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Preprint series, Vol. 43 (2005), 983

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ISSN 1318-4865

May 31, 2005

Ljubljana, May 31, 2005

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May 27, 2005

Abstract

Let Γ denote a bipartite Q-polynomial distance-regular graph with vertex set X, diameter $d \geq 3$ and valency $k \geq 3$. Let $\mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of column vectors with entries in $\mathbb{R}$ and rows indexed by X. For $z \in X$, let $\hat{z}$ denote the vector in $\mathbb{R}^X$ with a 1 in the z-coordinate, and 0 in all other coordinates. Fix $x, y \in X$ such that $\partial(x, y) = 2$, where ∂ denotes path-length distance. For $0 \leq i, j \leq d$ we define $w_{ij} = \sum \hat{z}$, where the sum is over all $z \in X$ such that $\partial(x, z) = i$ and $\partial(y, z) = j$. We define $W = \mathrm{span}\{w_{ij} \mid 0 \leq i, j \leq d\}$. In this paper we consider the space $MW = \mathrm{span}\{mw \mid m \in M, w \in W\}$, where M is the Bose-Mesner algebra of Γ. We observe MW is the minimal A-invariant subspace of $\mathbb{R}^X$ which contains W, where A is the adjacency matrix of Γ. We display a basis for MW that is orthogonal with respect to the dot product. We give the action of A on this basis. We show that the dimension of MW is $3d-3$ if Γ is 2-homogeneous, $3d-1$ if Γ is the antipodal quotient of the $2d$-cube, and $4d-4$ otherwise. We obtain our main result using Terwilliger's "balanced set" characterization of the Q-polynomial property.

1 Introduction

This paper is part of an ongoing effort to understand and classify the Q-polynomial bipartite distance-regular graphs [3]–[8]. We briefly summarize what is done so far. Throughout this introduction let Γ denote a Q-polynomial bipartite distance-regular graph with vertex set X, diameter $d \geq 3$ and valency $k \geq 3$ (see Section 2 for formal definitions). In [3] Caughman found the possible Q-polynomial orderings of the eigenvalues of Γ. In [4] he determined the irreducible modules for the Terwilliger algebra of Γ. In [5] he showed that if $d \geq 4$ and Γ is the quotient of an antipodal distance-regular graph, then Γ is the quotient of the $2d$-cube. In [6], [8] he considers the intersection numbers of Γ. It is known that, except for some special cases, these intersection numbers are determined by d and two complex scalars q and s^* [4, Lemma 15.1, Lemma 15.3]. In [6] he showed that q and s^* are real for $d \geq 4$ and in [8] he showed $s^* = 0$ for $d \geq 12$. In [7] he showed that with respect to any vertex, the distance-2 graph induced on the last subconstituent of Γ is distance-regular and Q-polynomial. See [2], [9]–[14], [16]–[18] for related topics.

In the present paper we obtain the following results about Γ. To state the results we use the following notation. Let $\mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of column vectors with entries in $\mathbb{R}$ and rows indexed by X. For $z \in X$, let $\hat{z}$ denote the vector in $\mathbb{R}^X$ with a 1 in the z-coordinate, and 0 in all other coordinates. We view $\mathbb{R}^X$ as a Euclidean space with inner product $\langle u, v\rangle = u^t v \ (u, v \in \mathbb{R}^X)$, where t denotes transpose. Fix $x, y \in X$ such that $\partial(x, y) = 2$, where ∂ denotes path-length distance. For $0 \leq i, j \leq d$ we define a vector $w_{ij} = \sum \hat{z}$, where the sum is over all $z \in X$ such that $\partial(x, z) = i$ and $\partial(y, z) = j$. We define $W = \text{span}\{w_{ij} \mid 0 \leq i, j \leq d\}$ and $MW = \text{span}\{mw \mid m \in M, w \in W\}$, where M denotes the Bose-Mesner algebra of Γ. We observe MW is the minimal A-invariant subspace of $\mathbb{R}^X$ that contains W, where A is the adjacency matrix of Γ. Our results are as follows.

We give an orthogonal basis for MW. We compute the action of A on this basis. We express the coefficients involved in terms of the intersection numbers of Γ and the dual eigenvalues for the given Q-polynomial structure. We show that the dimension of MW is $3d-3$ if Γ is 2-homogeneous, $3d-1$ if Γ is the antipodal quotient of the $2d$-cube, and $4d-4$ otherwise. We obtain our main result using Terwilliger's "balanced set" characterization of the Q-polynomial property. We remark that if Γ has intersection number $c_2 = 1$ then the results of this paper essentially follow from [19].

Our paper is organized as follows. In Sections 2–4 we give a brief introduction to the theory of distance-regular graphs. In Section 5 we define a certain partition of the vertex set of Γ. In Section 6 we use this partition to define the vectors w_{ij} and derive some of their properties. In Section 7 we define certain vectors that give an orthogonal basis of MW. In Sections 8–10 we study the action of A on this basis. In Section 11 we give more detailed information about this basis.

2 Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [1] for more background information.

Throughout this paper, $\Gamma = (X, R)$ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂, and diameter $d := \max\{\partial(x, y) | x, y \in X\}$. For a vertex $x \in X$ define $\Gamma_i(x)$ to be the set of vertices at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$. Let k denote a nonnegative integer. Then Γ is said to be *regular* with *valency* k, whenever $|\Gamma(x)| = k$ for all $x \in X$. We say Γ is *distance-regular*, whenever for all integers h, i, j $(0 \leq h, i, j \leq d)$, and all $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} := |\{z \mid z \in X,\, \partial(x,z) = i,\, \partial(y,z) = j\}|$$

is independent of x, y. The constants $p^h_{ij}\,(0 \le h,i,j \le d)$ are known as the *intersection numbers* of Γ. From now on we assume Γ is distance-regular. It is well known that the intersection numbers of Γ satisfy $p^h_{ij} = p^h_{ji}\,(0 \le h,i,j \le d)$. For convenience, set $c_i := p^i_{1i-1}\,(1 \le i \le d)$, $a_i := p^i_{1i}\,(0 \le i \le d)$, $b_i := p^i_{1i+1}\,(0 \le i \le d-1), k_i := p^0_{ii}\,(0 \le i \le d)$, and $c_0 = b_d = 0$. We observe Γ is regular with valency $k = b_0$, and that $a_0 = 0,\ c_1 = 1$. Moreover,

$$c_i + a_i + b_i = k \quad (0 \le i \le d). \tag{1}$$

By [1, p. 127],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \le i \le d) \qquad \text{and} \qquad k_h p^h_{ij} = k_j p^j_{ih} \quad (0 \le h,i,j \le d). \tag{2}$$

In the following lemma we cite some well known facts about the intersection numbers.

Lemma 2.1 ([1, p. 127, Lemma 4.1.7]) *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Then for all integers h,i,j $(0 \le h,i,j \le d)$ the following* (i), (ii) *hold.*

(i) *If one of h,i,j is greater than the sum of the other two, then $p^h_{ij} = 0$.*

(ii) *If one of h,i,j is equal to the sum of the other two, then $p^h_{ij} \ne 0$.*

Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra of matrices with entries in $\mathbb{R}$, whose rows and columns are indexed by X. For each integer i $(0 \le i \le d)$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \ne i \end{cases} \qquad (x, y \in X).$$

We refer to A_i as the i-th *distance matrix* of Γ. Let I and J denote the identity and the all ones matrix of $\mathrm{Mat}_X(\mathbb{R})$, respectively. Then

$$A_0 = I, \tag{3}$$

$$A_0 + A_1 + \cdots + A_d = J, \tag{4}$$

$$A_i^t = A_i \ \ (0 \le i \le d), \tag{5}$$

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \ \ (0 \le i,j \le d). \tag{6}$$

By (3), (5) and (6), the matrices $A_0, A_1, \ldots, A_d$ form the basis for a commutative semi-simple $\mathbb{R}$-algebra M, known as the *Bose-Mesner algebra* of Γ. By [15, Theorem 12.2.1], the algebra M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |X|^{-1} J, \tag{7}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{8}$$

$$E_i^t = E_i \ \ (0 \le i \le d), \tag{9}$$

$$E_i E_j = \delta_{ij} E_i \ \ (0 \le i,j \le d). \tag{10}$$

The matrices $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ, and E_0 is the *trivial* idempotent. Set $A := A_1$, and define the real numbers θ_i $(0 \leq i \leq d)$ by

$$A = \sum_{i=0}^{d} \theta_i E_i.$$

Then $AE_i = E_i A = \theta_i E_i$ $(0 \leq i \leq d)$, and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \ldots, \theta_d$ are distinct [1, p. 128]; they are known as the *eigenvalues* of Γ. For $0 \leq i \leq d$ we say the eigenvalue θ_i is *associated* with the primitive idempotent E_i.

