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Abstract. Let X be a Stein manifold, A a closed complex subvariety of X, Y a complex manifold and $f\colon X \to Y$ a continuous map such that $f|_A\colon A \to Y$ is holomorphic. After a homotopic deformation of the Stein structure outside a neighborhood of A in X we find a holomorphic map $\widetilde{f}\colon X \to Y$ which agrees with f on A and is homotopic to f relative to A. When $\dim_{\mathbb{C}} X = 2$ we must also change the $\mathcal{C}^\infty$ structure on $X\backslash A$.

1. Introduction

A classical theorem of H. Cartan asserts that every holomorphic function on a closed complex subvariety of a Stein manifold (or a Stein space) X extends to a holomorphic function on X. (For the theory of Stein manifolds see [13] and [17].) The analogous extension property fails in general for mappings $X \to Y$ to more general complex manifolds, unless the target manifold Y enjoys a certain holomorphic flexibility property introduced in [4] and [5]. Indeed, if Y is Kobayashi hyperbolic then the extension property fails if A contains more than one point due to Kobayashi distance decreasing property of holomorphic maps [18].

In this paper we show that the situation is completely different if we allow homotopic deformations of the Stein structure (and of the underlying smooth structure when $\dim_{\mathbb{C}} X = 2$) in the complement of the given subvariety. The following is a simplified version of theorem 3.1 in §3 below.

Theorem 1.1. *Let X be a Stein manifold with $\dim_{\mathbb{C}} X \neq 2$ and let $A \subset X$ be a closed complex subvariety. Given a continuous map $f\colon X \to Y$ to a complex manifold Y such that $f|_A\colon A \to Y$ is holomorphic, there is a homotopy $(J_t, f_t)_{t\in[0,1]}$ consisting of integrable complex structures J_t on X and continuous maps $f_t\colon X \to Y$ satisfying*

(i) *J_0 is the initial complex structure on X, $J_t = J_0$ in a neighborhood of A for each $t \in [0,1]$, and J_1 is a Stein structure on X;*
(ii) *$f_0 = f$, $f_t|_A = f|_A$ for every $t \in [0,1]$, and $f_1\colon X \to Y$ is J_1-holomorphic.*

This theorem is a relative version (with interpolation on a complex subvariety) of theorem 1.1 in [7] to the effect that every continuous map $f\colon X \to Y$ from a Stein manifold (X, J) of complex dimension $\neq 2$ to a complex manifold Y is homotopic to map $\widetilde{f}\colon X \to Y$ which is holomorphic with respect to a homotopic to J Stein

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structure $\widetilde{J}$ on X. In the exceptional case $\dim_{\mathbb{C}} X = 2$ (when X is a Stein surface) the analogous conclusion holds after changing the underlying smooth structure on X; see theorem 4.1 in §4 for the corresponding interpolation theorem.

The first author proved in [5] that for every complex manifold Y, the conclusion of theorem 1.1 holds for all data (X, A, f) and *without changing the Stein structure* on X if and only if Y satisfies the *convex approximation property* (CAP), introduced in [4], to the effect that every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to Y$. Among the conditions implying CAP we mention complex homogeneity and, more generally, the existence of a finite dominating family of holomorphic sprays. For a discussion of this subject and many further examples see [6].

It is possible to realize a Stein structure J_1 satisfying the conclusion of theorem 1.1 as a Stein domain $\Omega \subset X$ which contains the subvariety A and is diffeotopic to X relative to A. Here is a precise result; for $A = \emptyset$ this is theorem 1.2 in [7].

Theorem 1.2. *Let X be a Stein manifold with $\dim_{\mathbb{C}} X \neq 2$ and let $A \subset X$ be a closed complex subvariety. Given a continuous map $f\colon X \to Y$ to a complex manifold Y such that $f|_A\colon A \to Y$ is holomorphic, there exist a Stein domain $\Omega \subset X$ containing A, a holomorphic map $\widetilde{f}\colon \Omega \to Y$, and a diffeomorphism $h\colon X \to \Omega$ which is diffeotopic to id_X by a diffeotopy that is fixed on a neighborhood A, such that the map $\widetilde{f} \circ h\colon X \to Y$ is homotopic to f relative to A.*

Theorem 1.1 is an immediate consequence of theorem 1.2. Indeed, let $h_t\colon X \to h_t(X) \subset X$ be a diffeotopy satisfying $h_0 = id_X$, $h_1 = h\colon X \to \Omega$, and such that h_t is the identity map in a neighborhood of A for each $t \in [0,1]$. Let $J_t = h_t^*(J)$ denote the (unique) complex structure on X satisfying $dh_t \circ J_t = J \circ dh_t$ on TX. The homotopy $\{J_t\}_{t\in[0,1]}$ then satisfies theorem 1.1 (i), and the map $f_1 := \widetilde{f} \circ h\colon X \to Y$ is J_1-holomorphic and satisfies part (ii) of theorem 1.1.

Remark 1.3. A Stein structure J_1 satisfying the conclusion of theorem 1.1 will in general depend on the initial map f. However, as in [7] we can choose the same J_1 for all maps in a given family depending continuously on a parameter belonging to a compact Hausdorff space. We shall not formally state this more general version of theorem 1.1, but the reader will have no difficulty seeing how this is done by applying the parametric version of the main tools as in [7]. The analogous remark applies to theorem 1.2 in which the Stein domain $\Omega \subset X$ can be chosen the same for all maps in a compact family.

Theorems 1.1 and 1.2 are proved in §3 after we develop the main analytic ingredients in §2. In §4 we discuss the analogous result for Stein surfaces ($\dim_{\mathbb{C}} X = 2$).

2. The main lemma

An *almost complex structure* on an even dimensional smooth manifold X is a smooth endomorphism $J \in \mathrm{End}_{\mathbb{R}}(TM)$ satisfying $J^2 = -Id$. Such J gives rise to the conjugate differential d^c, defined on functions by $\langle d^c\rho, v\rangle = -\langle d\rho, Jv\rangle$ for $v \in TX$, and the Levi form operator dd^c. The structure J is *integrable* if every point of X admits an open neighborhood $U \subset X$ and a J-holomorphic coordinate map of maximal rank $z = (z_1, \ldots, z_n)\colon U \to \mathbb{C}^n$ ($n = \frac{1}{2}\dim_{\mathbb{R}} X$), i.e., satisfying $dz \circ J = idz$; for a necessary and sufficient integrability condition see [21].

