University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1 111 Ljubljana, Slovenia

Preprint series, Vol. 43 (2005), 991

STEIN STRUCTURES AND HOLOMORPHIC MAPPINGS

Franc Forstnerič   Marko Slapar

ISSN 1318-4865

September 23, 2005

Ljubljana, September 23, 2005

STEIN STRUCTURES AND HOLOMORPHIC MAPPINGS

FRANC FORSTNERIČ & MARKO SLAPAR

ABSTRACT. We prove that every continuous map from a Stein manifold X to another complex manifold Y can be made holomorphic by a homotopic deformation of both the map and the complex structure on X, and the latter can be chosen independent of a map in a compact family. In the absence of topological obstructions the holomorphic map may be chosen to have pointwise maximal rank. We show as a corollary that when TX is trivial then X is a Riemann domain over $\mathbb{C}^n$ in a Stein structure homotopic to the original one.

1. INTRODUCTION

The complex manifolds with richest function theoretic properties are undoubtedly the *Stein manifolds* (K. Stein [71]) which are characterized as the closed complex submanifolds of Euclidean spaces ([48], p. 224; [13]). Another characterization, more relevant to the purposes of this paper, is that a complex manifold is Stein precisely when it admits a strongly plurisubharmonic Morse exhaustion function (Grauert [41]). Such function enables one to solve $\overline{\partial}$-problems and thus create global holomorphic objects (Hörmander [51]); it also gives important differential topological information — since its Morse indices cannot exceed the complex dimension of the manifold, an n-dimensional Stein manifold is the interior of a handlebody without handles of index $> n$ (Lefshetz [59], Andreotti and Frankel [1], Milnor [61]).

In the converse direction, Eliashberg proved in his seminal work [10] that an almost complex manifold (X, J) with a correct handlebody structure and $\dim_{\mathbb{R}} X \neq 4$ admits an integrable Stein structure $\widetilde{J}$ homotopic to J; if $\dim_{\mathbb{R}} X = 4$ then X is homeomorphic to a Stein surface according to Gompf ([35], [36]), but the underlying smooth structure must be changed in general.

In this paper we establish the complementary theory of holomorphic mappings (morphisms). The following is a simplified statement of some of our main results.

Theorem 1.1. *Let (X, J) be a smooth almost complex manifold of real dimension $2n$ which admits a Morse exhaustion function without critical points of index $> n$. If $n \neq 2$, or if $n = 2$ and ρ has no critical points of index > 1, then every continuous map $f\colon X \to Y$ to a complex manifold Y is homotopic to a $\widetilde{J}$-holomorphic map $\widetilde{f}\colon X \to Y$, where $\widetilde{J}$ is a Stein structure on X homotopic to J. If $n = 2$, there is an orientation preserving homeomorphism $h\colon X \to X'$ onto a Stein surface X' and a holomorphic map $f'\colon X' \to Y$ such that $\widetilde{f} := f' \circ h\colon X \to Y$ is homotopic to f.*

Date: September 19, 2005.
2000 *Mathematics Subject Classification.* 32H02,32Q28,32Q30,32Q55,32Q60,32T15,57R17.
Key words and phrases. Stein manifolds, complex structures, holomorphic mappings.
Supported by grants P1-0291 and J1-6173, Republic of Slovenia.

Simple example show that in general we must change the complex structure on X even if the initial structure J is integrable Stein (example 6.2). The result for $\dim_{\mathbb{R}} X = 4$ can be equivalently expressed by saying that any map from X is homotopic to a holomorphic map with respect to a possibly *exotic Stein structure* on X, i.e., one whose underlying $\mathcal{C}^\infty$ structure on X may differ from the original one. Precise statements with additional information, including a parametric version (for a parameter in a compact Hausdorff space), and with uniform approximation on certain compact subsets of X, are given by theorem 6.1 for $n \neq 2$, and by theorem 7.1 for $n = 2$. As in Eliashberg's main theorem in [10] it is possible to choose the Stein structure $\widetilde{J}$ in theorem 1.1 such that the sublevel sets of ρ are strongly $\widetilde{J}$-pseudoconvex away from the critical points.

According to Eliashberg [10] and Gompf [35] we can change the initial almost complex structure J in theorem 1.1 to an integrable Stein structure on X (in dimension four we may also have to change the smooth structure on X), so the main content of our theorem is that one can homotopically change a given Stein structure to another Stein structure in which the given map $f\colon X \to Y$ admits a holomorphic representative in its homotopy class. The first reduction simplifies the proof only marginally, and for this reason we prefer to state the result in the context of almost complex manifolds. However, if J is integrable (but not necessarily Stein), we can realize a Stein structure $\widetilde{J}$ satisfying the conclusion of theorem 1.1 as a Stein domain $\Omega \subset X$ which is diffeomorphic (resp. homeomorphic) to X, and in this case there is a homotopy from J to $\widetilde{J}$ consisting of integrable structures. The precise statement is the following (compare with Theorem 1.3.6 of Eliashberg [10]).