Let E denote a primitive idempotent of Γ and let θ denote the associated eigenvalue. We define the real numbers θ^*_i $(0 \leq i \leq d)$ by

$$E = |X|^{-1} \sum_{i=0}^{d} \theta^*_i A_i.$$

We call the sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ the *dual eigenvalue sequence associated with* θ, E. The sequence is *trivial* whenever $E = E_0$ (in which case $\theta^*_0 = \theta^*_1 = \cdots = \theta^*_d = 1$). For convenience we let $\theta^*_{-1} = \theta^*_{d+1} = 0$.

Let $\mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of column vectors with entries in $\mathbb{R}$ and rows indexed by X. We observe $\text{Mat}_X(\mathbb{R})$ acts on $\mathbb{R}^X$ by left multiplication. For $z \in X$, let $\hat{z}$ denote the vector in $\mathbb{R}^X$ with a 1 in the z-coordinate, and 0 in all other coordinates. We view $\mathbb{R}^X$ as a Euclidean space with inner product

$$\langle u, v \rangle = u^t v \quad (u, v \in \mathbb{R}^X),$$

where t denotes transpose. Adopting this point of view we find $\{\hat{z} \mid z \in X\}$ is an orthonormal basis for $\mathbb{R}^X$.

In the following lemma, we cite a well known result about primitive idempotents and dual eigenvalue sequences.

Lemma 2.2 ([21, Lemma 1.1]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Pick any primitive idempotent E of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the associated dual eigenvalue sequence. Then the following* (i), (ii) *hold.*

(i) *For all $x, y \in X$,*

$$\langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta^*_i, \quad \text{where } i = \partial(x, y).$$

(ii) *The intersection numbers of Γ satisfy*

$$c_i \theta^*_{i-1} + a_i \theta^*_i + b_i \theta^*_{i+1} = \theta \theta^*_i \ (0 \leq i \leq d),$$

and $\theta^*_0 = \text{rank } E$.

3 The Q-polynomial property

In this section we recall the Q-polynomial property. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. The *Krein parameters* q^h_{ij} $(0 \leq h, i, j \leq d)$ of Γ are defined by

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q^h_{ij} E_h \quad (0 \leq i, j \leq d),$$

where $\circ$ denotes entrywise multiplication. We say Γ is *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents), whenever for all distinct integers i, j $(0 \leq i, j \leq d)$,

$$q^1_{ij} \neq 0 \text{ if and only if } |i - j| = 1.$$

Let E denote a nontrivial primitive idempotent of Γ. We say Γ is *Q-polynomial with respect to E* whenever there exists an ordering $E_0, E_1 = E, \ldots, E_d$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial. We have the following useful lemmas about the Q-polynomial property.

Lemma 3.1 ([1, Theorem 8.1.1]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the corresponding dual eigenvalue sequence. Suppose Γ is Q-polynomial with respect to E. Then $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are mutually distinct.*

Lemma 3.2 ([21, Theorem 3.3]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the corresponding dual eigenvalue sequence. Then the following* (i), (ii) *are equivalent.*

(i) Γ *is Q-polynomial with respect to E.*

(ii) $\theta^*_0 \neq \theta^*_i$ $(1 \leq i \leq d)$*; also for all integers h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$,*

$$\sum_{\substack{z \in X \\ \partial(x,z)=i \\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z \in X \\ \partial(x,z)=j \\ \partial(y,z)=i}} E\hat{z} \in \mathit{span}\{E\hat{x} - E\hat{y}\}.$$

Suppose (i), (ii) *hold. Then for all integers h, i, j $(1 \leq h \leq d)$, $(0 \leq i, j \leq d)$ and for all $x, y \in X$ such that $\partial(x, y) = h$,*

$$\sum_{\substack{z \in X \\ \partial(x,z)=i \\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z \in X \\ \partial(x,z)=j \\ \partial(y,z)=i}} E\hat{z} = p^h_{ij} \frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h} (E\hat{x} - E\hat{y}).$$

4 The bipartite case

Let Γ denote a distance-regular graph. Recall Γ is *bipartite* whenever $a_i = 0$ for $0 \le i \le d$. For the rest of this paper we assume Γ is bipartite. In order to avoid trivialities we assume the valency $k \ge 3$. In this section we recall some basic formula. Setting $a_i = 0$ in (1) we find

$$b_i + c_i = k \quad (0 \le i \le d). \tag{11}$$

The following two lemmas will be useful.

Lemma 4.1 *Let Γ denote a bipartite distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$. Then the following* (i)–(iv) *hold.*

(i) $p^1_{i,i-1} = p^1_{i-1,i} = k_i c_i / k \quad (1 \le i \le d)$;

(ii) $p^2_{i,i-2} = p^2_{i-2,i} = k_i c_{i-1} c_i / (k(k-1)) \quad (2 \le i \le d)$;

(iii) $p^2_{ii} = k_i(c_i(b_{i-1} - 1) + b_i(c_{i+1} - 1))/(k(k-1)) \quad (1 \le i \le d-1)$;

(iv) $p^2_{dd} = k_d(b_{d-1} - 1)/(k-1)$.

Proof. (i), (ii) Immediate from [1, Lemma 4.1.7] and since $b_1 = k - 1$.
(iii) By [1, p. 127, Equation (10)] and $a_i = 0$ $(0 \le i \le d)$, we obtain $b_{i-1}p^1_{i-1,i} + c_{i+1}p^1_{i,i+1} = k_i + (k-1)p^2_{ii}$. The result now follows from (i) above and (2).
(iv) We observe $p^2_{dd} = k_d - p^2_{d,d-2}$. The result now follows from (ii) above, $c_d = k$ and (11). ∎

Lemma 4.2 *Let Γ denote a bipartite distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$. Then the following* (i)–(iv) *hold.*

(i) $p^2_{i,i-2} \ne 0$, $p^2_{i-2,i} \ne 0$ $(2 \le i \le d)$;

(ii) $p^2_{00} = 0$ *and* $p^2_{ii} \ne 0$ $(1 \le i \le d-1)$;

(iii) $p^2_{dd} = 0$ *if and only if* $b_{d-1} = 1$;

(iv) $p^2_{ij} = 0$ *if* $|i-j| \notin \{0, 2\}$ $(0 \le i, j \le d)$.

Proof. (i) Immediate from Lemma 4.1(ii).
(ii) It is clear that $p^2_{00} = 0$. Suppose there exists an integer i $(1 \le i \le d-1)$ such that $p^2_{ii} = 0$. Then $b_{i-1} = c_{i+1} = 1$ by Lemma 4.1(iii). Recall $b_{i-1} \ge b_i$ and $c_{i+1} \ge c_i$ by [1, Prop. 4.1.6(i)], implying $b_i = c_i = 1$. But now $k = 2$ in view of (11), a contradiction.
(iii) Immediate from Lemma 4.1(iv).
(iv) If $|i-j| \ge 3$, then $p^2_{ij} = 0$ by Lemma 2.1(i). If $|i-j| = 1$, then $p^2_{ij} = 0$; otherwise Γ has a cycle of odd length, contradicting our assumption that Γ is bipartite. ∎

5 The subsets D^i_j

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. In this section we define a certain partition of X that we will find useful.