We assume familiarity with standard complex analytic notions such as (strong) plurisubharmonicity and (strong) pseudoconvexity (see [13], [17]). Since we shall deal with several different (almost) complex structures, we shall write J-holomorphic, J-Stein, strongly J-plurisubharmonic, strongly J-pseudoconvex, etc., whenever a confusion might arise.

If (X, J) is a Stein manifold and $K \subset L \subset X$, with K compact, we shall say that K is *J-holomorphically convex in L* if for every $p \in L \backslash K$ there is a J-holomorphic function on (a neighborhood of) L such that $|f(p)| > \sup_{x \in K} |f(x)|$. When this holds with $L = X$ we say that K is $\mathcal{H}(X, J)$-convex.

The following lemma is the main ingredient in the proof of theorems 1.1 and 1.2.

Lemma 2.1. *Let (X, J_X) be a Stein manifold with $\dim_{\mathbb{C}} X = n \neq 2$. Let $\rho \colon X \to \mathbb{R}$ be a smooth strongly plurisubharmonic exhaustion function, let $c' < c$ be regular values of ρ, $K = \{x \in X \colon \rho(x) \leq c'\}$, and $L = \{x \in X \colon \rho(x) \leq c\}$. Let A be a closed complex subvariety of X. Assume that J is an almost complex structure on X which is integrable in an open neighborhood $U \subset X$ of $A \cup K$, it agrees with J_X in a neighborhood of A, and K is a strongly J-pseudoconvex domain with J-Stein interior. Let Y be a complex manifold with a distance function d_Y induced by a Riemannian metric.*

Given a continuous map $f \colon X \to Y$ which is J-holomorphic in a neighborhood of K and such that $f|_A \colon A \to Y$ is holomorphic, there exists for every $\epsilon > 0$ a homotopy of pairs (J_t, f_t) ($t \in [0,1]$), where J_t is an almost complex structure on X and $f_t \colon X \to Y$ is a continuous map, satisfying the following:

(i) *J_t agrees with $J_0 = J$ in a neighborhood of $A \cup K$ for all $t \in [0,1]$,*
(ii) *J_1 is integrable in a neighborhood of $A \cup L$,*
(iii) *L is a strongly J_1-pseudoconvex domain with J_1-Stein interior and K is J_1-holomorphically convex in L,*
(iv) *$f_0 = f$ and $f_t|_A = f|_A$ for each $t \in [0,1]$,*
(v) *for every $t \in [0,1]$ the map f_t is J-holomorphic in a neighborhood of K and satisfies $\sup_{x \in K} d_Y\big(f_t(x), f(x)\big) < \epsilon$, and*
(vi) *the map $f_1 \colon X \to Y$ is J_1-holomorphic in a neighborhood of L.*

If J is integrable on X then all structures J_t ($t \in [0,1]$) can be chosen integrable.

The situation is illustrated on fig. 1: J is integrable in $U \supset A \cup K$ (shown with the dashed line), $f|_A$ is holomorphic with respect to the complex structure induced by J_X, and f is J-holomorphic in a neighborhood of K. The pair (J_1, f_1) enjoys the analogous properties on the larger set L.

Proof. We may assume that $K = \{\rho \leq -1\}$ and $L = \{\rho \leq 0\}$.

The set K, being strongly J-pseudoconvex with a J-Stein interior, admits a basis of J-Stein neighborhoods. Also, since K is J_X-holomorphically convex in X and $J = J_X$ in a neighborhood of A, it follows that $A \cap K$ is holomorphically convex in A with respect to the complex structure on A induced by J. Theorem 2.1 in [5], applied to the set $A \cup K$ in the complex manifold $(U, J|_U)$, shows that $A \cup K$ admits a fundamental basis of open J-Stein neighborhoods $V_j \subset U$ such that K is J-holomorphically convex in V_j. Replacing U by such a neighborhood we shall assume that U is J-Stein and K is $\mathcal{H}(U, J)$-convex.

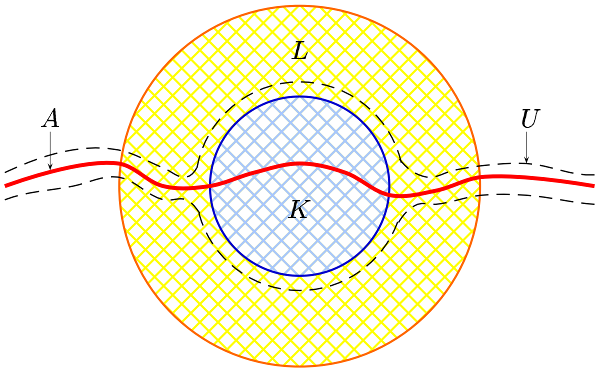

FIGURE 1. The main lemma.

Theorem 3.1 in the same paper [5] furnishes a J-holomorphic map $f'\colon U' \to Y$ in an open neighborhood $U' \subset U$ of $A \cup K$ such that $f'|_A = f|_A$, $f'|_K$ is uniformly as close as desired to $f|_K$, and f' is homotopic to f by a homotopy which is fixed on A and consists of maps that are holomorphic in a neighborhood of K and uniformly close to f on K. (The size of U' depends on the desired rate of approximation of $f|_K$ by $f'|_K$.) Using this homotopy we can patch f' with f outside a suitably chosen neighborhood of $A \cup K$ without changing f' sufficiently close to $A \cup K$. Replacing f by this new map f' and shrinking the neighborhood $U \supset A \cup K$ we may therefore assume that the map $f\colon X \to Y$ is J-holomorphic in a J-Stein domain $U \supset A \cup K$.

Let $g_1, \ldots, g_r$ be J_X-holomorphic functions on X such that

$$A = \{x \in X\colon g_1(x) = 0, \ldots, g_r(x) = 0\}.$$

We may assume that $\sum_{j=1}^r |g_j|^2 < 1$ on K. For every $\delta > 0$ the function

$$\phi_\delta = (\rho + 1) + \delta \cdot \log\left(\sum_{j=1}^r |g_j|^2\right)$$

is strongly J_X-plurisubharmonic on X, $\phi_\delta < 0$ on K, and $A = \{\phi_\delta = -\infty\}$. A generic choice of δ insures that $\Sigma_\delta = \{x \in L\colon \phi_\delta(x) = 0\}$ is a smooth strongly J_X-pseudoconvex hypersurface intersecting bL transversely.