Theorem 1.2. *Let X be a complex manifold of dimension n which admits a Morse exhaustion function without critical points of index $> n$. Let $f\colon X \to Y$ be continuous map to a complex manifold Y. If $n \neq 2$ then there exist an open Stein domain $\Omega \subset X$, a diffeomorphism $h\colon X \to h(X) = \Omega$ diffeotopic to id_X, and a holomorphic map $f'\colon \Omega \to Y$ such that $f' \circ h\colon X \to Y$ is homotopic to f. If $n = 2$, the same is true if ρ has no critical points of index > 1; in the presence of critical points of index 2 the conclusion holds with h being a homeomorphism.*

If $h_t\colon X \to h_t(X) \subset X$ ($t \in [0,1]$) is a diffeotopy from $h_0 = id_X$ to $h_1 = h\colon X \to \Omega$ as in theorem 1.2 then $J_t := h_t^*(J)$ is a homotopy of integrable complex structures on X from $J_0 = J$ to the Stein structure $\widetilde{J} = h^*(J|_{T\Omega})$ on X, and $\widetilde{f} = f' \circ h\colon X \to Y$ is a $\widetilde{J}$-holomorphic map homotopic to f, so we get the conclusion of theorem 1.1. However, assuming that J is integrable Stein, we do not know whether a homotopy from J to $\widetilde{J}$ can be realized through Stein structures.

With the possible exception of dimension four, we also find holomorphic maps of maximal rank (immersions resp. submersions) provided that there are no topological obstructions. The following is a simplified version of theorem 6.3 below.

Theorem 1.3. *Let (X, J) be a smooth almost complex manifold of real dimension $2n \neq 4$ which admits a Morse exhaustion function $\rho\colon X \to \mathbb{R}$ without critical points of index $> n$. If $f\colon X \to Y$ is a continuous map such that there exists a complex vector bundle map $\iota\colon TX \to f^*(TY)$ of fiberwise maximal rank then there is a Stein structure $\widetilde{J}$ on X, homotopic to J, and a $\widetilde{J}$-holomorphic map $\widetilde{f}\colon X \to Y$ of pointwise maximal rank which is homotopic to f. The analogous conclusion holds if $n = 2$ and ρ has no critical points of index > 1.*

The conclusion of theorem 1.3 fails in general with a fixed Stein structure on X, unless Y satisfies a certain holomorhic flexibility condition introduced (for submersions) in [24]. Results on this subject for maps to Euclidean spaces $Y = \mathbb{C}^N$ can be found in [45] (for immersions) and in [23] (for submersions). Theorem 1.3 is an analogue in the holomorphic category of the Smale-Hirsch h-principle for smooth immersions ([70], [50], [45]) and of the Gromov-Phillips h-principle for smooth submersions ([42], [67]).

An important source of Stein manifolds, especially so from the historical point of view, are the *holomorphically complete Riemann domains* $\pi\colon X \to \mathbb{C}^n$, π being a locally biholomorphic map. These arise as the envelopes of holomorphy of domains in, or over, $\mathbb{C}^n$. Every such manifold is holomorphically parallelizable, but the converse has been a long standing open problem:

Does every n-dimensional Stein manifold X with a trivial complex tangent bundle admit a locally biholomorphic map $\pi\colon X \to \mathbb{C}^n$ with $n = \dim_{\mathbb{C}} X$?

In 1967 Gunning and Narasimhan [47] gave a positive answer for open Riemann surfaces (these are precisely the one dimensional Stein manifolds). In 2003 the first author proved that every parallelizable Stein manifold X^n admits a holomorphic submersion to $\mathbb{C}^{n-1}$ [23]; its level sets define a foliation of X by open Riemann surfaces, and the remaining problem is to find a holomorphic function on X whose restriction to each leaf has no critical points on the leaf. Applying theorem 1.3 with $Y = \mathbb{C}^n$ ($n = \dim_{\mathbb{C}} X$) gives the following partial answer to the above question.

Corollary 1.4. *If (X, J) is as in theorem 1.3 and if the tangent bundle (TX, J) is trivial then there is a homotopic to J Stein structure $\widetilde{J}$ on X and a $\widetilde{J}$-holomorphic immersion $\pi\colon X \to \mathbb{C}^n$ with $n = \dim_{\mathbb{C}} X$.*

Thus a suitable choice of a Stein structure makes X a Riemann domain over $\mathbb{C}^n$.

Note that parallelizable Stein manifolds are much more common than parallelizable smooth manifold; indeed, every smooth complex hypersurface in a complex Euclidean space is such, and so is every closed complex submanifold $X \subset \mathbb{C}^N$ with trivial complex normal bundle $T\mathbb{C}^N|_X/TX$ [18].

Before proceeding we recall some basic notions of the *handlebody theory*; precise treatments can be found in texts on differential topology such as [33], [37], [61]. Let X be a smooth compact n-manifold with boundary ∂X. Let D^k denote the closed unit ball in $\mathbb{R}^k$. A *k-handle H* attached to X is a copy of $D^k \times D^{n-k}$ smoothly attached to ∂X along $\partial D^k \times D^{n-k}$, with the corners smoothed, which gives a larger manifold with boundary. The central disc $D^k \times \{0\}^{n-k}$ is the *core* of H. A *handle decomposition* of a smooth (open or closed) manifold X is a representation of X as an increasing union of compact domains with boundary $X_j \subset X$ such that X_{j+1} is obtained by attaching a handle to X_j. (In the case of open manifolds one takes the interior of the resulting handlebody.) The *Morse theory* [61] tells us that every smooth manifold admits a handlebody representation, and the *Kirby calculus* [37] provides effective ways of finding sufficiently simple representations and telling whether two representations give the same manifold.

We now discuss more carefully the case $\dim_{\mathbb{R}} X = 4$ which is especially interesting. A *complex surface* will mean a complex 2-dimensional manifold. The following is a consequence of theorem 1.1 (with $n = 2$) and of Corollary 3.2 and Theorem 3.3 of Gompf ([35], p. 648).