Definition 5.1 *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$. For all integers i, j we define $D^i_j = D^i_j(x, y)$ by*

$$D^i_j = \{z \in X \mid \partial(x, z) = i \text{ and } \partial(y, z) = j\}.$$

We observe $D^i_j = \emptyset$ unless $0 \leq i, j \leq d$. In the following two lemmas we derive some properties of the sets D^i_j.

Lemma 5.2 *With reference to Definition 5.1, the following* (i), (ii) *hold for $0 \leq i, j \leq d$.*

(i) *$|D^i_j| = p^2_{ij}$;*

(ii) *$D^i_j = \emptyset$ if and only if $p^2_{ij} = 0$.*

PROOF. (i) Immediate from the definition of p^2_{ij} and D^i_j.
(ii) Immediate from (i) above. ∎

Lemma 5.3 *With reference to Definition 5.1, the following* (i)–(iii) *hold for $v \in D^1_1$.*

(i) *For $1 \leq i \leq d-1$ and $u \in D^{i-1}_{i+1} \cup D^{i+1}_{i-1}$ we have $\partial(u, v) = i$.*

(ii) *For $1 \leq i \leq d-1$ and $u \in D^i_i$ we have $\partial(u, v) \in \{i-1, i+1\}$.*

(iii) *For $u \in D^d_d$ we have $\partial(u, v) = d-1$.*

PROOF. (i) Assume $u \in D^{i-1}_{i+1}$. Then $\partial(x, u) = i-1$ and $\partial(y, u) = i+1$. The result now follows from the triangle inequality. If $u \in D^{i+1}_{i-1}$ the proof is similar.
(ii) Observe $\partial(u, v) \in \{i-1, i, i+1\}$ by the triangle inequality, and $\partial(u, v) \neq i$ since $a_i = 0$.
(iii) Similar to the proof of (ii) above. ∎

The following lemma will be useful.

Lemma 5.4 ([9, Lemma 15]) *With reference to Definition 5.1, the following* (i)–(iii) *hold.*

(i) *For $1 \leq i \leq d-1$, each $z \in D^{i-1}_{i+1}$ (resp. D^{i+1}_{i-1}) is adjacent to*
  (a) precisely c_{i-1} vertices in D^{i-2}_i (resp. D^i_{i-2}),
  (b) precisely b_{i+1} vertices in D^i_{i+2} (resp. D^{i+2}_i),
  (c) precisely $b_{i-1} - b_{i+1}$ vertices in D^i_i,

and no other vertices in X.

(ii) *For $1 \leq i \leq d-1$, each $z \in D_i^i$ is adjacent to*

(a) precisely	$\lvert\Gamma(z) \cap D_{i-1}^{i-1}\rvert$	*vertices in* D_{i-1}^{i-1},
(b) precisely	$c_i - \lvert\Gamma(z) \cap D_{i-1}^{i-1}\rvert$	*vertices in* D_{i+1}^{i-1},
(c) precisely	$c_i - \lvert\Gamma(z) \cap D_{i-1}^{i-1}\rvert$	*vertices in* D_{i-1}^{i+1},
(d) precisely	$k - 2c_i + \lvert\Gamma(z) \cap D_{i-1}^{i-1}\rvert$	*vertices in* D_{i+1}^{i+1},

and no other vertices in X.

(iii) *Each $z \in D_d^d$ is adjacent to precisely k vertices in D_{d-1}^{d-1}, and no other vertices in X.*

6 The vectors w_{ij}, w_{ii}^+ and w_{ii}^-

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. In this section we define certain vectors in $\mathbb{R}^X$ that are associated with the sets D_j^i from Definition 5.1.

Definition 6.1 *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$. With reference to Definition 5.1, for all integers i, j we define a vector $w_{ij} = w_{ij}(x, y)$ by*

$$w_{ij} = \sum_{z \in D_j^i} \hat{z}.$$

Observe $w_{ij} = 0$ unless $0 \leq i, j \leq d$. We define

$$W = \text{span}\{w_{ij} \mid 0 \leq i, j \leq d\}.$$

The following two lemmas follow immediately from Definition 6.1 and Lemma 5.2.

Lemma 6.2 *With reference to Definition 6.1, the following* (i), (ii) *hold for $0 \leq i, j \leq d$.*

(i) $\|w_{ij}\|^2 = p_{ij}^2$;

(ii) $w_{ij} = 0$ *if and only if* $p_{ij}^2 = 0$. ∎

Lemma 6.3 *With reference to Definition 6.1, the vectors $\{w_{ij} \mid 0 \leq i, j \leq d,\ p_{ij}^2 \neq 0\}$ form an orthogonal basis for W.* ∎

We now define a subspace $W^\perp$ of $\mathbb{R}^X$.

Definition 6.4 *With reference to Definition 6.1, consider the subspace*

$$MW = \text{span}\{mw \mid m \in M,\ w \in W\},$$

where M is the Bose-Mesner algebra of Γ. We observe MW is the minimal A-invariant subspace of $\mathbb{R}^X$ that contains W, where A is the adjacency matrix of Γ. We let $W^\perp$ denote the orthogonal complement of W in MW. We observe

$$MW = W + W^\perp \qquad \textit{(orthogonal direct sum).}$$

Our goal is to give an orthogonal basis for MW, in the case where Γ is Q-polynomial. In light of Lemma 6.3 it suffices to give an orthogonal basis for $W^{\perp}$. Towards this purpose we make a definition.

Definition 6.5 *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$. With reference to Definition 5.1, for an integer i we define vectors $w_{ii}^{+} = w_{ii}^{+}(x, y)$ and $w_{ii}^{-} = w_{ii}^{-}(x, y)$ by*

$$w_{ii}^{+} = \sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1| \, \hat{z}, \qquad\qquad w_{ii}^{-} = \sum_{z \in D_i^i} |\Gamma(z) \cap D_{i-1}^{i-1}| \, \hat{z}.$$

We observe $w_{ii}^{+} = w_{ii}^{-} = 0$ unless $1 \leq i \leq d$. Furthermore, $w_{11}^{+} = w_{11}$, $w_{11}^{-} = 0$, $w_{dd}^{+} = c_2 w_{dd}$, $w_{dd}^{-} = k w_{dd}$, and $w_{22}^{+} = w_{22}^{-}$.

In the next two sections we will consider the following situation.

Definition 6.6 *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and adjacency matrix A. Fix vertices $x, y \in X$ such that $\partial(x, y) = 2$. Let the vectors $w_{ij}, w_{ii}^{+}, w_{ii}^{-}$ be as in Definitions 6.1 and 6.5. Let the subspaces W, $W^{\perp}$ be as in Definitions 6.1 and 6.4.*

7 The vectors $\widetilde{w}_{ii}$

With reference to Definition 6.6, in this section we define some vectors that will give an orthogonal basis for $W^{\perp}$ when Γ is Q-polynomial. We will need the following lemma.

Lemma 7.1 *With reference to Definition 6.6, the following* (i), (ii) *hold for $2 \leq i \leq d-1$.*

(i) $\langle w_{ii}^{+}, w_{ii} \rangle = k_i c_i (b_{i-1} - 1)/k_2$;

(ii) $\|w_{ii}^{+}\|^2 = k_i c_i (c_2(b_{i-1} - 1) - (c_2 - 1) b_i)/k_2$.