We wish to smoothen the corner of the set $\{x \in L\colon \phi_\delta(x) \le 0\}$ along $\Sigma_\delta \cap bL$ so that the new domain will have J-Stein interior and smooth strongly J-pseudoconvex boundary. Let $\tau_\delta = rmax(\rho, \phi_\delta)$, where $rmax$ denotes a regularized maximum function (see lemma 5 in [2]). The function τ_δ is smooth and strongly J_X-plurisubharmonic on X (since $rmax$ preserves this property), it equals ρ near A (since $\phi_\delta|_A = -\infty$), and equals ϕ_δ on $\{x \in L\colon \phi_\delta \ge 0\}$ (since $\rho \le 0$ on L). The set $E_\delta = \{x \in L\colon \tau_\delta(x) \le 0\}$ has smooth strongly J_X-pseudoconvex boundary which coincides with bL in a neighborhood of $A \cap bL$, and it coincides with Σ_δ in $\{\rho \le c\}$ for some $c < 0$ close to 0. (The set E_δ is shown as D_0 in fig. 2 below.) We have $K \subset \mathrm{Int} E_\delta$ for every $\delta > 0$. As δ decreases to 0, E_δ shrinks down to $K \cup (A \cap L)$.

We claim that for a sufficiently small $\delta > 0$ the set E_δ has J-Stein interior and strongly J-pseudoconvex boundary bE_δ. Since E_δ is contained in the J-Stein manifold U, it suffices to verify the latter property; the first one will then follow from the general theory. Recall that $J = J_X$ in an open set $V \supset A$. The part of bE_δ which belongs to V is strongly J-pseudoconvex since $J = J_X$ in V. The remaining part $bE_\delta \cap (L\backslash V)$ converges to $bK\backslash V$ in the $\mathcal{C}^\infty$ topology as δ decreases to 0 as is seen from the definition of ϕ_δ. Since bK is strongly J-pseudoconvex by the assumption, it follows that $bE_\delta\backslash V$ is also such provided that $\delta > 0$ is chosen sufficiently small. This establishes the claim.

We now fix a $\delta > 0$ satisfying the above requirements and drop the δ from our notation; thus $\tau = \tau_\delta$ and $E_\delta = E$. For $t \in [0, 1]$ we set

$$\rho_t = (1-t)\tau + t\rho, \quad D_t = \{x \in X : \rho_t(x) \le 0\}.$$

The function ρ_t, being a convex combination of two strongly J_X-plurisubharmonic functions $\rho_0 = \tau$ and $\rho_1 = \rho$, is itself strongly J_X-plurisubharmonic. The sets D_t are strongly J_X-pseudoconvex with smooth boundaries, except at points where $d\rho_t = 0$. We have $E = D_0 \subset D_t \subset D_1 = L$ for every $t \in [0, 1]$; as t increases from 0 to 1, D_t monotonically increases from D_0 to $D_1 = L$ (fig. 2).

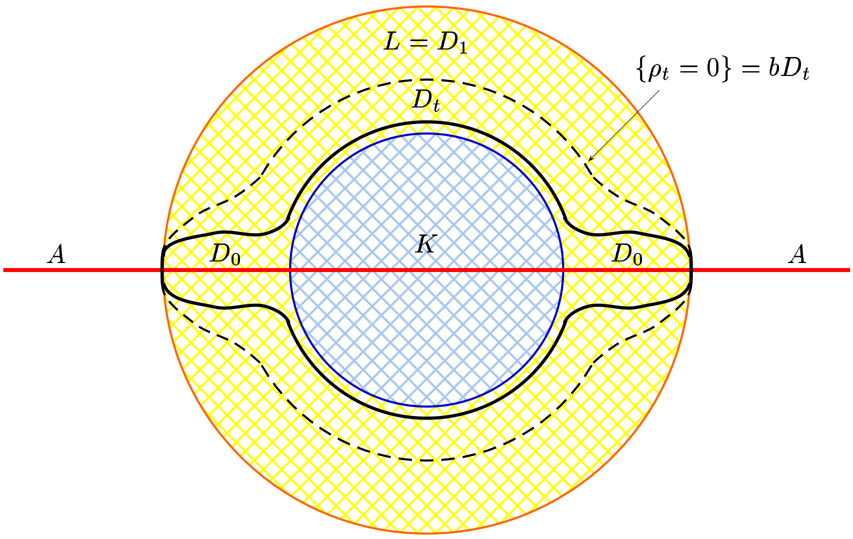

FIGURE 2. The sets $D_t = \{\rho_t \le 0\}$.

Our next goal is to show that the domain $L = D_1$ can be obtained (up to a diffeomorphism) from the domain D_0 by attaching handles of indices $\le n$. To this end we investigate the singular points of the hypersurfaces $bD_t = \{\rho_t = 0\}$ for $t \in [0, 1]$. By the construction, all these boundaries coincide on $\{\rho = 0, \tau = 0\} = bL \cap bD_0$ which is a relative neighborhood of $A \cap bL$ in bL. Since the boundaries $bD_0 = \{\tau = 0\}$ and $bL = \{\rho = 0\}$ are smooth, all nonsmooth points of bD_t are contained in the open set $\Omega = \{\rho < 0,\ \tau > 0\} = \mathrm{Int}L\backslash D_0$. The defining equation of $D_t \cap \Omega$ is $\tau \le t(\tau - \rho)$ and, after dividing by $\tau - \rho > 0$,

$$D_t \cap \Omega = \{x \in \Omega : \sigma(x) = \frac{\tau(x)}{\tau(x) - \rho(x)} \le t\}.$$

The critical point equation $d\sigma = 0$ is equivalent to

$$(\tau - \rho)d\tau - \tau(d\tau - d\rho) = \tau d\rho - \rho d\tau = 0.$$

Generic choices of ρ and τ insure that these equations have at most finitely many solutions $p_1, \ldots, p_m \in \Omega$, all nondegenerate (Morse) and belonging to pairwise distinct level sets of σ, and there are no solution on $b\Omega$. A calculation gives the following relationship between the Levi forms of these functions at a critical point p_j of σ:

$$\big(\tau(p_j) - \rho(p_j)\big)^2 \mathcal{L}_\sigma(p_j) = \tau(p_j)\mathcal{L}_\rho(p_j) - \rho(p_j)\mathcal{L}_\tau(p_j).$$

(In local holomorphic coordinates $z = (z_1, \ldots, z_n)$ at p, and with $w \in \mathbb{C}^n$, we have $\mathcal{L}_\sigma(p) \cdot w = \sum_{j,k=1}^n \frac{\partial \sigma}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k$, and $\mathcal{L}_\sigma(p) > 0$ means that this expression is positive for every $w \neq 0$.) Since $\tau(p_j) > 0$, $-\rho(p_j) > 0$ and the functions τ and σ are strongly plurisubharmonic, we obtain $\mathcal{L}_\sigma(p_j) > 0$. It follows that the Morse index of σ at p_j is $\leq n = \dim_{\mathbb{C}} X$. (If not, the $\mathbb{R}$-linear subspace of $T_{p_j}X$ corresponding to all negative eigenvalues of the real Hessian of σ at p_j would have real dimension at least $n+1$ and hence would contain a complex line $\lambda \subset T_{p_j}X$; the restriction of $\mathcal{L}_\sigma(p_j)$ to this line would therefore be negative, a contradiction.)

