University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1 111 Ljubljana, Slovenia

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PARTIAL CUBES ARE DISTANCE GRAPHS

Melita Gorše Pihler   Janez Žerovnik

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Melita Gorše Pihler[a], Janez Žerovnik[b,c,*]

[a]FEECS, University of Maribor, Smetanova 17, SI-2000 Maribor, Slovenia
[b]FME, University of Maribor, Smetanova 17, SI-2000 Maribor, Slovenia
[c]IMFM, Jadranska 19, SI-1000 Ljubljana, Slovenia
email: janez.zerovnik@imfm.uni-lj.si

Abstract

Chatrand, Kubicki and Schultz [Aequationes Math. **55** *(1998) 129-145] have recently conjectured that all bipartite graphs are distance graphs. Here we show that all graphs of a large subclass of bipartite graphs, i.e. partial cubes, are distance graphs.*

Keywords: distance graph, partial cube, hypercube, isometric subgraph, embedding.

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*The corresponding author.

1 Introduction

If two graphs G_1 and G_2 are isomorphic, then there exists a one-to-one mapping $\phi : V(G_1) \to V(G_2)$ with the property that vertices u and v are adjacent in G_1 if and only if ϕu and ϕv are adjacent in G_2. Of course, ϕ is an isomorphism. In fact, if G_1 and G_2 are connected, then ϕ preserves the distance between every pair of vertices of G_1 (not only pairs of adjacent vertices), that is, if u and v are any two vertices of G_1, then $d_{G_1}(u,v) = d_{G_2}(\phi u, \phi v)$.

Let G_1 and G_2 be connected graphs of order n. There are $n!$ one-to-one mappings from $V(G_1)$ to $V(G_2)$. If G_1 and G_2 are isomorphic, then the number of isomorphisms among these $n!$ mappings is the order of the automorphism group Aut G_1 of G_1. For a one-to-one mapping ϕ and each pair u, v of vertices of G_1 it is of interest to compare $d_{G_1}(u,v)$ with $d_{G_2}(\phi u, \phi v)$. For this reason, we define the ϕ-distance between G_1 and G_2 as

$$d_\phi(G_1, G_2) = \sum |d_{G_1}(u,v) - d_{G_2}(\phi u, \phi v)|, \tag{1}$$

where the sum is taken over all $\binom{n}{2}$ unordered pairs u, v of vertices of G_1. Of course, if $d_\phi(G_1, G_2) = 0$ then ϕ is an isomorphism and $G_1 \cong G_2$, while if $G_1 \not\cong G_2$, then $d_\phi(G_1, G_2) > 0$ for every one-to-one mapping ϕ. This suggests defining the distance $d(G_1, G_2)$ between G_1 and G_2 by

$$d(G_1, G_2) = \min\{d_\phi(G_1, G_2)\}, \tag{2}$$

where the minimum is taken over all one-to-one mappings ϕ from $V(G_1)$ to $V(G_2)$. Thus, $d(G_1, G_2) = 0$ if and only if $G_1 \cong G_2$. Hence $d(G_1, G_2)$ can be interpreted as a measure of the similarity of G_1 and G_2, where then the smaller the value of $d(G_1, G_2)$, the more similar the structure of G_1 is to that of G_2.

This distance defined on the space of all connected graphs of a fixed order produces a metric space.

Let S be a set of connected graphs having the same order. Then the distance graph $D(S)$ of S has vertex set S and two vertices G_1 and G_2 of $D(S)$ are adjacent if and only if $d(G_1, G_2) = 1$. Further, we say that a graph G is a **distance graph** if there exists a set S of graphs having fixed order such that $D(S) \cong G$.

In [1] it has been conjectured:

Conjecture 1 *A graph G is a distance graph if and only if G is bipartite.*

The conjecture is based on the fact that every distance graph is bipartite, and that several classes of bipartite graphs are shown to be distance graphs, for example every even cycle is a distance graph, every tree is a distance graph, the graph $K_{2,n}$ is a distance graph for every positive integer n, and the graph $K_{3,3}$ is a distance graph [1].

In this note we first show that hypercubes are distance graphs and consequently partial cubes are distance graphs. (Partial cubes are induced subgraphs of hypercubes [3].) We also show some other distance graphs.