PROOF. (i) Observe that $\langle w_{ii}^{+}, w_{ii} \rangle = \sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1|$. Hence $\langle w_{ii}^{+}, w_{ii} \rangle$ is equal to the number of ordered pairs (v, z), where $v \in D_1^1$, $z \in D_i^i$, and $\partial(v, z) = i - 1$. In order to find this number, we fix $v \in D_1^1$ and observe $|\Gamma_i(x) \cap \Gamma_{i-1}(v)| = p^1_{i,i-1}$. By Lemma 5.3 we find D_{i-2}^i is contained in $\Gamma_i(x) \cap \Gamma_{i-1}(v)$, and $\Gamma_i(x) \cap \Gamma_{i-1}(v)$ is contained in $D_{i-2}^i \cup D_i^i$. Therefore, $D_i^i \cap \Gamma_{i-1}(v) = \big(\Gamma_i(x) \cap \Gamma_{i-1}(v)\big) \setminus D_{i-2}^i$. Since $|D_{i-2}^i| = p^2_{i,i-2}$ by Lemma 5.2(i), we have $|D_i^i \cap \Gamma_{i-1}(v)| = p^1_{i,i-1} - p^2_{i,i-2}$. Finally, since $|D_1^1| = c_2$, we have $\langle w_{ii}^{+}, w_{ii} \rangle = c_2(p^1_{i,i-1} - p^2_{i,i-2})$. The result now follows from Lemma 4.1(i),(ii) and from (2), (11).

(ii) Observe that

$$\|w_{ii}^{+}\|^2 = \sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1|^2 = \sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1| + \sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1|(|\Gamma_{i-1}(z) \cap D_1^1| - 1).$$

By (i) above, $\sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1| = \langle w_{ii}^{+}, w_{ii} \rangle = k_i c_i (b_{i-1} - 1)/k_2$. Furthermore, the number $\sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1|(|\Gamma_{i-1}(z) \cap D_1^1| - 1)$ is equal to the number of ordered triples

(v_1, v_2, z), where $v_1, v_2 \in D_1^1$, $v_1 \neq v_2$, $z \in D_i^i$, and $\partial(v_1, z) = \partial(v_2, z) = i - 1$. In order to find this number we fix $v_1, v_2 \in D_1^1$, $v_1 \neq v_2$. From the proof of (i) above we find $|D_i^i \cap \Gamma_{i-1}(v_1)| = p^1_{i,i-1} - p^2_{i,i-2}$. By Lemma 5.3 we find $\Gamma_{i-1}(v_1) \cap \Gamma_{i+1}(v_2)$ is contained in D_i^i, and that $|\Gamma_{i-1}(v_1) \cap \Gamma_{i+1}(v_2)| = p^2_{i+1,i-1}$. Therefore, $|D_i^i \cap \Gamma_{i-1}(v_1) \cap \Gamma_{i-1}(v_2)| = p^1_{i,i-1} - p^2_{i,i-2} - p^2_{i+1,i-1}$. Finally, we can choose $v_1, v_2 \in D_1^1$, $v_1 \neq v_2$, in $c_2(c_2 - 1)$ different ways, implying $\sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^1|(|\Gamma_{i-1}(z) \cap D_1^1| - 1) = c_2(c_2 - 1)(p^1_{i,i-1} - p^2_{i,i-2} - p^2_{i+1,i-1})$. Hence we have

$$\|w_{ii}^+\|^2 = \frac{k_i c_i (b_{i-1} - 1)}{k_2} + c_2(c_2 - 1)(p^1_{i,i-1} - p^2_{i,i-2} - p^2_{i+1,i-1}).$$

The result now follows from Lemma 4.1(i),(ii) and (2). ∎

Definition 7.2 *With reference to Definition 6.6, for* $2 \leq i \leq d - 1$ *we define*

$$\widetilde{w}_{ii} = w_{ii}^+ - \lambda_i w_{ii},$$

where

$$\lambda_i = \frac{k_i c_i (b_{i-1} - 1)}{k_2 p^2_{ii}}.$$

Definition 7.3 *With reference to Definition 6.6, for* $2 \leq i \leq d - 1$ *we define*

$$\Delta_i = (b_{i-1} - 1)(c_{i+1} - 1) - (c_2 - 1)p^i_{2i}.$$

We now prove that the vectors w_{ii} and $\widetilde{w}_{ii}$ $(2 \leq i \leq d - 1)$ are orthogonal.

Lemma 7.4 *With reference to Definitions 6.6, 7.2 and 7.3, the following* (i), (ii) *hold for* $2 \leq i \leq d - 1$.

(i) $\langle \widetilde{w}_{ii}, w_{ii} \rangle = 0$;

(ii) $\|\widetilde{w}_{ii}\|^2 = k_i b_i c_i \Delta_i / (k_2 p^i_{2i})$.

Proof. (i) In the expression $\langle \widetilde{w}_{ii}, w_{ii} \rangle$ eliminate $\widetilde{w}_{ii}$ using Definition 7.2 and evaluate the result using Lemma 6.2(i) and Lemma 7.1(i).

(ii) By Definition 7.2 and (i) above we have

$$\|\widetilde{w}_{ii}\|^2 = \langle \widetilde{w}_{ii}, w_{ii}^+ - \lambda_i w_{ii} \rangle = \langle \widetilde{w}_{ii}, w_{ii}^+ \rangle = \|w_{ii}^+\|^2 - \lambda_i \langle w_{ii}, w_{ii}^+ \rangle.$$

The result now follows from Lemma 7.1, Lemma 4.1(iii) and (2). ∎

Lemma 7.5 *With reference to Definitions 6.6, 7.2 and 7.3, the following* (i)–(iii) *are equivalent for* $2 \leq i \leq d - 1$.

(i) *The vectors* w_{ii}^+ *and* w_{ii} *are linearly dependent;*

(ii) $\widetilde{w}_{ii} = 0$;

(iii) $\Delta_i = 0$.

Proof. (i) → (ii) Observe w_{ii} and $\widetilde{w}_{ii}$ are linearly dependent in view of Definition 7.2. The result now follows from Lemma 7.4(i) and since $w_{ii} \neq 0$.

(ii) → (i) Immediate from Definition 7.2.

(ii) ↔ (iii) Immediate from Lemma 7.4(ii). ∎

8 The action - part I

With reference to Definition 6.6, in this section we compute the images of the vectors $w_{ij}, w_{ii}^+, w_{ii}^-$ under the action of A.

Lemma 8.1 *With reference to Definition 6.6, the following* (i), (ii) *hold.*

(i) $Aw_{02} = w_{13} + w_{11}$,
$Aw_{i-1,i+1} = b_i w_{i-2,i} + c_i w_{i,i+2} + c_i w_{ii} - w_{ii}^-$ $(2 \le i \le d-2)$,
$Aw_{d-2,d} = b_{d-1} w_{d-3,d-1} + c_{d-1} w_{d-1,d-1} - w_{d-1,d-1}^-$.

(ii) $Aw_{20} = w_{31} + w_{11}$,
$Aw_{i+1,i-1} = b_i w_{i,i-2} + c_i w_{i+2,i} + c_i w_{ii} - w_{ii}^-$ $(2 \le i \le d-2)$,
$Aw_{d,d-2} = b_{d-1} w_{d-1,d-3} + c_{d-1} w_{d-1,d-1} - w_{d-1,d-1}^-$.