Choose numbers $t_0 = 0 < t_1 < t_2 < \ldots < t_m = 1$ which are regular values of $\sigma|_\Omega$ such that σ has exactly one critical point $p_j \in \Omega$ with $t_{j-1} < \rho(p_j) < t_j$ for $j = 1, 2, \ldots, m$. Let k_j denote the Morse index of σ at p_j; we have seen that $k_j \leq n$ for all j. By Morse theory [20] the domain D_{t_j} is diffeomorphic to a smooth handlebody obtained by attaching a handle of index k_j to $D_{t_{j-1}}$ and smoothing the corners (fig. 3).

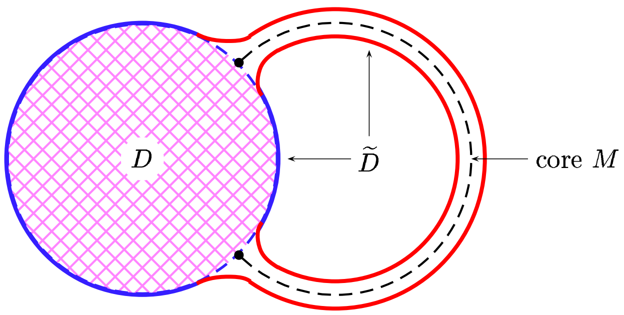

FIGURE 3. A handlebody $\widetilde{D}$.

Recall that a k-handle attached to a compact smoothly bounded domain $D \subset X$ is a diffeomorphic image of $\Delta_k \times \Delta_{2n-k} \subset \mathbb{R}^k \times \mathbb{R}^{2n-k}$, where Δ_k denotes the closed unit ball in $\mathbb{R}^k$. The set $b\Delta_k \times \Delta_{2n-k} = S^{k-1} \times \Delta_{2n-k}$ gets attached to bD, the image of $\Delta_k \times \{0\}^{2n-k}$ is called the *core disc* (or simply the core) of the handle, and the union of D with the handle, suitably smoothed at the corners, is a *handlebody* $\widetilde{D}$ shown on fig. 3. (In practice one often glues a handle to a thickening of D.) The Morse theory [20] tells us that every smooth manifold is obtained by successive gluing of handles, i.e., it admits a *handlebody decomposition.*

We are now ready to complete the proof of lemma 2.1. Define $W_0 := D_0$. By what has been said, D_{t_1} is diffeomorphic to a handlebody $W_1 \subset X$ obtained by attaching to W_0 a handle of index k_1. Since W_0 is strongly J-pseudoconvex and $k_1 \leq n \neq 2$, Eliashberg's results from [3] show that the core disc M of the handle can be chosen J-totally real in X and such that its boundary sphere bM is a J-Legendrian (complex tangential) submanifold of bW_0. (See lemma 3.1 in [7] for further details of this construction. It is here that the hypothesis $\dim_{\mathbb{C}} X \neq 2$ is needed; in the exceptional case $\dim_{\mathbb{C}} X \neq 2$ and $k_1 = 2$ it is in general impossible to find an *embedded* totally real core disc M for the 2-handle as shown by the gauge theory; see [3], [10] and [7]. We shall discuss this in §4 below.)

After a small homotopic deformation of J in a neighborhood of the core disc M away from W_0 we can make J integrable near $W_0 \cup M$, and the handlebody W_1 (a thickening of $W_0 \cup M$) can be chosen such that bW_1 is smooth strongly J-pseudoconvex, $\mathrm{Int} W_1$ is J-Stein, and W_0 is J-holomorphically convex in W_1. If the complex structure J is already integrable on X then the same can be accomplished without a homotopic correction of J as was shown in [3] and [7].

In addition, lemma 5.1 in [7] shows that we can choose W_1 sufficiently thin around $W_0 \cup M$ such that there exists a map $g_1 \colon X \to Y$ which is J-holomorphic in a neighborhood of W_1 and satisfies the following properties:

(a) $\sup_{x \in W_0} d_Y\big(f(x), g_1(x)\big) < \frac{\epsilon}{m}$,
(b) $g_1|_A = f|_A$,
(c) g_1 is homotopic to f by a homotopy $\{g_t\}_{t\in[0,1]}$ consisting of maps defined near W_1 which agree with f on A, they are holomorphic in a neighborhood of W_0, and each of them is $\frac{\epsilon}{m}$-close to f on W_0.

To obtain the interpolation conditions in (b) and (c) (which are not included in lemma 5.1 in [7]) the reader should observe that the proof of that lemma relies on theorem 3.2 in [5], p. 1924, which includes interpolation of the given map on a complex subvariety.

Using the homotopy $\{g_t\}$ we can patch all these maps with f outside a certain neighborhood of W_0 in order to get a homotopy of global maps $X \to Y$.

We now proceed to the next set D_{t_2}. By the same argument as above, D_{t_2} is diffeomorphic to a handlebody obtained from D_{t_1} by attaching a handle of index k_2. As D_{t_1} is diffeomorphic to W_1, D_{t_2} is also diffeomorphic to a handlebody $W_2 \subset D_{t_2}$ obtained by attaching a handle of index k_2 to W_1. By repeating the above arguments we modify J near the core (J-totally real) disc of the handle and then choose W_2 to be strongly J-pseudoconvex, with J-Stein interior, and such that W_1 is J-holomorphically convex in W_2. After shrinking W_2 around the union of W_1 with the core of the handle we also get a map $g_2 \colon X \to Y$ which is holomorphic in a neighborhood of W_2, it agrees with f on A, it satisfies $\sup_{x\in W_1} d_Y\big(g_2(x), g_1(x)\big) < \frac{\epsilon}{m}$, and is homotopic to g_1 by a homotopy $\{g_t\}_{t\in[1,2]}$ which is fixed on A, holomorphic near W_1 and uniformly $\frac{\epsilon}{m}$-close to g_1 on W_1.