2 Partial cubes are distance graphs

We will show that hypercubes are distance graphs. To this aim we first recall some results from [1].

Theorem 2 *[1] Let G_1 and G_2 be two connected graphs of the same order having sizes p_1 and p_2, respectively, such that the size of a greatest common subgraph is s. Then*

$$d(G_1, G_2) \geq p_1 + p_2 - 2s. \tag{3}$$

Corollary 3 *[1] If G_1 and G_2 are connected graphs of the same order having sizes p_1 and p_2, respectively, such that $d(G_1, G_2) = 1$, then $|p_1 - p_2| = 1$ and one of G_1 and G_2 is a subgraph of the other.*

Theorem 4 *[1] Let G_1 and G_2 be connected graphs of the same order having sizes p_1 and p_2, respectively, with $p_1 \leq p_2$. Then $d(G_1, G_2) = 1$ if and only if $G_1 \subseteq G_2$, $p_2 = p_1 + 1$, and there exists a one-to-one mapping $\phi : V(G_1) \to V(G_2)$ such that for some 2-element subset $\{x, y\}$, it follows that $xy \notin E(G_1)$, $\phi x \phi y \in E(G_2)$, $d_{G_1}(x, y) = 2$, and if $\{u, v\} \neq \{x, y\}$, then $d_{G_1}(u, v) = d_{G_2}(\phi u, \phi v)$.*

Corollary 5 *[1] Let G_1 and G_2 be connected graphs of the same order having sizes p_1 and p_2, respectively, such that diam $G_1 = 2$ and $p_2 \geq p_1$. Then $d(G_1, G_2) = 1$ if and only if $G_1 \subseteq G_2$ and $p_2 = p_1 + 1$.*

Let $G_{m_1 m_2 \ldots m_n}$ ($m_i \in \{0, 1\}$) be a graph, depicted on Figure 1.

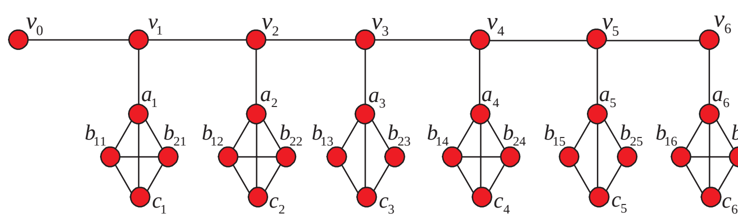

Figure 1: The graph G_{110101} .

Theorem 6 *Let m_1, m_2, ..., $m_n \in \{0, 1\}$ and m'_1, m'_2, ..., $m'_n \in \{0, 1\}$. Graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m'_1 m'_2 \ldots m'_n}$ are isomorphic if and only if $m_i = m'_i$ for $i = 1$, 2, ..., n.*

Proof Suppose that graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m'_1 m'_2 \ldots m'_n}$ are isomorphic. Any isomorphism φ from $V(G_{m_1 m_2 \ldots m_n})$ onto $V(G_{m'_1 m'_2 \ldots m'_n})$ maps v_0 into v'_0, because v_0 is the only vertex which has degree 1. Since φ preserves adjacency and nonadjacency, it follows that φ maps v_1 into v'_1 (only v_1 is adjacent to v_0), v_2 into v'_2 (from all vertices which are adjacent to v_1 only v_2 has degree 3), v_3 into

v_3' (from all vertices which are adjacent to v_2 have only v_1 and v_3 degree 3, so φ maps $\{v_1, v_3\}$ onto $\{v_1', v_3'\}$; we have seen that φ maps v_1 into v_1', therefore φ maps v_3 into v_3'), ..., v_n into v_n' (from all vertices which are adjacent to v_{n-1} has only v_n degree 2).

Let $i \in \{1, 2, \ldots, n\}$. From all vertices which are adjacent to v_i has only a_i degree 4, so φ maps a_i into a_i'. b_{1i}, b_{2i} and c_i are adjacent to a_i, so φ maps $\{b_{1i}, b_{2i}, c_i\}$ onto $\{b_{1i}', b_{2i}', c_i'\}$.