PROOF. (i) For $z \in X$ and for each equation we show that the z-coordinate of both sides agree. For the first equation this is routinely checked, so consider the second equation. By Definition 6.5, the z-coordinate of w_{ii}^- is $|\Gamma(z) \cap D_{i-1}^{i-1}|$ if $z \in D_i^i$, and 0 if $z \notin D_i^i$. For $0 \le r, s \le d$ the z-coordinate of w_{rs} is 1 if $z \in D_s^r$, and 0 if $z \notin D_s^r$. The z-coordinate of $Aw_{i-1,i+1}$ is $|\Gamma(z) \cap D_{i+1}^{i-1}|$. Moreover, by Lemma 5.4 we find $|\Gamma(z) \cap D_{i+1}^{i-1}| = b_i$ if $z \in D_i^{i-2}$, $|\Gamma(z) \cap D_{i+1}^{i-1}| = c_i$ if $z \in D_{i+2}^i$, $|\Gamma(z) \cap D_{i+1}^{i-1}| = c_i - |\Gamma(z) \cap D_{i-1}^{i-1}|$ if $z \in D_i^i$, and $|\Gamma(z) \cap D_{i+1}^{i-1}| = 0$ for all other $z \in X$. By these comments, the z-coordinate of both sides agree. This proves the second equation and the third equation is similarly obtained.

(ii) Similar to the proof of (i) above. ∎

Lemma 8.2 *With reference to Definition 6.6, the following equations hold:*

$Aw_{11} = c_2(w_{02} + w_{20}) + w_{22}^-$,
$Aw_{22} = (b_1 - b_3)(w_{13} + w_{31}) + (k-2)w_{11} + w_{33}^-$,
$Aw_{ii} = (b_{i-1} - b_{i+1})(w_{i-1,i+1} + w_{i+1,i-1}) + (k - 2c_{i-1})w_{i-1,i-1} + w_{i-1,i-1}^- + w_{i+1,i+1}^-$ $(3 \le i \le d-1)$,
$Aw_{dd} = (k - 2c_{d-1})w_{d-1,d-1} + w_{d-1,d-1}^-$.

PROOF. For $z \in X$ and for each equation we show that the z-coordinate of both sides agree. Consider the third equation. By Definition 6.5, for $1 \le r \le d$ the z-coordinate of w_{rr}^- is $|\Gamma(z) \cap D_{r-1}^{r-1}|$ if $z \in D_r^r$, and 0 if $z \notin D_r^r$. For $0 \le r, s \le d$ the z-coordinate of w_{rs} is 1 if $z \in D_s^r$, and 0 if $z \notin D_s^r$. The z-coordinate of Aw_{ii} is $|\Gamma(z) \cap D_i^i|$. By Lemma 5.4 we find $|\Gamma(z) \cap D_i^i| = b_{i-1} - b_{i+1}$ if $z \in D_{i+1}^{i-1} \cup D_{i-1}^{i+1}$, $|\Gamma(z) \cap D_i^i| = k - 2c_{i-1} + |\Gamma(z) \cap D_{i-2}^{i-2}|$ if $z \in D_{i-1}^{i-1}$, and $|\Gamma(z) \cap D_i^i| = 0$ for all other $z \in X \setminus D_{i+1}^{i+1}$. By these comments, the z-coordinate of both sides agree. This proves the third equation and the other equations are similarly obtained. ∎

Lemma 8.3 *With reference to Definition 6.6, the following equations hold:*

$Aw_{22}^+ = c_2(c_2 - 1)(w_{13} + w_{31}) + (b_2 + c_2(c_2 - 2))w_{11} + c_2 w_{33}^+$,
$Aw_{ii}^+ = c_2(c_i - c_{i-1})(w_{i-1,i+1} + w_{i+1,i-1}) + c_2(c_i - 2c_{i-1})w_{i-1,i-1} + b_i w_{i-1,i-1}^+ + c_2 w_{i-1,i-1}^- + c_i w_{i+1,i+1}^+$ $(3 \le i \le d-1)$.

Proof. For $z \in X$ and for each equation we show that the z-coordinate of both sides agree. Consider the second equation. By Definition 6.5, the z-coordinate of $w^-_{i-1,i-1}$ is $|\Gamma(z) \cap D^{i-2}_{i-2}|$ if $z \in D^{i-1}_{i-1}$, and 0 if $z \notin D^{i-1}_{i-1}$. Similarly, by Definition 6.5, for $1 \le r \le d$ the z-coordinate of w^+_{rr} is $|\Gamma_{r-1}(z) \cap D^1_1|$ if $z \in D^r_r$, and 0 if $z \notin D^r_r$. For $0 \le r, s \le d$ the z-coordinate of w_{rs} is 1 if $z \in D^r_s$, and 0 if $z \notin D^r_s$. The z-coordinate of Aw^+_{ii} is

$$\sum_{v \in \Gamma(z) \cap D^i_i} |\Gamma_{i-1}(v) \cap D^1_1| = \sum_{u \in D^1_1} |\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i|. \tag{12}$$

Observe that if $\Gamma(z) \cap D^i_i \neq \emptyset$ then $z \in D^{i+1}_{i-1} \cup D^{i-1}_{i+1} \cup D^{i-1}_{i-1} \cup D^{i+1}_{i+1}$. In view of this we split the argument into four cases: $z \in D^{i+1}_{i-1}$, $z \in D^{i-1}_{i+1}$, $z \in D^{i-1}_{i-1}$, and $z \in D^{i+1}_{i+1}$.

First assume $z \in D^{i+1}_{i-1}$. Recall $|D^1_1| = c_2$ and pick any $u \in D^1_1$. Observe that, by Lemma 5.3(i),(ii), $\Gamma_{i-1}(u) \cap \Gamma(z)$ is contained in $D^i_{i-2} \cup D^i_i$, and $\partial(u, z) = i$. Hence $|\Gamma_{i-1}(u) \cap \Gamma(z)| = c_i$. By Lemma 5.4(i), z has exactly c_{i-1} neighbours in D^i_{i-2}. Furthermore, each of these neighbours is at distance $i-1$ from u by Lemma 5.3(i). Thus $|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i| = c_i - c_{i-1}$. By these comments and (12) we find the z-coordinate of Aw^+_{ii} is $c_2(c_i - c_{i-1})$.

Next assume $z \in D^{i-1}_{i+1}$. Interchanging x, y in the previous case we routinely find the z-coordinate of Aw^+_{ii} is $c_2(c_i - c_{i-1})$.

Next assume $z \in D^{i+1}_{i+1}$. For $u \in D^1_1$, if $|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i| \neq 0$, then $\partial(u, z) = i$. In this case we have, by Lemma 5.3, $|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i| = |\Gamma_{i-1}(u) \cap \Gamma(z)| = c_i$. Therefore,

$$\sum_{u \in D^1_1} |\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i| = \sum_{u \in D^1_1 \cap \Gamma_i(z)} c_i \; = \; c_i |\Gamma_i(z) \cap D^1_1|. \tag{13}$$

By these comments and (12), (13), the z-coordinate of Aw^+_{ii} is $c_i|\Gamma_i(z) \cap D^1_1|$.