Continuing inductively we obtain after m steps a a handlebody $W_m \subset L$ which is diffeomorphic to L, with an almost complex structure J (homotopic to the original one) which is integrable in a neighborhood of W_1, such that W_m is strongly J-pseudoconvex with J-Stein interior; we also obtain a map $g_m \colon X \to Y$ which is J-holomorphic in a neighborhood of W_m, it agrees with f on A, and it satisfies $\sup_{x\in D_0} d_Y\big(f(x), g_m(x)\big) < \epsilon$. Furthermore, there is a homotopy of maps $X \to Y$ from f to g_m which is fixed on A, each map in the family is holomorphic in a neighborhood of D_0 and uniformly ϵ-close to f on D_0 (and hence on K).

Our construction of the handlebodies $W_1, \ldots, W_m$ insures that there is a diffeomorphism $h \colon X \to X$ such that $h(L) = W_m$ and h is diffeotopic to id_X by a diffeotopy that is fixed in an open neighborhood of $A \cup K$. (We may even insure that $h(D_{t_j}) = W_j$ for $j = 0, 1, \ldots, m$.)

Let $J_1 = h^*(J)$ (i.e., $dh \circ J_1 = J \circ dh$) and $f_1 = g_m \circ h \colon X \to Y$. The almost complex structure J_1 is integrable in a neighborhood of $A \cup L$ (since J is integrable near W_m), and J_1 coincides with J (and hence with J_X) near the subvariety A (since h is the identity near A). If $\{h_t\}_{t\in[0,1]}$ is a diffeotopy on X from $h_0 = id_X$ to $h_1 = h$ which is fixed near A then $J_t = h_t^*(J)$ is a homotopy of almost complex structures which is fixed in a neighborhood of A and connects $J_0 = J$ to J_1. If J is integrable on X then so is J_t for every $t \in [0, 1]$ since conjugation by a diffeomorphism preserves integrability. This verifies properties (i) and (ii) in lemma 2.1.

By the construction the set $L = h^{-1}(W_m)$ is strongly J_1-pseudoconvex and has J_1-Stein interior (since W_m enjoys these properties with respect to the structure J). Since W_j was chosen J-holomorphically convex in W_{j+1} for $j = 0, 1, \ldots, m-1$ and K is J-holomorphically convex in U and hence in W_0, we see that K is J-holomorphically convex in W_m; hence K is J_1-holomorhically convex in L and (iii) holds.

The map $f_1 = g_m \circ h^{-1} \colon X \to Y$ is J_1-holomorphic in a neighborhood of L (since g_m is J-holomorphic in a neighborhood of $W_m = h(L)$), so (vi) holds. By the construction we also have $\sup_{x\in K} d_Y\big(f(x), f_1(x)\big) < \epsilon$. A homotopy from $f = f_0$ to f_1 satisfying properties (iv) and (v) is obtained by combining the individual homotopies obtained in the construction. This completes the proof. □

Remark 2.2. H. Hamm proved in [14] and [15] that for every n-dimensional Stein space X and closed complex subvariety $A \subset X$ the pair (X, A) is homotopically equivalent to a relative CW complex of dimension $\leq n = \dim_{\mathbb{C}} X$. (The absolute version with $A = \emptyset$ is a well known theorem of Lefshetz [19], Abraham and Fraenkel [1] and Milnor [20].) In his proof Hamm used Morse theory for manifolds with boundary. The essential step is the following (see [15], pp. 2–5):

Assume that X is an n-dimensional Stein space, $A \subset X$ is a closed complex subvariety and $X\backslash A$ is regular (without singularities). Let $K \subset L$ be sublevel sets of a real analytic, strongly plurisubharmonic Morse exhaustion function on X. Then $(A \cap L) \cup K$ admits a thickening $D \subset L$ such that $A \cup L$ is obtained from $A \cup D$ by attaching handles of index $\leq n$.

The geometric device in the proof of our lemma 2.1 (using the family of domains D_t which increase from D_0 to $D_1 = L$) accomplishes this step by only using the classical Morse theory for manifolds without boundary.

3. Proof of theorems 1.1 and 1.2

Theorem 1.1 follows from the following more precise result.

Theorem 3.1. *Let (X, J_X) be a Stein manifold with $\dim_{\mathbb{C}} X \neq 2$, let $K \subset X$ be a compact $\mathcal{H}(X, J_X)$-convex subset with smooth strongly J_X-pseudoconvex boundary, and let $A \subset X$ be a closed complex subvariety of X. Assume that J is an almost complex structure on X which is integrable in an open neighborhood of $A \cup K$, it agrees with J_X in a neighborhood of A, and such that K is a strongly J-pseudoconvex with J-Stein interior. Let Y be a complex manifold with a distance function d_Y induced by a Riemannian metric.*

Given a continuous map $f\colon X \to Y$ which is J-holomorphic in a neighborhood of K and such that $f|_A\colon A \to Y$ is holomorphic, there exists for every $\epsilon > 0$ a homotopy of pairs (J_t, f_t) ($t \in [0,1]$), where J_t is an almost complex structure on X and $f_t\colon X \to Y$ is a continuous map, satisfying the following:

(i) *$J_0 = J$, and J_t agrees with J in a neighborhood of $A \cup K$ for every $t \in [0,1]$,*
(ii) *the structure J_1 is integrable Stein on X and K is $\mathcal{H}(X, J_1)$-convex,*
(iii) *$f_0 = f$, and $f_t|_A = f|_A$ for every $t \in [0,1]$,*
(iv) *for each $t \in [0,1]$ the map f_t is J-holomorphic in a neighborhood of K and satisfies $\sup_{x\in K} d_Y\big(f_t(x), f(x)\big) < \epsilon$, and*
(v) *the map $f_1\colon X \to Y$ is J_1-holomorphic.*

If J is integrable on X then J_t can be chosen integrable for every $t \in [0,1]$.

We emphasize that the almost complex structure J on X is not assumed to be integrable except near $A \cup K$, and it need not be homotopic to J_X.