1. If $m_i = 0$, then b_{1i} and b_{2i} have degree 2. Since c_i' has degree 3, it follows that φ maps $\{b_{1i}, b_{2i}\}$ onto $\{b_{1i}', b_{2i}'\}$. Therefore b_{1i}' and b_{2i}' have degree 2, so $m_i' = 0$.

2. If $m_i = 1$, then b_{1i}, b_{2i} and c_i have degree 3. It follows that φ maps $\{b_{1i}, b_{2i}, c_i\}$ onto $\{b_{1i}', b_{2i}', c_i'\}$. Therefore b_{1i}' and b_{2i}' have degree 3, so $m_i' = 1$.

Hence $m_i = m_i'$ for $i = 1, 2, ..., n$.

For the converse we assume that $m_i = m_i'$ for $i = 1, 2, ..., n$. Then it is easy to see that graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m_1' m_2' \ldots m_n'}$ are isomorphic. □

Theorem 7 *Let $m_1, m_2, ..., m_n \in \{0,1\}$ and $m_1', m_2', ..., m_n' \in \{0,1\}$. Then $d(G_{m_1 m_2 \ldots m_n}, G_{m_1' m_2' \ldots m_n'}) = 1$ if and only if the corresponding tuples differ in precisely one position.*

Proof Let $p_1 = |E(G_{m_1 m_2 \ldots m_n})|$ and $p_2 = \left|E(G_{m_1' m_2' \ldots m_n'})\right|$.

Suppose that $d(G_{m_1 m_2 \ldots m_n}, G_{m_1' m_2' \ldots m_n'}) = 1$. By Corollary 3, $|p_1 - p_2| = 1$ and one of the graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m_1' m_2' \ldots m_n'}$ is a subgraph of the other. We may, without loss of generality, assume that $p_2 = p_1 + 1$. We can assume that the graph $G_{m_1 m_2 \ldots m_n}$ is a subgraph of $G_{m_1' m_2' \ldots m_n'}$. In other words, we can get $G_{m_1' m_2' \ldots m_n'}$ from $G_{m_1 m_2 \ldots m_n}$ by adding one edge. By construction of the graphs $G_{m_1 m_2 \ldots m_n}$, the corresponding tuples differ in precisely one place.

For the converse we suppose that corresponding tuples of graphs $G_{m_1 m_2 \ldots m_n}$ and $G_{m_1' m_2' \ldots m_n'}$ differ in precisely one place. Without loss of generality, we may assume that there exists such $j \in \{1, 2, ..., n\}$ that $m_i = m_i'$ for $i = 1, 2, ..., j-1, j+1, ..., n$ and $m_j = 0$, $m_j' = 1$. Then graph $G_{m_1 m_2 \ldots m_n}$ is a subgraph of $G_{m_1' m_2' \ldots m_n'}$, $p_2 = p_1 + 1$ and the identity mapping ϕ, which maps vertices from $V(G_{m_1 m_2 \ldots m_n})$ onto corresponding vertices in $V(G_{m_1' m_2' \ldots m_n'})$ has the following properties: (1) $b_{1j} b_{2j} \notin E(G_{m_1 m_2 \ldots m_n})$ and $\phi b_{1j} \phi b_{2j} \in E(G_{m_1' m_2' \ldots m_n'})$, (2) $d_{G_{m_1 m_2 \ldots m_n}}(b_{1j}, b_{2j}) = 2$, (3) if $\{u, v\} \neq \{b_{1j}, b_{2j}\}$ then $d_{G_{m_1 m_2 \ldots m_n}}(u, v) = d_{G_{m_1' m_2' \ldots m_n'}}(\phi u, \phi v)$. By Theorem 4, $d(G_{m_1 m_2 \ldots m_n}, G_{m_1' m_2' \ldots m_n'}) = 1$. □

Theorem 8 *Every hypercube is a distance graph.*

Proof By Theorem 7 and the definition of hypercubes, Q_n is the distance graph of the collection $\{G_{m_1 m_2 \ldots m_n} | \; m_i \in \{0,1\}\}$. □

Theorem 9 *Every induced subgraph of a hypercube is a distance graph.*

Proof Let G be an induced subgraph of a hypercube Q_n ($n \in \mathbb{N}$) and let u and v be arbitrary vertices of the graph G.