Finally, assume $z \in D^{i-1}_{i-1}$. Let $u \in D^1_1$ and observe that $\partial(u, z) \in \{i-2, i\}$ by Lemma 5.3(ii). We now evaluate $|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i|$. If u is at distance $i-2$ from z, then, by Lemma 5.4(ii),

$$|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i| = |\Gamma(z) \cap D^i_i| = k - 2c_{i-1} + |\Gamma(z) \cap D^{i-2}_{i-2}|. \tag{14}$$

If u is at distance i from z, then $|\Gamma_{i-1}(u) \cap \Gamma(z)| = c_i$. Observe that, by Lemmas 5.3 and 5.4(ii), $|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^{i-2}_{i-2}| = |\Gamma(z) \cap D^{i-2}_{i-2}|$ and $|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^{i-2}_i| = c_{i-1} - |\Gamma(z) \cap D^{i-2}_{i-2}|$. Hence

$$|\Gamma_{i-1}(u) \cap \Gamma(z) \cap D^i_i| = c_i + |\Gamma(z) \cap D^{i-2}_{i-2}| - 2c_{i-1}. \tag{15}$$

We now evaluate the z-coordinate of Aw^+_{ii}. Observe there are $|\Gamma_{i-2}(z) \cap D^1_1|$ vertices in D^1_1 which are at distance $i-2$ from z. The other $c_2 - |\Gamma_{i-2}(z) \cap D^1_1|$ vertices from D^1_1 are at distance i from z by Lemma 5.3(ii). Thus, by (12), (14) and (15), the z-coordinate of Aw^+_{ii} is

$$|\Gamma_{i-2}(z) \cap D^1_1| \big(k - 2c_{i-1} + |\Gamma(z) \cap D^{i-2}_{i-2}|\big) + \big(c_2 - |\Gamma_{i-2}(z) \cap D^1_1|\big)\big(c_i + |\Gamma(z) \cap D^{i-2}_{i-2}| - 2c_{i-1}\big) =$$

$$b_i |\Gamma_{i-2}(z) \cap D^1_1| + c_2 |\Gamma(z) \cap D^{i-2}_{i-2}| + c_2(c_i - 2c_{i-1}).$$

By these comments, the z-coordinate of both sides agree. This proves the second equation and the first equation is similarly obtained. ∎

9 Dependencies

With reference to Definition 6.6, in this section we show that if Γ is Q-polynomial, then the vectors w_{ii}, w_{ii}^+ and w_{ii}^- are linearly dependent for $2 \le i \le d-1$.

Theorem 9.1 *With reference to Definition 6.6, assume Γ is Q-polynomial with respect to a primitive idempotent E. Let $\theta_0^*, \theta_1^*, \dots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Then for $2 \le i \le d-1$ we have*

$$w_{ii}^- = \alpha_i w_{ii} + \beta_i w_{ii}^+,$$

where

$$\alpha_i = c_i \frac{(\theta_0^* - \theta_i^*)(\theta_3^* - \theta_{i+1}^*) - (\theta_1^* - \theta_{i-1}^*)(\theta_2^* - \theta_i^*)}{(\theta_0^* - \theta_i^*)(\theta_{i-1}^* - \theta_{i+1}^*)} \quad \text{and} \quad \beta_i = \frac{\theta_1^* - \theta_3^*}{\theta_{i-1}^* - \theta_{i+1}^*}.$$

Proof. Observe that the denominators of the above expressions are nonzero by Lemma 3.1. The result holds for $i = 2$ since $\alpha_2 = 0$, $\beta_2 = 1$, and since $w_{22}^- = w_{22}^+$ by Definition 6.5. Next assume $3 \le i \le d-1$ and pick a vertex $v \in D_i^i$. By Lemma 3.2, we find

$$\sum_{\substack{z \in X \\ \partial(x,z)=1 \\ \partial(v,z)=i-1}} E\hat{z} - \sum_{\substack{z \in X \\ \partial(x,z)=i-1 \\ \partial(v,z)=1}} E\hat{z} = c_i \frac{\theta_1^* - \theta_{i-1}^*}{\theta_0^* - \theta_i^*}(E\hat{x} - E\hat{v}). \tag{16}$$

To finish the proof take the inner product of (16) with $E\hat{y}$ and evaluate the result using Lemma 2.2(i). ∎

For the rest of this paper we will consider the following situation.

Definition 9.2 *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $d \ge 3$, valency $k \ge 3$ and adjacency matrix A. Assume Γ is Q-polynomial with respect to a primitive idempotent E, and let $\theta_0^*, \theta_1^*, \dots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Fix vertices $x, y \in X$ such that $\partial(x,y) = 2$. Let the vectors $w_{ij}, w_{ii}^+, w_{ii}^-, \widetilde{w}_{ii}$ be as in Definitions 6.1, 6.5 and 7.2. Let the subspaces W and $W^\perp$ be as in Definitions 6.1 and 6.4. Let the scalars Δ_i, α_i and β_i be as in Definition 7.3 and Theorem 9.1.*

10 The action - part II

With reference to Definition 9.2, in this section we obtain the action of A on the vectors w_{ij}, w_{ii}^+ and $\widetilde{w}_{ii}$ for the case in which Γ is Q-polynomial. We will use Theorem 9.1.

Lemma 10.1 *With reference to Definition 9.2, the following* (i)–(iv) *hold.*

(i) $Aw_{02} = w_{13} + w_{11}$,
$Aw_{i-1,i+1} = b_i w_{i-2,i} + c_i w_{i,i+2} + (c_i - \alpha_i) w_{ii} - \beta_i w_{ii}^+ \quad (2 \le i \le d-2)$,
$Aw_{d-2,d} = b_{d-1} w_{d-3,d-1} + (c_{d-1} - \alpha_{d-1}) w_{d-1,d-1} - \beta_{d-1} w_{d-1,d-1}^+$.

(ii) $Aw_{20} = w_{31} \ + \ w_{11}$,
$Aw_{i+1,i-1} = b_i w_{i,i-2} \ + \ c_i w_{i+2,i} \ + \ (c_i - \alpha_i) w_{ii} \ - \ \beta_i w^+_{ii} \ \ (2 \leq i \leq d-2)$,
$Aw_{d,d-2} = b_{d-1} w_{d-1,d-3} \ + \ (c_{d-1} - \alpha_{d-1}) w_{d-1,d-1} \ - \ \beta_{d-1} w^+_{d-1,d-1}$.

(iii) $Aw_{11} = c_2(w_{02} + w_{20}) \ + \ \alpha_2 w_{22} \ + \ \beta_2 w^+_{22}$,
$Aw_{22} = (b_1 - b_3)(w_{13} + w_{31}) \ + \ (k-2) w_{11} \ + \ \alpha_3 w_{33} \ + \ \beta_3 w^+_{33}$,
$Aw_{ii} = (b_{i-1} - b_{i+1})(w_{i-1,i+1} + w_{i+1,i-1}) \ + \ (k - 2c_{i-1} + \alpha_{i-1}) w_{i-1,i-1} +$
$\beta_{i-1} w^+_{i-1,i-1} \ + \ \alpha_{i+1} w_{i+1,i+1} \ + \ \beta_{i+1} w^+_{i+1,i+1} \ \ (3 \leq i \leq d-2)$,
$Aw_{d-1,d-1} = b_{d-2}(w_{d-2,d} + w_{d,d-2}) \ + \ (k - 2c_{d-2} + \alpha_{d-2}) w_{d-2,d-2} +$
$\beta_{d-2} w^+_{d-2,d-2} \ + \ k w_{dd}$,
$Aw_{dd} = (k - 2c_{d-1} + \alpha_{d-1}) w_{d-1,d-1} \ + \ \beta_{d-1} w^+_{d-1,d-1}$.

(iv) $Aw^+_{22} = c_2(c_2 - 1)(w_{13} + w_{31}) \ + \ (b_2 + c_2(c_2 - 2)) w_{11} \ + c_2 w^+_{33}$,
$Aw^+_{ii} = c_2(c_i - c_{i-1})(w_{i-1,i+1} + w_{i+1,i-1}) \ + \ c_2(c_i - 2c_{i-1} + \alpha_{i-1}) w_{i-1,i-1} +$
$(b_i + c_2 \beta_{i-1}) w^+_{i-1,i-1} \ + \ c_i w^+_{i+1,i+1} \ \ (3 \leq i \leq d-1)$.