Corollary 1.5. *Let X be a smooth, closed, oriented 4-manifold. There exists a smooth, finite wedge of circles $\Gamma \subset X$ such that for every continuous map $f\colon X\backslash\Gamma \to Y$ to a complex manifold Y there is a (possibly exotic) Stein structure on $X\backslash\Gamma$ and a holomorphic map $\widetilde{f}\colon X\backslash\Gamma \to Y$ homotopic to f. If $X = \mathbb{CP}^2$, this holds after removing a single point (and in this case any Stein structure on $\mathbb{CP}^2\backslash\{p\}$ is exotic). The analogous result holds for a smooth, open, oriented 4-manifold after removing a suitably chosen smooth 1-complex.*

The point is that a wedge of circles $\Gamma \subset X$ may be chosen such that $X\backslash\Gamma$ admits a handle decomposition without 3-handles and 4-handles. The projective plane $\mathbb{CP}^2$ has a single 4-cell (and no 3-cells) in its handlebody decomposition, hence removing a point leaves only cells of index ≤ 2.

A well known consequence of the *Seiberg-Witten theory* is that certain smooth, orientable 4-manifolds with a correct handlebody structure (i.e., without handles of index > 2) do not admit any Stein structure. Indeed, a closed, orientable real surface S which is smoothly embedded in a Stein surface X (or in a compact Kähler surface with $b^+(X) > 1$) and is not a null-homologous 2-sphere satisfies the *generalized adjunction inequality*

$$[S]^2 + |c_1(X)\cdot S| \leq -\chi(S).$$

(See Chapter 11 in [37] and the papers [55], [60], [66], [62], [21]. This can also be explained by the Heegaard-Floer theory developed recently by Ozsváth and Szabo, see e.g. [65].) For a 2-sphere we obtain $[S]^2 \leq -2$, and hence the smooth (even complex) manifold $X = S^2 \times \mathbb{R}^2 = \mathbb{CP}^1 \times \mathbb{C}$ does not admit any Stein structure. (Each embedded sphere $S^2 \times \{c\} \subset X$ ($c \in \mathbb{R}^2$) generating $H_2(X,\mathbb{Z}) = \mathbb{Z}$ has self-intersection number zero.) Similarly, $\mathbb{CP}^2\backslash\{p\}$ has no nonexotic Stein structure because of embedded projective lines with self-intersection index $+1$.

Nevertheless, there is a bounded Stein domain in $\mathbb{C}^2$ which is *homeomorphic* to $S^2 \times \mathbb{R}^2$. This is a very special case of Gompf's main result in [36] (Theorem 2.4) to the effect that for every tamely topologically embedded CW 2-complex M in a complex surface X there is a topological isotopy of X, uniformly close to the identity on X, carrying M onto a complex $M' \subset X$ with a *Stein thickening*, i.e., open domain $\Omega \subset X$ which is Stein in the complex structure inherited from X and is homeomorphic to the interior of a handlebody with core M. (An embedded CW complex in X is said to be *tame* if it admits a thickening inside X.) Combining Gompf's methods [36] with those of the present paper we obtain the following.

Corollary 1.6. *Let $M \subset X$ be a tame, topologically embedded CW 2-complex in a complex surface X and let $U \subset X$ be an open neighborhood of M. For every continuous map $f\colon M \to Y$ to a complex manifold Y there exist a topological isotopy $h_t\colon X \to X$, with $h_0 = id_X$ and $h_t(M) \subset U$ for all $t \in [0,1]$, a Stein thickening $\Omega \subset U$ of the CW complex $h_1(M)$, and a holomorphic map $\widetilde{f}\colon \Omega \to Y$ such that the map $\widetilde{f} \circ h_1\colon M \to Y$ is homotopic to $f\colon M \to Y$.*

Gompf showed that the necessary adjustment of the initial 2-complex M in X is quite mild from the topological point of view, and all essential data of the topological embedding $M \hookrightarrow X$ can be preserved. Stein domains constructed in this way will typically have nonsmooth boundaries in X and may be chosen to realize uncountably many distinct diffeomorphism types. In certain special cases when the 2-cells in M satisfy certain framing conditions it is possible to find Stein thickenings of $\mathcal{C}^0$-small smooth perturbation of M in X with the diffeomorphism type of a smooth handlebody with core M; in this direction see the recent paper by Costantino [7].

It seems appropriate to compare our results with the classical *Oka-Grauert theory* where similar questions are studied for a fixed Stein structure on the source Stein manifold X. A complex manifold Y is said to enjoy the *basic Oka property* if for every Stein manifold X, each homotopy class of maps $X \to Y$ admits a holomorphic representative. We can also talk about the *parametric Oka property*, the *Oka property with approximation* (on compact holomorphically convex subsets), and the *Oka property with interpolation* (on closed complex subvarieties). By the classical results of Oka [64] and Grauert [38], [39], [40] (see also Cartan [4]) all these properties hold if Y is complex homogeneous. Generalizations were given by many authors, see e.g. [46], [49], [29], [30], [22], [57], [58]. It was recently shown by the first author ([25], [26]) that all basic (and also the 1-parametric) Oka type properties of Y are equivalent to the following *convex approximation property*:

CAP: *Any holomorphic map from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) to Y can be approximated uniformly on K by entire maps $\mathbb{C}^n \to Y$.*

Thus we can view the Oka-Grauert theory on the one hand, and the results of the present paper on the other hand, as two facets of the following

Generalized Oka principle: *Every map $X \to Y$ from a Stein manifold X to a complex manifold Y is homotopic to a holomorphic map provided that Y satisfies* CAP, *or that we are free to change the Stein structure on X.*

