tomaž Pisanski for his insight and his helpful remarks on the early version of this work. Several helpful discussions with Tom Tucker and Mark Watkins are also acknowledged.

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