Proof. Immediate from Lemma 8.1, Lemma 8.2, Lemma 8.3 and Theorem 9.1, and since $w^-_{dd} = k w_{dd}$. ∎

Theorem 10.2 *With reference to Definition 9.2, the following* (i)–(iv) *hold.*

(i) $Aw_{02} = w_{13} \ + \ w_{11}$,
$Aw_{i-1,i+1} = b_i w_{i-2,i} \ + \ c_i w_{i,i+2} \ + \ (c_i - \alpha_i - \beta_i \lambda_i) w_{ii} \ - \ \beta_i \widetilde{w}_{ii} \ \ (2 \leq i \leq d-2)$,
$Aw_{d-2,d} = b_{d-1} w_{d-3,d-1} \ + \ (c_{d-1} - \alpha_{d-1} - \beta_{d-1}\lambda_{d-1}) w_{d-1,d-1} \ - \ \beta_{d-1} \widetilde{w}_{d-1,d-1}$.

(ii) $Aw_{20} = w_{31} \ + \ w_{11}$,
$Aw_{i+1,i-1} = b_i w_{i,i-2} \ + \ c_i w_{i+2,i} \ + \ (c_i - \alpha_i - \beta_i \lambda_i) w_{ii} \ - \ \beta_i \widetilde{w}_{ii} \ \ (2 \leq i \leq d-2)$,
$Aw_{d,d-2} = b_{d-1} w_{d-1,d-3} \ + \ (c_{d-1} - \alpha_{d-1} - \beta_{d-1}\lambda_{d-1}) w_{d-1,d-1} \ - \ \beta_{d-1} \widetilde{w}_{d-1,d-1}$.

(iii) $Aw_{11} = c_2(w_{02} + w_{20}) \ + \ (\alpha_2 + \beta_2 \lambda_2) w_{22} \ + \ \beta_2 \widetilde{w}_{22}$,
$Aw_{22} = (b_1 - b_3)(w_{13} + w_{31}) \ + \ (k-2) w_{11} \ + \ (\alpha_3 + \beta_3 \lambda_3) w_{33} \ + \ \beta_3 \widetilde{w}_{33}$,
$Aw_{ii} = (b_{i-1} - b_{i+1})(w_{i-1,i+1} + w_{i+1,i-1}) \ + \ (k - 2c_{i-1} + \alpha_{i-1} + \beta_{i-1}\lambda_{i-1}) w_{i-1,i-1} +$
$\beta_{i-1} \widetilde{w}_{i-1,i-1} \ + \ (\alpha_{i+1} + \beta_{i+1}\lambda_{i+1}) w_{i+1,i+1} \ + \ \beta_{i+1} \widetilde{w}_{i+1,i+1} \ \ (3 \leq i \leq d-2)$,
$Aw_{d-1,d-1} = b_{d-2}(w_{d-2,d} + w_{d,d-2}) \ + \ (k - 2c_{d-2} + \alpha_{d-2} + \beta_{d-2}\lambda_{d-2}) w_{d-2,d-2} +$
$\beta_{d-2} \widetilde{w}_{d-2,d-2} \ + \ k w_{dd}$,
$Aw_{dd} = (k - 2c_{d-1} + \alpha_{d-1} + \beta_{d-1}\lambda_{d-1}) w_{d-1,d-1} \ + \ \beta_{d-1} \widetilde{w}_{d-1,d-1}$.

(iv) $A\widetilde{w}_{22} = \big(c_2(c_2 - 1) - \lambda_2(b_1 - b_3)\big)(w_{13} + w_{31}) \ + \ \big(b_2 + c_2(c_2 - 2) - \lambda_2(k-2)\big) w_{11} +$
$\big(c_2 \lambda_3 - \lambda_2(\alpha_3 + \beta_3 \lambda_3)\big) w_{33} \ + \ (c_2 - \lambda_2 \beta_3) \widetilde{w}_{33}$,
$A\widetilde{w}_{ii} = \big(c_2(c_i - c_{i-1}) - \lambda_i(b_{i-1} - b_{i+1})\big)(w_{i-1,i+1} + w_{i+1,i-1}) +$
$\big(\lambda_{i-1}(b_i + c_2 \beta_{i-1}) + c_2(c_i - 2c_{i-1} + \alpha_{i-1}) -$
$\lambda_i(k - 2c_{i-1} + \alpha_{i-1} + \beta_{i-1}\lambda_{i-1})\big) w_{i-1,i-1} +$
$(b_i + c_2 \beta_{i-1} - \lambda_i \beta_{i-1}) \widetilde{w}_{i-1,i-1} +$
$\big(c_i \lambda_{i+1} - \lambda_i(\alpha_{i+1} + \beta_{i+1}\lambda_{i+1})\big) w_{i+1,i+1} \ + \ (c_i - \lambda_i \beta_{i+1}) \widetilde{w}_{i+1,i+1} \ \ (3 \leq i \leq d-2)$,
$A\widetilde{w}_{d-1,d-1} = \big(c_2(c_{d-1} - c_{d-2}) - \lambda_{d-1} b_{d-2}\big)(w_{d-2,d} + w_{d,d-2}) +$
$\big(\lambda_{d-2}(b_{d-1} + c_2 \beta_{d-2}) + c_2(c_{d-1} - 2c_{d-2} + \alpha_{d-2}) -$
$\lambda_{d-1}(k - 2c_{d-2} + \alpha_{d-2} + \beta_{d-2}\lambda_{d-2})\big) w_{d-2,d-2} +$
$(b_{d-1} + c_2 \beta_{d-2} - \lambda_{d-1}\beta_{d-2}) \widetilde{w}_{d-2,d-2} \ + \big(c_{d-1} c_2 - k \lambda_{d-1}\big) w_{dd}$.

Proof. Immediate from Lemma 10.1 and Definition 7.2. ∎

We have the following important result.

Corollary 10.3 *With reference to Definition 9.2, the vectors* $\{\widetilde{w}_{ii} \mid 2 \leq i \leq d-1,\ \Delta_i \neq 0\}$ *form an orthogonal basis for* $W^{\perp}$.

Proof. Let $W' = \mathrm{span}\{\widetilde{w}_{ii} \mid 2 \leq i \leq d-1,\ \Delta_i \neq 0\}$. We show $W^{\perp} = W'$. We first show $W^{\perp} \subseteq W'$. By Theorem 10.2, the subspace $W + W'$ is A-invariant. Since MW is the minimal A-invariant subspace that contains W, we have $MW \subseteq W + W'$. Recall $W^{\perp}$ is the orthogonal complement of W in MW. By construction W and W' are orthogonal, so W' is the orthogonal complement of W in $W + W'$ By these comments $W^{\perp} \subseteq W'$.

Next we show $W' \subseteq W^{\perp}$. Since MW is A-invariant and $\beta_i \neq 0$ $(2 \leq i \leq d-1)$ by Lemma 3.1, we have $\widetilde{w}_{ii} \in MW$ $(2 \leq i \leq d-1)$ by Theorem 10.2(i). But now Lemma 7.4 implies $\widetilde{w}_{ii} \in W^{\perp}$ for $2 \leq i \leq d-1$, and hence $W' \subseteq W^{\perp}$. Now $W^{\perp} = W'$ and the result follows. ∎

11 A basis for MW

With reference to Definition 9.2, in Lemma 6.3 and Corollary 10.3 we gave an orthogonal basis for W and $W^{\perp}$, respectively. In this section we give more detailed information about these bases. We will consider three cases. In order to describe these cases we recall a definition.

With reference to Definition 9.2, Γ is said to be 2-*homogeneous* in the sense of Nomura [20] whenever for all integers i $(2 \leq i \leq d-1)$ and for all $x, y, z \in X$ with $\partial(x, y) = 2$, $\partial(x, z) = \partial(y, z) = i$, the number $|D_1^1(x, y) \cap \Gamma_{i-1}(z)|$ is independent of x, y, z. By [9, Theorem 17] we find Γ is 2-homogeneous if and only if (i) for all $x, y \in X$ such that $\partial(x, y) = 2$, the partition of X given by Definition 5.1 is equitable [15, Section 5.1]; and (ii) the corresponding parameters of this partition do not depend on x, y.

We use the following lemma.