Our proofs combine the construction of Stein manifolds, due to Eliashberg and Gompf, with the holomorphic approximation techniques developed in [52], [23] and [24] (theorem 4.1 below). To prove theorems 1.1 and 1.3 (see §6) we construct a sequence (X_j, J_j, f_j) $(j = 0, 1, 2, \ldots)$ where

(i) $X_0 \subset X_1 \subset X_2 \subset \cdots \subset \cup_j X_j = X$ are smoothly bounded, relatively compact domains, chosen as sublevel sets of a Morse exhaustion function $\rho\colon X \to \mathbb{R}$ without critical points of index $> \dim_{\mathbb{C}} X$,
(ii) J_j is an almost complex structure on X which is integrable in a neighborhood of $\overline{X}_j$ and is homotopic to J_{j-1} by a homotopy that is fixed on $\overline{X}_{j-1}$,
(iii) (X_j, J_j) is a Stein manifold with strongly pseudoconvex boundary ∂X_j which is Runge in (X_{j+1}, J_{j+1}), and
(iv) $f_j\colon X \to Y$ is a continuous map which is J_j-holomorphic in a neighborhood of $\overline{X}_j$ and which approximates f_{j-1} uniformly on $\overline{X}_{j-1}$.

The complex structure $\widetilde{J}$ on X which agrees with J_j on X_j then makes X into a Stein manifold, and the limit map $\widetilde{f} = \lim_{j\to\infty} f_j\colon X \to Y$ is $\widetilde{J}$-holomorphic. In the inductive step we apply the methods of Eliashberg [10] to modify J_j to an integrable Stein structure J'_j in a handlebody $X'_{j+1} \subset X$, obtained by attaching to $\overline{X}_j$ a handle of correct index (equal to the next Morse index of ρ) whose core is a totally real disc M_j attached to ∂X_j along a Legendrian sphere $\partial M_j \subset \partial X_j$. The map $f_j\colon X \to Y$ is then approximated uniformly on X_j by a map $f'_{j+1}\colon X \to Y$ which is J'_j-holomorphic in a thin neighborhood of $\overline{X}_j \cup M_j$, and X'_{j+1} is trimmed to a Stein domain $X''_{j+1} \supset \overline{X}_j$ with a strongly pseudoconvex boundary and contained in the region of holomorphicity of f'_{j+1}. Finally we apply a diffeomorphism $X \to X$, diffeotopic to id_X, which maps a suitable higher sublevel set X_{j+1} of ρ onto the handlebody X''_{j+1} and which is fixed on $\overline{X}_j$, to get $(X_{j+1}, J_{j+1}, f_{j+1})$.

In dimension four we follow Gompf [35], using *kinky handles* of index 2 and performing an inductive procedure which cancels all superfluous loops caused by kinks, thereby creating *Casson handles* which are homeomorphic, but not diffeomorphic, to the standard index two handle $D^2 \times D^2$ according to Freedman [32]. To construct a holomorphic map, we perform the Casson tower construction simultaneously at a possibly increasing number of places. The details are found in §7.

A similar construction can be carried out inside a given complex manifold X to prove theorem 1.2 and corollary 1.6.

The point of view of allowing changes of the (almost) complex structure when considering complex curves in X was pioneered by M. Gromov [44], but to our knowledge it had not been discussed before in holomorphic mapping problems when the source manifold is not a Riemann surface. A result similar in spirit to those in the present paper, concerning embeddings of finite bordered Riemann surfaces X into $\mathbb{C}^2$, has been proved in [5]: For a fixed topological type of X, a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^2$ exists for a nonempty open set of complex structures on X. It is unknown whether all open Riemann surfaces embed in $\mathbb{C}^2$; for results in this direction see [34], [16], [17].

We wish to add that Eells and Sampson [9] used variational methods to construct *harmonic maps* in a given homotopy class of maps between Riemannian manifolds. It would be very interesting to find a similar approach for holomorphic maps.

Organization of the paper. In §2 we recall the relevant notions from Stein geometry and contact geometry. Sections §3 – §5 contain preparatory lemmas. A main analytic ingredient is a Hörmander-Wermer type approximation theorem for maps to arbitrary complex manifolds (theorem 4.1 in §4). In §5 we prove lemma 5.1 which provides an approximate extension of a holomorphic map to an attached handle. The main results of the paper are presented and proved in sections §6 (for $\dim_{\mathbb{R}} X \neq 4$) and §7 (for $\dim_{\mathbb{R}} X = 4$).

2. Preliminaries

An *almost complex structure* on an even dimensional smooth manifold X is a smooth endomorphism $J \in \mathrm{End}_{\mathbb{R}}(TM)$ satisfying $J^2 = -Id$. The complex structure operator J gives rise to the conjugate differential d^c, defined on functions by $\langle d^c\rho, v\rangle = -\langle d\rho, Jv\rangle$ for $v \in TX$, and the Levi form operator dd^c. The structure J is *integrable* if every point of X admits an open neighborhood $U \subset X$ and a J-holomorphic coordinate map of maximal rank $z = (z_1, \ldots, z_n) \colon U \to \mathbb{C}^n$ ($n = \frac{1}{2}\dim_{\mathbb{R}} X$), i.e., satisfying $dz \circ J = idz$; for a necessary and sufficient integrability condition see Newlander and Nirenberg [63].

If $h \colon X \to X'$ is a diffeomorphism and J' is an almost complex structure on X', we denote by $J = h^*(J')$ the (unique) almost complex structure on X satisfying $dh \circ J = J' \circ dh$; i.e., such that h is a biholomorphism. Similarly we denote by $J' = h_*(J)$ the push-forward of an almost complex structure J by h. A map $f' \colon X' \to Y$

to a complex manifold Y is J'-holomorphic if and only if $f = f' \circ h\colon X \to Y$ is J-holomorphic with $J = h^*(J')$.