If vertices u and v are adjacent in G then u and v are also adjacent in hypercube Q_n, because G is a subgraph of Q_n. Therefore the distance between the corresponding graphs from the proof of Theorem 8 is 1.

If the vertices u and v are not adjacent in G then u and v are also not adjacent in hypercube Q_n, because G is an induced subgraph of Q_n. Therefore the distance between the corresponding graphs from the proof of Theorem 8 is more than 1.

Thus the subset of corresponding graphs from the proof of Theorem 8 has distance graph isomorphic to G. □

Theorem 10 *Every graph which can be isometrically embedded into a hypercube is a distance graph.*

Proof Let G be a graph which can be isometrically embedded in a hypercube Q_n ($n \in \mathbb{N}$). Then G is isomorphic to an isometric subgraph of Q_n. Every isometric subgraph is also an induced subgraph, therefore G is isomorphic to an induced subgraph of Q_n. By Theorem 9, G is a distance graph. □

Using results [2, 3] on sufficient conditions for the existence of isometric embeddings into hypercubes Theorem 10 has the following three corollaries:

Corollary 11 *Every bipartite graph for which relation Θ is transitive ($\Theta^* = \Theta$) is a distance graph.*

We say e is in relation Θ to f, if

$$d(x,u) + d(y,v) \neq d(x,v) + d(y,u), \tag{4}$$

where $e = xy$ and $f = uv$ are two edges of connected graph G.

Corollary 12 *Let G be a bipartite graph such that for any edge vw of G, the set of vertices that are closer v than w is closed under taking shortest paths. Then G is a distance graph.*

Corollary 13 *Let G be a bipartite graph and for any edge xy of G, if $a, b, c \in V(G)$ such that $d(a,x) < d(a,y)$, $d(b,x) < d(b,y)$ and $d(a,b) = d(a,c) + d(b,c)$ then $d(c,x) < d(c,y)$. It follows that G is a distance graph.*

3 Some more results

At first sight one might guess that a subgraph of a hypercube is a distance graph if and only if it is an isometric subgraph of a hypercube. The next example shows that it is not true.

Let G be a graph, depicted on Figure 2.

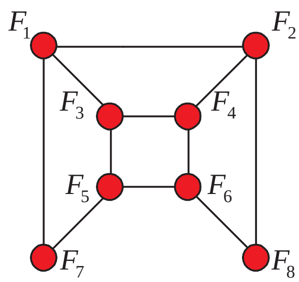

Figure 2: The graph G.

For $i \in \{1, 2, ..., 8\}$, consider the graphs $F_i = K_6 - H_i$, where each H_i is shown in Figure 3. Since the diameter of each graph in $S = \{F_1, F_2, ..., F_8\}$ is 2, Corollary 5 implies that $D(S) \cong G$.

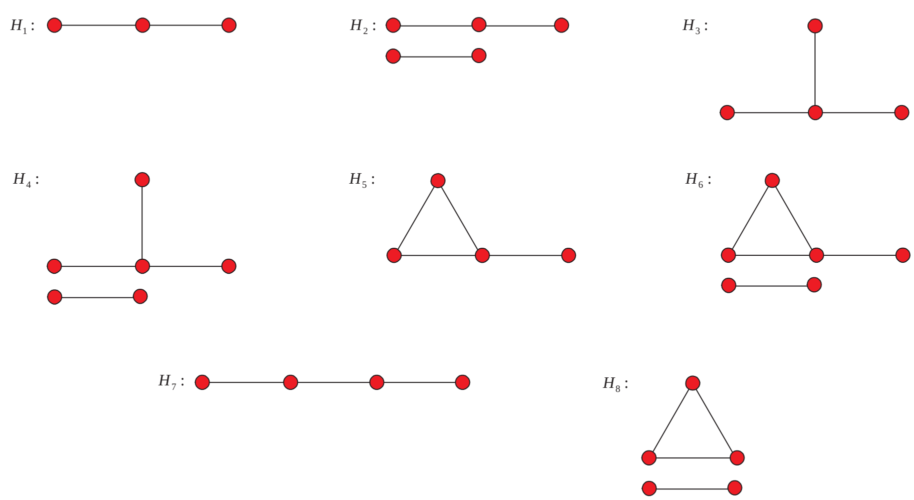

Figure 3: Graphs H_i ($i = 1, 2, \ldots, 8$).