Lemma 11.1 *With reference to Definition 9.2, the following* (i)–(iii) *hold.*

(i) *Assume* Γ *is 2-homogeneous. Then* $\Delta_i = 0$ *for* $2 \leq i \leq d-1$ *and* $p^2_{dd} = 0$.

(ii) *Assume* Γ *is the antipodal quotient of the* $2d$*-cube. Then* $\Delta_i = 0$ *for* $2 \leq i \leq d-2$ *and* $\Delta_{d-1} \neq 0$, $p^2_{dd} \neq 0$.

(iii) *Assume* Γ *is neither 2-homogeneous nor the antipodal quotient of the* $2d$*-cube. Then* $\Delta_i \neq 0$ *for* $2 \leq i \leq d-1$ *and* $p^2_{dd} \neq 0$.

Proof. (i) By [9, Theorem 13], $\Delta_i = 0$ for $2 \leq i \leq d-1$. Furthermore, by [9, Theorem 42], Γ is an antipodal 2-cover. Hence $b_{d-1} = 1$, so $p^2_{dd} = 0$ by Lemma 4.2(iii).

(ii) The intersection numbers of the antipodal quotient of the $2d$-cube are given in [1, p. 264]. The result now follows straightforward from Definition 7.3 and Lemma 4.2(iii).

(iii) First assume $d=3$. Using Definition 7.3 and Lemma 4.1(iii) we find $\Delta_2 = b_2(b_2-1)/c_2$. By [9, Theorem 13] and since Γ is not 2-homogeneous we find $\Delta_2 \neq 0$, implying $b_2 \neq 1$. Combining this with Lemma 4.1(iv) we find $p^2_{33} \neq 0$, and the result follows.

Next assume $d \geq 4$. Observe that Γ is not the d-cube, since the d-cube is 2-homogeneous. By [8, Lemma 3.2, Lemma 3.3], there exist $q, s^* \in \mathbb{R}$ such that

$$|q| > 1, \quad s^* q^i \neq 1 \ (2 \leq i \leq 2d+1), \tag{17}$$

$$c_i = \frac{h(q^i-1)(1-s^*q^{d+i+1})}{1-s^*q^{2i+1}}, \quad b_i = \frac{h(q^d-q^i)(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \ (1 \leq i \leq d-1), \tag{18}$$

$$k = c_d = h(q^d-1), \tag{19}$$

where

$$h = \frac{1-s^*q^3}{(q-1)(1-s^*q^{d+2})}.$$

By direct computation we obtain

$$b_{d-1} - 1 = \frac{(q^{d-1}-1)(1-s^*q^{d+1})(1+s^*q^{d+1})}{(1-s^*q^{2d-1})(1-s^*q^{d+2})}. \tag{20}$$

Similarly, by (18) and (19), we obtain also

$$\Delta_i = \frac{q^2(q^{i-1}-1)(q^i-1)(1-s^*q^{i+1})(1-s^*q^{i+2})(1-s^*q^3)(1-s^*q^{2d+1})(1+s^*q^{d+1})}{(q^2-1)(1-s^*q^{2i-1})(1-s^*q^{2i+3})(1-s^*q^{d+2})^2(1-s^*q^{d+3})}. \tag{21}$$

Assume for a moment $\Delta_i = 0$ for $2 \leq i \leq d-1$. Then, by [9, Theorem 13] and by the definition of the 2-homogeneous property, Γ is 2-homogeneous. Hence there exists i $(2 \leq i \leq d-1)$ such that $\Delta_i \neq 0$. Therefore, $1+s^*q^{d+1} \neq 0$ by (21). But now, by (17) and (21), we have $\Delta_i \neq 0$ for $2 \leq i \leq d-1$. Finally, by (17) and (20), $b_{d-1}-1 \neq 0$. Using Lemma 4.2(iii) we find $p^2_{dd} \neq 0$. This completes the proof. ∎

In what follows we treat the three cases of Lemma 11.1 separately.

Theorem 11.2 *With reference to Definition 9.2, assume Γ is 2-homogeneous. Then the following* (i),(ii) *hold.*

(i) *The vectors $\{w_{i-1,i+1}, w_{i+1,i-1}, w_{ii} \mid 1 \leq i \leq d-1\}$ form an orthogonal basis for W.*

(ii) $W^\perp = 0$.

Proof. (i) The result follows from Lemmas 11.1(i), 6.2(ii), 4.2 and 6.3.

(ii) The result follows from Lemmas 11.1(i) and 7.5, and Corollary 10.3. ∎

Corollary 11.3 *With reference to Definition 9.2, assume Γ is 2-homogeneous. Then the following* (i)–(iii) *hold.*

(i) *The dimension of W is $3d-3$.*

(ii) *The dimension of $W^\perp$ is 0.*

(iii) *The dimension of* MW *is* $3d-3$.

PROOF. Immediate from Theorem 11.2 and since $W^{\perp}$ is the orthogonal complement of W in MW. ∎

We now look at the case when Γ is the antipodal quotient of the $2d$-cube.

Theorem 11.4 *With reference to Definition 9.2, assume* Γ *is the antipodal quotient of the* $2d$*-cube. Then the following* (i),(ii) *hold.*

(i) *The vectors* $\{w_{i-1,i+1}, w_{i+1,i-1} \mid 1 \le i \le d-1\}$ *and* $\{w_{ii} \mid 1 \le i \le d\}$ *form an orthogonal basis for* W.

(ii) *The vector* $\widetilde{w}_{d-1,d-1}$ *forms an orthogonal basis for* $W^{\perp}$.

PROOF. (i) The result follows from Lemmas 11.1(ii), 6.2(ii), 4.2 and 6.3.
(ii) The result follows from Lemmas 11.1(ii) and 7.5, and Corollary 10.3. ∎

Corollary 11.5 *With reference to Definition 9.2, assume* Γ *is the antipodal quotient of the* $2d$*-cube. Then the following* (i)–(iii) *hold.*

(i) *The dimension of* W *is* $3d-2$.

(ii) *The dimension of* $W^{\perp}$ *is* 1.

(iii) *The dimension of* MW *is* $3d-1$.

PROOF. Immediate from Theorem 11.4 and since $W^{\perp}$ is the orthogonal complement of W in MW. ∎

Finally, let us consider the case when Γ is neither 2-homogeneous nor the antipodal quotient of the $2d$-cube.

Theorem 11.6 *With reference to Definition 9.2, assume* Γ *is neither* 2*-homogeneous nor the antipodal quotient of the* $2d$*-cube. Then the following* (i),(ii) *hold.*

(i) *The vectors* $\{w_{i-1,i+1}, w_{i+1,i-1} \mid 1 \le i \le d-1\}$ *and* $\{w_{ii} \mid 1 \le i \le d\}$ *form an orthogonal basis for* W.

(ii) *The vectors* $\{\widetilde{w}_{ii} \mid 2 \le i \le d-1\}$ *form an orthogonal basis for* $W^{\perp}$.

PROOF. (i) The result follows from Lemmas 11.1(iii), 6.2(ii), 4.2 and 6.3.
(ii) The result follows from Lemmas 11.1(iii) and 7.5, and Corollary 10.3. ∎

Corollary 11.7 *With reference to Definition 9.2, assume* Γ *is neither* 2*-homogeneous nor the antipodal quotient of the* $2d$*-cube. Then the following* (i)–(iii) *hold.*

(i) *The dimension of* W *is* $3d-2$.

(ii) *The dimension of* $W^{\perp}$ *is* $d-2$.

(iii) *The dimension of* MW *is* $4d-4$.

PROOF. Immediate from Theorem 11.6 and since $W^{\perp}$ is the orthogonal complement of W in MW. ∎

Acknowledgement: The author would like to thank to Paul Terwilliger for proposing the problem and for many valuable suggestions.

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