An integrable structure J on a smooth manifold X is said to be *Stein* if (X, J) is a Stein manifold; this is the case if and only if there is a *strongly J-plurisubharmonic Morse exhaustion function* $\rho\colon X \to \mathbb{R}$, i.e., $\langle dd^c\rho, v \wedge Jv\rangle > 0$ for every $0 \neq v \in TX$ (Grauert [41]). The $(1,1)$-form $\omega = dd^c\rho = 2i\partial\overline{\partial}\rho$ is then a symplectic form on X, defining a J-invariant Riemannian metric $g(v,w) = \langle \omega, v \wedge Jw\rangle$ $(v, w \in TX)$. The Morse indices of such function ρ are $\leq n = \frac{1}{2}\dim_{\mathbb{R}} X$ and hence X is the interior of a handlebody without handles if index $> n$ [1], [61].

A real subbundle V of the tangent bundle TX is said to be *J-real*, or *totally real*, if $V_x \cap JV_x = \{0\}$ for every $x \in X$; in this case we may, and shall, identify its complexification $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ with the J-complex linear subbundle $V \oplus JV$ of TX. An immersion $G\colon D \to X$ of a smooth manifold D into X is J-real (or *totally real*) if $dG_x(T_xD)$ is a J-real subspace of $T_{G(x)}X$ for every $x \in D$.

Let W be a relatively compact domain with smooth boundary $\Sigma = \partial W$ in an almost complex manifold (X, J). The set $\xi = T\Sigma \cap J(T\Sigma)$ is a corank one J-complex linear subbundle of $T\Sigma$. Assume now that ρ is a smooth function in a neighborhood of $\Sigma = \partial W$ such that $\Sigma = \{\rho = 0\}$, $d\rho \neq 0$ on Σ and $\rho < 0$ on W. Let $\eta := d^c\rho|_{T\Sigma}$, a one-form on Σ with $\ker\eta = \xi$. We say that Σ is *strongly J-pseudoconvex*, or simply *J-convex*, if $\langle dd^c\rho, v \wedge Jv\rangle > 0$ for all $0 \neq v \in \xi$; this condition is independent of the choice of ρ. (We shall omit J when it is clear which almost complex structure do we have in mind.) This implies that $\eta \wedge (d\eta)^{n-1} \neq 0$ on Σ $(n = \dim_{\mathbb{C}} X)$ which means that η is a *contact form* and (Σ, ξ) is a *contact manifold* (see [8], [10], [45], pp. 338-340). A smooth function $\rho\colon X \to \mathbb{R}$ all of whose level sets are J-convex outside of the critical points is said to be J-convex.

An immersion $g\colon S \to \Sigma$ of a smooth manifold S into a contact manifold (Σ, ξ) is *Legendrian* if $dg(TS) \subset \xi$. In the case at hand another common expression is a *complex tangential immersion.*

Let J_{st} denote the standard complex structure on $\mathbb{C}^n$. For a fixed $k \in \{1, \ldots, n\}$ let $z = (z_1, \ldots, z_n) = (x' + iy', x'' + iy'')$, with $z_j = x_j + iy_j$, denote the coordinates on $\mathbb{C}^n$ corresponding to the decomposition

$$\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k} = \mathbb{R}^k \oplus i\mathbb{R}^k \oplus \mathbb{R}^{n-k} \oplus i\mathbb{R}^{n-k}.$$

Let $D = D^k \subset \mathbb{R}^k$ be the closed unit ball in $\mathbb{R}^k$ and $S = S^{k-1} = \partial D$ its boundary sphere. Identifying D^k with its image in $\mathbb{R}^k \oplus \{0\}^{2n-k} \subset \mathbb{C}^n$ we obtain the core of the standard index k handle

$$H_\delta = (1+\delta)D^k \times \delta D^{2n-k} \subset \mathbb{C}^n, \quad \delta > 0. \tag{2.1}$$

A *standard handlebody of index k* in $\mathbb{C}^n$ is a set $K_{\lambda,\delta} = Q_\lambda \cup H_\delta$ for some $0 < \lambda < 1$ and $0 < \delta < \frac{2\lambda}{1-\lambda}$ (fig. 1), where

$$Q_\lambda = \big\{z = (x' + iy', z'') \in \mathbb{C}^k \oplus \mathbb{C}^{n-k} : |y'|^2 + |z''|^2 \leq \lambda(|x'|^2 - 1)\big\}. \tag{2.2}$$

The condition $\lambda < 1$ insures that Q_λ is strongly pseudoconvex, and the bound on δ implies $(1+\delta)\partial D^k \times \delta D^{2n-k} \subset Q_\lambda$. An important tool in this theory is the following result of Eliashberg (§3 in [10]; see also [28]).

Lemma 2.1. (Eliashberg) *For every $\epsilon > 0$ and $\lambda > 1$ there exist a number $\delta \in (0, \epsilon)$ and a smoothly bounded, strongly pseudoconvex handlebody $L \subset \mathbb{C}^n$ with core $Q_\lambda \cup D^k$ such that $K_{\lambda,\delta} \subset L \subset K_{\lambda,\epsilon}$ (fig. 1).*

Eliashberg's construction gives an L of the form

$$L = \{(x' + iy', z'') \in \mathbb{C}^n : |y'|^2 + |z''|^2 \le h(|x'|^2)\}$$

where $h\colon [0,\infty] \to [\delta^2, \infty]$ is a smooth, increasing, convex function chosen so that L is a round tube of constant radius δ around $D^k \subset \mathbb{C}^n$ over a slightly smaller ball rD^k, $r<1$, and it equals Q_λ over $r'D^k$ for some $r'>1$ close to 1.