Theorem 1.1 corresponds to the special case $K = \emptyset$ and $J = J_X$ in theorem 3.1.

Proof. Choose a smooth strongly J_X-plurisubharmonic exhaustion function $\rho\colon X \to \mathbb{R}$ such that $K = \{x \in X\colon \rho(x) \le 0\}$ and $d\rho \ne 0$ on $bK = \{\rho = 0\}$. (Such ρ exists since K is strongly J_X-pseudoconvex and $\mathcal{H}(X, J_X)$-convex.)

Choose a sequence $c_0 = 0 < c_1 < c_2 \ldots$ consisting of regular values of ρ, with $\lim_{j\to\infty} c_j = +\infty$. Let $K_j = \{x \in X\colon \rho(x) \le c_j\}$. Set $f_0 = f$ and $J_0 = J$. Applying lemma 2.1 we inductively construct sequences of maps $f_j\colon X \to Y$ and of almost complex structures J_j satisfying the following for $j = 1, 2, \ldots$:

(a) J_j is integrable in a neighborhood of $A \cup K_j$, it agrees with J_X in a neighborhood of A, and it agrees with J_{j-1} in a neighborhood of K_{j-1},
(b) K_j is strongly J_j-pseudoconvex with J_j-Stein interior, and K_{j-1} is J_j-holomorphically convex in K_j,
(c) there is a homotopy of almost complex structures $J_{j,s}$ ($s \in [0,1]$), with $J_{j,0} = J_{j-1}$ and $J_{j,1} = J_j$, which is fixed in a neighborhood of $A \cup K_{j-1}$,
(d) the map $f_j\colon X \to Y$ is J_j-holomorphic in a neighborhood of K_j and $f_j|_A = f|_A$, and
(e) there is a homotopy $f_{j,s}\colon X \to Y$ ($s \in [0,1]$) which is fixed on A such that $f_{j,0} = f_{j-1}$, $f_{j,1} = f_j$, and for every $s \in [0,1]$ the map $f_{j,s}$ is J_j-holomorphic in a neighborhood of K_{j-1} and satisfies
$$\sup_{x\in K_{j-1}} d_Y\big(f_{j,s}(x), f_{j-1}(x)\big) < 2^{-j-1}\epsilon.$$

Indeed, assuming that we have already constructed the above sequences up to $j-1$, it suffices to apply lemma 2.1 with $K = K_{j-1}$, $L = K_j$, $f = f_{j-1}$, $J = J_{j-1}$, and ϵ replaced by $2^{-j-1}\epsilon$ to get the next complex structure J_j and the next map f_j satisfying the stated properties.

Condition (a) insures that $\widetilde{J} = \lim_{j\to\infty} J_j$ is an integrable complex structure on X which agrees with J in a neighborhood of $A \cup K$. Note that X is exhausted by the sequence of strongly $\widetilde{J}$-pseudoconvex domains K_j with $\widetilde{J}$-Stein interior. Property (b) implies that K_j is $\mathcal{H}(X, \widetilde{J})$-convex for $j = 0, 1, 2, \ldots$ and hence the manifold $(X, \widetilde{J})$ is Stein. By combining the individual homotopies furnished by (c) we obtain a homotopy of almost complex structures on X which connects J to $\widetilde{J}$ and is fixed in a neighborhood of $A \cup K$.

Properties (d) and (e) insure that the sequence $f_j\colon X\to Y$ converges uniformly on compacts in X to a $\widetilde{J}$-holomorphic map $\widetilde{f}=\lim_{j\to\infty} f_j\colon X\to Y$ satisfying $\widetilde{f}|_A=f|_A$ and $\sup_{x\in K} d_Y\big(\widetilde{f}(x),f(x)\big)<\epsilon$. Furthermore, condition (e) implies that the homotopies $f_{j,s}$ $(s\in[0,1], j=1,2,\ldots)$ can be assembled into a homotopy from f to $\widetilde{f}$ which is fixed on A, holomorphic on K, and ϵ-close to f on K.

Changing the notation so that $\widetilde{J}$ is denoted J_1 and $\widetilde{f}$ is denoted f_1 we obtain the conclusion of theorem 3.1. □

Remark 3.2. The Stein structure J_X on X was used in the above proof only to insure that for every $j=1,2,\ldots$ there is a thickening $D_{j-1}\subset K_j$ of the set $K_{j-1}\cup(A\cap K_j)$ such that $A\cup K_j$ is obtained (up to a diffeomorphism) by attaching handles of index $\le \dim_{\mathbb{C}} X$ to $A\cup D_{j-1}$. (In the proof of lemma 2.1 this was shown using the notation $K_j=L$, $K_{j-1}=K$ and $D_{j-1}=D_0$.) This leads to a proof of theorem 1.1 under the weaker conditions that (X,J) is an almost complex manifold of real dimension $2n\neq 4$ such that J is integrable in a neighborhood of a closed Stein subvariety $A\subset X$, and X is exhausted by an increasing sequence of compact strongly J-pseudoconvex domains $K_0\subset K_1\subset\ldots\subset\cup_{j=0}^{\infty}K_j=X$ such that every pair $(A\cup K_j, A\cup K_{j-1})$ satisfies the above topological condition.

Proof of theorem 1.2. We shall use the same tools as in the proof of theorem 3.1, but will change the induction procedure. Unlike in theorem 3.1, the complex structure on X will be unchanged during the entire proof.

Let $K_0\subset K_1\subset\cdots\subset\cup_{j=0}^{\infty}K_j=X$ be an exhaustion of X by compact, smoothly bounded, strongly pseudoconvex sets as in the proof of theorem 3.1. Set $f_0=f$. We shall assume that f_0 is holomorphic in a neighborhood of K_0 (choosing $K_0=\emptyset$ if so desired.) Let d_Y be a distance function on Y.