We have checked all subgraphs of 3-cube, which are not isometric subgraphs of a hypercube and we found that they are all distance graphs. It seems reasonable to work on the subproblem:

Problem 14 *Prove (or disprove) that every subgraph of a hypercube is a distance graph.*

We have also considered some complete bipartite graphs. For example, it is well known that [1]:

1. the graph $K_{1,n}$ is a distance graph for every positive integer n (because every tree is a distance graph),
2. the graph $K_{2,n}$ is a distance graph for every positive integer n,
3. $K_{3,3}$ is a distance graph.

Next examples will prove that $K_{3,4}$ and $K_{3,5}$ are distance graphs.

For $i = 1, 2, ..., n$ let $U_i = K_8 - L_i$, where graphs L_i are given in Figure 4. Since the diameter of each graph in $S = \{U_1, U_2, ..., U_7\}$ is 2, Corollary 5 implies that $D(S) \cong K_{3,4}$, where the bipartite sets of $D(S)$ are $\{U_1, U_2, U_3\}$ and $\{U_4, U_5, U_6, U_7\}$.

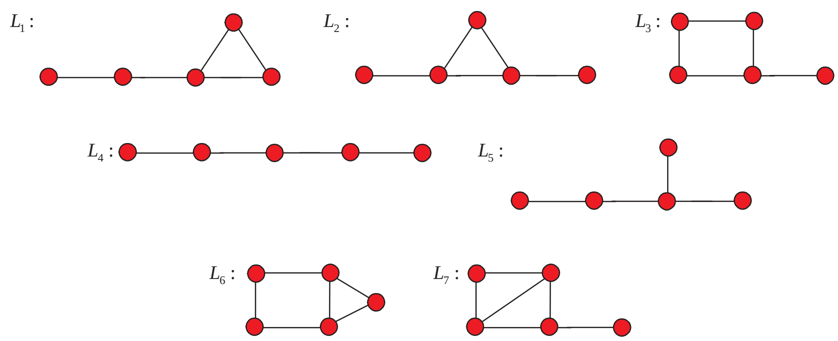

Figure 4: Graphs L_i $(i = 1, 2, \ldots, 7)$.

For $i = 1, 2, ..., n$ let $R_i = K_9 - Z_i$, where graphs Z_i are given in Figure 5. Since the diameter of each graph in $S = \{R_1, R_2, ..., R_8\}$ is 2, Corollary 5 implies that $D(S) \cong K_{3,5}$, where the bipartite sets of $D(S)$ are $\{R_1, R_2, R_3\}$ and $\{R_4, R_5, R_6, R_7, R_8\}$.

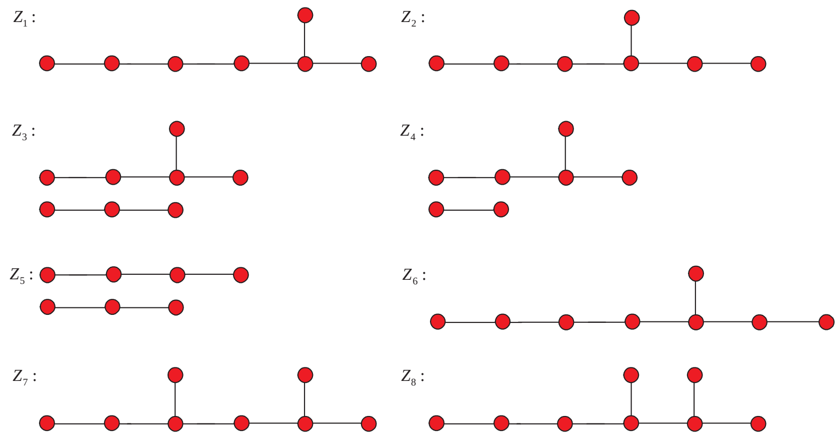

Figure 5: Graphs Z_i ($i = 1, 2, \ldots, 8$).

These examples are reason for our next working problem:

Problem 15 *Prove that the graph $K_{3,n}$ is a distance graph for every positive integer n.*

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