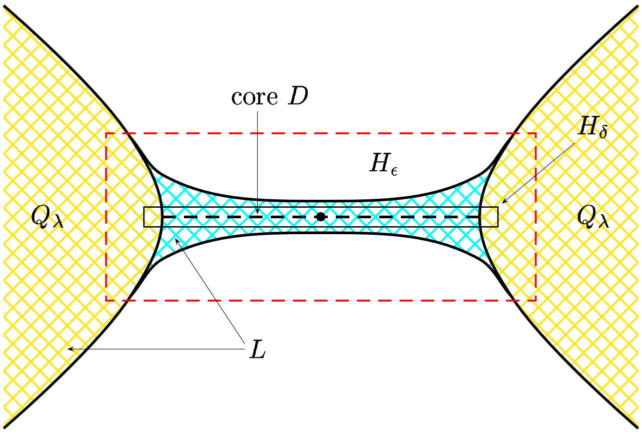

FIGURE 1. A strongly pseudoconvex handlebody L

We introduce the following (trivial) bundles over the disc $D \subset \mathbb{R}^k \oplus \{0\}^{2n-k}$:

$$\begin{aligned}
\nu' &= \mathrm{Span}\Big\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_k}\Big\}\Big|_D = D \times \big(\{0\}^k \oplus \mathbb{R}^k \oplus \{0\}^{2n-2k}\big),\\
\nu'' &= \mathrm{Span}\Big\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} : j = k+1, \ldots, n\Big\}\Big|_D = D \times \big(\{0\}^{2k} \oplus \mathbb{R}^{2n-2k}\big),\\
\nu &= \nu' \oplus \nu'' = D \times \big(\{0\}^k \oplus \mathbb{R}^{2n-k}\big).
\end{aligned}$$

Thus $\nu' = J_{st}(TD)$, $T^{\mathbb{C}}D = TD \oplus \nu'$, and $T\mathbb{C}^n|_D = TD \oplus \nu = T^{\mathbb{C}}D \oplus \nu''$.

Let $v \to S$ denote the (trivial) real line bundle over S spanned by the vector field $\sum_{j=1}^k x_j \frac{\partial}{\partial x_j}$. Over S we then have further decompositions

$$TD|_S = v \oplus TS, \quad \nu'|_S = J_{st}(v) \oplus J_{st}(TS), \quad TD|_S \oplus \nu'|_S \simeq v^{\mathbb{C}} \oplus T^{\mathbb{C}}S.$$

Note that $T^{\mathbb{C}}S$ is a trivial complex vector bundle.

Given a smooth embedding (or immersion) $G\colon D \to X$ of the disc $D = D^k \subset \mathbb{C}^n$ to a smooth $2n$-dimensional manifold X, a *normal framing* over G is a homomorphism $\beta\colon \nu \to TX|_{G(D)}$ such that $dG_x \oplus \beta_x \colon T_xD \oplus \nu_x = T_x\mathbb{C}^n \to T_{G(x)}X$ is a linear isomorphism for every $x \in D$.

3. Totally real discs attached to strongly pseudoconvex domains

In this section we show how one finds embedded or immersed totally real discs attached from the outside to a strongly pseudoconvex domain along a Legendrian sphere. Lemma 3.1 below is due to Eliashberg and is implicitly contained in the proof of Theorem 1.3.6 in [10]. We state it in the form which will be used in this paper. We wish to express our sincere thanks to Yakov Eliashberg who gave us decisive help in understanding the critical case $k = n > 2$ (personal communication, June 2005).

Let W be an open, relatively compact domain with smooth strongly pseudoconvex boundary $\Sigma = \partial W$ in an almost complex manifold (X, J). Choose a defining function ρ for Σ which is strongly J-plurisubharmonic near $\Sigma = \{\rho = 0\}$ and $\rho < 0$ on W. Let $w \subset TX|_\Sigma$ be the orthogonal complement of $T\Sigma$ with respect to the metric associated to the symplectic form $dd^c\rho$ (see §2); thus w is spanned by the gradient of ρ with respect to this metric. Then $Jw \subset T\Sigma$ and we have orthogonal decompositions $TX|_\Sigma = w \oplus T\Sigma = w \oplus Jw \oplus \xi$, where $\xi = T\Sigma \cap J(T\Sigma)$.

Let $D = D^k$, $S = S^{k-1} = \partial D$ and v be as in §2. (Thus v is the orthogonal complement of TS in $T\mathbb{R}^k|_S$.) An *embedding of a pair* $G\colon (D,S) \to (X\backslash W, \Sigma)$ is a smooth embedding $G\colon D \hookrightarrow X\backslash W$ such that $G(S) = G(D) \cap \Sigma$ and G is transverse to Σ along $G(S)$. Such G is said to be *normal to* Σ if $dG_x(v_x) = w_{G(x)}$ for every $x \in S$, i.e., G maps the direction orthogonal to $S \subset \mathbb{R}^k$ into the direction orthogonal to $\Sigma \subset X$. The analogous definition applies to immersions.