Given an $\epsilon>0$ we shall inductively construct a sequence of compact, smoothly bounded, strongly pseudoconvex sets $\emptyset=O_{-1}\subset O_0\subset O_1\subset\ldots\subset X$, a sequence of smooth diffeomorphisms $h_j\colon X\to X$, and a sequence of maps $f_j\colon X\to Y$ satisfying the following properties for $j=1,2,\ldots$:

(i) $h_j(K_j)=O_j$, and h_j is diffeotopic to h_{j-1} by a diffeotopy which is fixed in a neighborhood of $A\cup K_{j-1}$,
(ii) O_{j-1} is holomorphically convex in O_j,
(iii) f_j is holomorphic in an open neighborhood of O_j and satisfies $f_j|_A=f|_A$,
(iv) there is a homotopy $f_{j,s}\colon X\to Y$ $(s\in[0,1])$ such that $f_{j,0}=f_{j-1}$, $f_{j,1}=f_j$, the homotopy is fixed on A, each map $f_{j,s}$ is holomorphic in a neighborhood of O_{j-1}, and
$$\sup_{x\in O_{j-1}} d_Y\big(f_{j,s}(x),f_{j-1}(x)\big)<2^{-j-1}\epsilon,\quad s\in[0,1].$$

We begin by setting $O_0=K_0$, $h_0=id_X$ and $f_{0,s}=f_0$ for all $s\in[0,1]$. Suppose inductively that we have already constructed our sequences up to an index $j\in\mathbb{Z}_+$; thus the map $f_j\colon X\to Y$ is holomorphic on A and in an open neighborhood of O_j. Property (i) implies that h_j equals the identity map in a neighborhood of $A\cup K_0$. Hence $O_j\cap A=K_j\cap A$, and this set is holomorphically convex in A since K_j is $\mathcal{H}(X)$-convex. The set O_j, being strongly pseudoconvex, admits a basis of open Stein (strongly pseudoconvex) neighborhoods in X. In this situation theorem 3.1 in [5] applies and furnishes a map $f'_j\colon X\to Y$ which is holomorphic in an open neighborhood $V_j \supset A \cup O_j$ and which approximates f_j as close as desired uniformly on O_j. Replacing f_j by f'_j we may therefore assume that f_j is holomorphic in an open set $V_j \supset A \cup O_j$.

Applying lemma 2.1 with $f = f_j$, $K = K_j$ and $L = K_{j+1}$ we find a compact domain $D_j \subset K_{j+1}$ with strongly pseudoconvex boundary (denoted D_0 in lemma 2.1) such that $(A \cap K_{j+1}) \cup K_j \subset D_j$, K_{j+1} is obtained from D_j by attaching finitely many handles of index $\leq n = \dim_{\mathbb{C}} X$, and $h_j(D_j) \subset V_j$. The last inclusion is trivially satisfied in a neighborhood of A where h_j coincides with the identity map, while outside this neighborhood D_j can be chosen as close as desired to K_j; since $h_j(K_j) = O_j \subset V_j$, the inclusion follows.

Set $O'_j = h_j(D_j)$. If the above approximations were chosen sufficiently close then O'_j is a compact set with smooth strongly pseudoconvex boundary (since bO'_j coincides with bD_j near the subvariety A, and elsewhere bO'_j is $\mathcal{C}^\infty$-close to the strongly pseudoconvex hypersurface $h_j(bK_j) = bO_j$). Note that O_j is holomorphically convex in O'_j provided that D_j is chosen in a sufficiently small neighborhood of $(A \cap K_{j+1}) \cup K_j$. Applying the diffeomorphism h_j to the above sets we see that $h_j(K_{j+1})$ is diffeomorphic to a handlebody O_{j+1} obtained from $O'_j = h_j(D_j)$ by attaching finitely many handles of index $\leq n$.

We now proceed as in the proof of theorem 3.1. By Lemma 5.1 in [7] the above handles can be chosen such that the resulting handlebody O_{j+1} has smooth strongly pseudoconvex boundary, O'_j is holomorphically convex in O_{j+1}, and there is a map $f_{j+1}\colon X \to Y$ which is holomorphic in a neighborhood of O_{j+1}, it agrees with f_j on A, and $\sup_{x \in O_j} d_Y\big(f_{j+1}(x), f_j(x)\big) < 2^{-j-2}\epsilon$. The same lemma provides a homotopy from f_j to f_{j+1} satisfying property (iv) for the index $j+1$.

Since O_{j+1} is constructed from O'_j by using the topological data provided by the pair $D_j \subset K_{j+1}$ and all handles used in the construction of O_{j+1} are contained in $X \backslash A$, there exists a diffeomorphism $g_j\colon X \to X$ which maps $h_j(K_{j+1})$ onto O_{j+1} and which is diffeotopic to id_X by a diffeotopy which is fixed (equal the identity map) in a neighborhood of $A \cup O'_j$. The map $h_{j+1} = g_j \circ h_j\colon X \to X$ is a diffeomorphism of X which maps K_{j+1} onto O_{j+1} and is diffeotopic to h_j by a diffeotopy which is fixed near $A \cup K_j$. The induction may now continue.

Properties (i)–(iv) insure that $\Omega = \cup_{j=0}^{\infty} O_j \subset X$ is a Stein domain which contains $A \cup K_0$, the sequence f_j converges uniformly on compacts in Ω to a holomorphic map $\widetilde{f} = \lim_{j\to\infty} f_j\colon \Omega \to Y$ satisfying $\widetilde{f}|_A = f|_A$ and $\sup_{x \in K_0} d_Y\big(\widetilde{f}(x), f(x)\big) < \epsilon$. Also, there is a homotopy of maps $\Omega \to Y$ from $f|_\Omega$ to $\widetilde{f}$ which is holomorphic on K_0 and ϵ-close to f_0 on K_0. Property (i) also gives a diffeomorphism $h = \lim_{j\to\infty} h_j\colon X \to h(X) = \Omega$ which is diffeotopic to id_X and equals the identity map in a neighborhood of A. It follows that the map $\widetilde{f} \circ h\colon X \to Y$ is homotopic to f, thereby completing the proof of theorem 1.2.

4. The case $\dim_{\mathbb{C}} X = 2$

The proof of lemma 2.1 (and hence of theorems 1.1 and 1.2) breaks down when X is a Stein surface ($\dim_{\mathbb{C}} X = 2$), the reason being that a certain framing obstruction may arise when trying to attach a 2-handle with an embedded, totally real core disc, attached along a Legendrian knot to a given strongly pseudoconvex boundary in X. This obstruction in the proof has been pointed out by Eliashberg [3], and it was subsequently confirmed by results of the Seiberg-Witten theory that it cannot be removed in general. In particular, there exist smooth, orientable, almost complex 4-manifolds (X, J) with a handlebody decomposition without handles of index > 2 which do not admit any Stein structure; perhaps the simplest such example is the manifold $X = S^2 \times \mathbb{R}^2 = \mathbb{CP}^1 \times \mathbb{C}$. (Many other examples can be found in [10].) A precise obstruction for the existence of a Stein structure is provided by the *generalized adjunction inequality* which states that for every closed, orientable, smoothly embedded 2-surface S in a Stein manifold X, with the only exception of a null-homologous 2-sphere, we have

$$[S]^2 + |c_1(X)\cdot S| \le -\chi(S).$$

(See Chapter 11 in [12], or [23], for a proof, references to the original papers and further results.)