Lemma 3.1. *Let W be an open, relatively compact domain with smooth strongly pseudoconvex boundary $\Sigma = \partial W$ in an almost complex manifold (X, J). Let $1 \le k \le n = \frac{1}{2}\dim_\mathbb{R} X$, $D = D^k$, $S = S^{k-1} = \partial D$. Given a smooth embedding $G_0\colon (D,S) \to (X\backslash W, \Sigma)$, there is a regular homotopy of immersions $G_t\colon (D,S) \to (X\backslash W, \Sigma)$ $(t \in [0,1])$ which is $\mathcal{C}^0$ close to G_0 such that the immersion $G_1\colon D \to X\backslash W$ is J-real and normal to Σ, and $g_1 := G_1|_S\colon S \hookrightarrow \Sigma$ is a Legendrian embedding. If $k < n$ or $k = n \neq 2$, there exists an isotopy of embeddings G_t with these properties. If the structure J is integrable and Σ is real-analytic then G_1 can be chosen real analytic.*

Remark 3.2. In the special case $k = n = 2$ it is impossible in general to find an isotopy of embeddings $\{G_t\}$ with these properties. Indeed, there exists no totally real 2-disc in $\mathbb{C}^2\backslash B$ attached to the round ball $B \subset \mathbb{C}^2$ along a Legendrian curve, for the resulting configuration would have strongly pseudoconvex neighborhoods diffeomorphic to $S^2 \times \mathbb{R}^2$ (Eliashberg [10]), a contradiction since $S^2 \times \mathbb{R}^2$ does not admit any nonexotic Stein structures. On the other hand, a configuration with these properties exists in every dimension $n \neq 2$.

Proof. The scheme of proof is illustrated on fig. 2. First we find a regular homotopy of the initial disc $G_0\colon D \hookrightarrow X\backslash W$ to a new disc G_1 which is attached to ∂W along a Legendrian sphere with a correct normal framing (or, equivalently, with a correct homotopy property of the deformation). Then G_1 is deformed by a regular homotopy which is fixed near the boundary to a totally real disc G_2, using the h-principle for totally real immersions. Finally it is shown that, unless $k = n = 2$, all steps can be done by isotopies of embeddings.

We now begin with the actual proof. Set $g_0 = G_0|_S\colon S \hookrightarrow \partial W$. By a correction of G_0 along S (keeping g_0 fixed) we may assume that it is normal to Σ, i.e., such that $l_0 := dG_0|_v$ maps v to $w|_{g_0(S)}$. Choose a complex vector bundle isomorphism

$$\phi_0\colon T\mathbb{C}^n|_D = D \times \mathbb{C}^n \to TX|_{G_0(D)}, \quad \phi_0 \circ J_{st} = J \circ \phi_0$$

covering G_0. Using the coordinates on $\mathbb{C}^n$ introduced in §2 we set $\tau = \sum_{j=1}^k x_j \frac{\partial}{\partial x_j}$ (this field is radial to $S = \partial D$). Let $\tilde\tau$ be the unique nonvanishing vector field on $\mathbb{C}^n$ over S satisfying $\phi_0(\tilde\tau_x) = \ell_0(\tau_x)$ for every $x \in S$. By dimension reasons

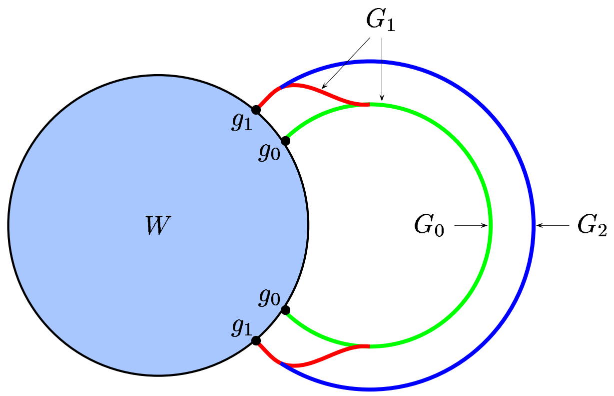

FIGURE 2. Deformations of an attached disc

there exists a map $A\colon D \to GL_n(\mathbb{C})$ satisfying $A_x \tau_x = \widetilde{\tau}_x$ for $x \in S$. Replacing ϕ_0 by $\phi_0 \circ A$ we may (and shall) assume from now on that $\phi_0|_v = \ell_0$. A further homotopic correction of ϕ_0 insures that $\phi_0(T^{\mathbb{C}}S \oplus \nu''|_S) = \xi|_{g_0(S)}$, thereby providing a trivialization of the latter bundle.

Write $\phi_0 = \phi_0' \oplus \phi_0''$ where $\phi_0' = \phi_0|_{T^{\mathbb{C}}D}$ and $\phi_0'' = \phi_0|_{\nu''}$. Setting $\psi_0 := \phi_0|_{T^{\mathbb{C}}S}$ we thus have

$$\phi_0'|_{T^{\mathbb{C}}D|_S} = \ell_0^{\mathbb{C}} \oplus \psi_0\colon v^{\mathbb{C}} \oplus T^{\mathbb{C}}S \to TX|_{g_0(S)} = w^{\mathbb{C}} \oplus \xi|_{g_0(S)}.$$

Note that $\psi_0 \oplus \phi_0''\colon T^{\mathbb{C}}S \oplus \nu''|_S \to \xi|_{g_0(S)}$ is a complex vector bundle isomorphism. Furthermore, there is a homotopy of real vector bundle monomorphisms $\iota_s\colon TD \hookrightarrow TX|_{G_0(D)}$ $(s \in [0,1])$ satisfying

$$\iota_0 = dG_0, \quad \iota_1 = \phi_0|_{TD}, \quad \iota_s|_v = \ell_0\colon v \to w|_{g_0(S)} \quad (s \in [0,1]).$$