On the other hand, Gompf proved that there always exist *exotic Stein structures* on any such 4-manifold X [10], [11]. More precisely, given a smooth, almost complex 4-manifold (X, J) with a Morse exhaustion function without critical points of Morse index > 2, there exist a Stein surface (X', J') and an orientation preserving *homeomorphism* $h\colon X \to X'$ such that the class determined by the almost complex structure J' via h agrees with the class of J (see [10]).

Keeping the same hypotheses on (X, J), the authors have shown in §7 of [7] that for any continuous map $f\colon X \to Y$ to a complex manifold Y, a Stein surface (X', J') and a homeomorphism $h\colon X \to X'$ in Gompf's theorem can be chosen such that there exists a J'-holomorphic map $f'\colon X' \to Y$ with the property that the map $\widetilde{f} = f' \circ h\colon X \to Y$ is homotopic to f. If in addition the almost complex structure J on X is integrable (not necessarily Stein), one can realize such (X', J') as an open J-Stein domain $\Omega \subset X$ which is homeomorphic to X (theorem 1.2 in [7]; without considering mappings this is again due to Gompf [11]).

The constructions in [10], [11] and [7] use *kinky discs* and *Casson handles* at every place where a framing obstruction arises in the construction, together with the famous result of Freedman to the effect that a Casson handle is homeomorphic to a standard index two handle $\Delta_2 \times \Delta_2 \subset \mathbb{R}^4$ [8], [9]. By using the same tools, together with the methods explained in this paper, one can prove the following interpolation theorem which is the analogue of theorem 1.2 in the case $\dim_{\mathbb{C}} X = 2$.

Theorem 4.1. *Let X be a Stein surface and $A \subset X$ a closed complex subvariety. Given a continuous map $f\colon X \to Y$ to a complex manifold Y such that $f|_A\colon A \to Y$ is holomorphic, there exist a Stein domain $\Omega \subset X$ containing A, a holomorphic map $\widetilde{f}\colon \Omega \to Y$, and an orientation preserving homeomorphism $h\colon X \to \Omega$ which is homeotopic to id_X by a homeotopy that is fixed on a neighborhood A, such that the map $\widetilde{f} \circ h\colon X \to Y$ is homotopic to f relative to A.*

This is proved by modifying the proof of theorem 1.2 in §3 above, where the necessary modification is explained in details in the proof of theorem 1.2 in [7] (p. 32 in §7 of [7]). To avoid unnecessary repetitions we shall indicate the essential point of this modification and refer the reader to [7] for further details.

Let J denote the Stein structure on X. We assume the notation used in the proof of theorem 1.2 in §3 above. In that proof it is explained how one obtains a strongly pseudoconvex handlebody O_{j+1} by attaching handles of index $\le n$ to a strongly pseudoconvex domain O'_j. Each of the handles must have an embedded totally real core disc, attached to the previous strongly pseudoconvex hypersurface along a Legendrian knot; this enables us to choose the next handlebody to be strongly pseudoconvex, and to approximate the holomorphic map by a map which is holomorphic on a neighborhood of the new (larger) handlebody.

When $\dim_{\mathbb{C}} X = 2$, a framing problem may arise for handles of index 2, and a required totally real core disc M does not exist in general. As explained in [7] (and before that in [10]), the problem can be resolved by choosing an embedded core disc M which is attached to the given strongly pseudoconvex domain $W \subset X$ along a Legendrian knot $bM \subset bW$, and then adding finitely many (positive) ***kinks*** to M. More precisely, we remove from M finitely many small pairwise disjoint discs and glue along each of the resulting circles an immersed disc with one positive double point. (Fig. 4, borrowed from [7], shows a kink with a trivializing disc Δ which will be attached at the next step in order to cancel the superfluous loop at the double point p. A model kink used in [7] is provided by an explicit immersed Lagrangian sphere in $\mathbb{C}^2$ due to Weinstein [24].)

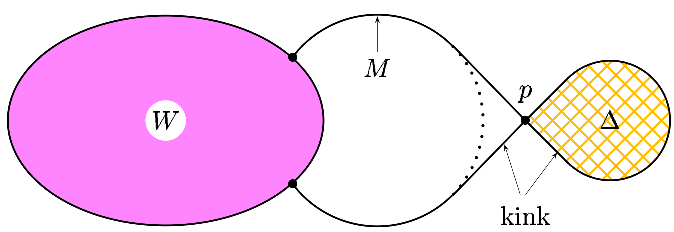

FIGURE 4. A kinky disc M with a trivializing 2-cell Δ

As explained in [7], kinking the core disc sufficiently many times gives an immersed disc which can be deformed to a totally real immersed disc $M' \subset X \backslash \mathrm{IntW}$, attached to bW along a Legendrian knot $bM' \subset bW$. It is then possible to find a thin strongly pseudoconvex neighborhood $W' \subset X$ of $W \cup M'$ and a holomorphic map $W' \to Y$ which approximates the given initial map $f\colon X \to Y$ uniformly on W (see [7]). The manifold W' does not have the correct topology (it is not even homeomorphic to the domain obtained by attaching to W a standard handle with an embedded core disc). The problem is corrected in the next stage of the construction by attaching to W' a trivializing 2-disc Δ at each of the kinky points in order to cancel the extra loop. Unfortunately the framing obstruction arises at this disc as well, requiring us to place another kink on Δ which will require a new trivializing disc, etc. The ensuing procedure is always infinite, it can be carried out in a small neighborhood of the initial kinky point in M, and (the good point!) it converges to an attached Casson handle which is homeomorphic to the standard 2-handle $\Delta_2 \times \Delta_2$ (Freedman [8], [9]). Performing this construction inside the Stein manifold X gives a Stein domain $\Omega \subset X$ which is homeomorphic, but in general not diffeomorphic to X due to the presence of Casson handles. A more precise description of this construction can be found in [7] (and also in [11] if one is not interested in holomorphic maps). To insure that Ω contains the given subvariety $A \subset X$ we follow the proof of theorem 1.2 with these modifications.

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