Consider the pair (g_0, ψ_0) consisting of the embedding $g_0\colon S \hookrightarrow \Sigma$ covered by the $\mathbb{C}$-linear embedding $\psi_0\colon T^{\mathbb{C}}S \hookrightarrow \xi|_{g_0(S)}$ of the complexified tangent bundle of S (a trivial complex vector bundle of rank $k-1$) to the contact subbundle $\xi \subset T\Sigma$ over $g_0(S)$. By the *Legendrization theorem* of Gromov ([45], p. 339, (B')) and Duchamp [8] there exists a Legendrian embedding $g_1\colon S \hookrightarrow \Sigma$ such that $\psi_1 = d^{\mathbb{C}}g_1$ (the complexified differential) is homotopic to ψ_0 by a homotopy consisting of $\mathbb{C}$-linear vector bundle embeddings $\psi_t\colon T^{\mathbb{C}}S \hookrightarrow \xi$ $(t \in [0,1])$. The Legendrization theorem also holds in the category of embeddings by an argument due to Eliasherg (Note 2.3.2. on p. 33 in [10]), but this will not be used here.

Let $\mathrm{Hom}_{inj}(TS, T\Sigma)$ denote the space of all fiberwise injective real vector bundle maps (monomorphism) $TS \hookrightarrow T\Sigma$. Consider the path in $\mathrm{Hom}_{inj}(TS, T\Sigma)$ beginning at dg_0 and ending at dg_1, consisting of the homotopy $\iota_s|_{TS}$ $(s \in [0,1])$ followed by the homotopy $\psi_t|_{TS}$ $(t \in [0,1])$ (left and top side of the square in fig. 3). By Hirsch's one parametric h-principle for immersions ([50], [45]) there is a regular homotopy of immersions $g_t\colon S \to \Sigma$ $(t \in [0,1])$ from g_0 to g_1 such that the above path can be deformed in the space $\mathrm{Hom}_{inj}(TS, T\Sigma)$ (with fixed ends) to the path $dg_t\colon TS \hookrightarrow T\Sigma|_{g_t(S)}$ $(t \in [0,1])$. We may insure that ψ_t covers the base map g_t

for all $t \in [0,1]$. This gives a two parameter homotopy $\theta_{t,s} \in \mathrm{Hom}_{inj}(TS, T\Sigma)$ for $(t,s) \in [0,1]^2$ satisfying the following conditions (fig. 3):

(i) $\theta_{t,0} = dg_t$ (bottom side),
(ii) $\theta_{t,1} = \psi_t|_{TS}$ (top side),
(iii) $\theta_{0,s} = \iota_s|_{TS}$ (left side; hence $\theta_{0,0} = dg_0$ and $\theta_{0,1} = \psi_0|_{TS}$),
(iv) $\theta_{1,s} = dg_1$ (right side), and
(v) $\theta_{t,s}$ covers g_t for every $t, s \in [0,1]$.

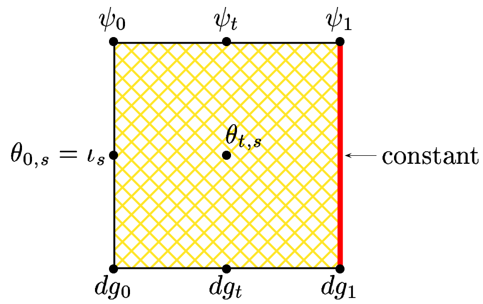

FIGURE 3. The homotopy $\theta_{t,s}$

We can extend g_t to a regular homotopy of immersions $G_t\colon (D,S) \to (X\backslash W, \Sigma)$ ($t \in [0,1]$) which are normal to Σ, beginning at $t=0$ with the given map G_0. Let $\ell_t := dG_t|_v \colon v \to w|_{g_t(S)}$. By the homotopy lifting theorem there exists a homotopy of $\mathbb{C}$-linear vector bundle isomorphisms ϕ_t covering G_t,

$$\phi_t = \phi'_t \oplus \phi''_t \colon T\mathbb{C}^n|_D = T^{\mathbb{C}}D \oplus \nu'' \to TX|_{G_t(D)}, \quad t \in [0,1],$$

beginning at $t=0$ with the given map ϕ_0, such that over $S = \partial D$ we have

$$\phi'_t = \ell_t^{\mathbb{C}} \oplus \psi_t, \quad t \in [0,1],$$

and $dG_1 = \phi_1$ on $TD|_S$.

Set $\widetilde{\theta}_{t,s} = \ell_t \oplus \theta_{t,s} \colon TD|_S \hookrightarrow TX|_{g_t(S)}$ for $t, s \in [0,1]$ (a real vector bundle monomorphism over g_t). From the above properties (i)–(v) of $\theta_{t,s}$ we obtain

(i') $\widetilde{\theta}_{t,0} = \ell_t \oplus dg_t = dG_t|_{TD|_S}$ (bottom side),
(ii') $\widetilde{\theta}_{t,1} = \ell_t \oplus \psi_t|_{TS} = \phi_t|_{TD|_S}$ (top side),
(iii') $\widetilde{\theta}_{0,s} = \iota_s|_{TD|_S}$ (left side),
(iv') $\widetilde{\theta}_{1,s} = \ell_1 \oplus dg_1 = dG_1|_{TD|_S}$ (right side), and
(v') $\widetilde{\theta}_{t,s}$ covers g_t for every $t, s \in [0,1]$.

We wish to extend the monomorphisms $\widetilde{\theta}_{t,s} \colon TD|_S \hookrightarrow TX|_{g_t(S)}$ to real vector bundle monomorphisms $\Theta_{t,s} \colon TD \to TX$ ($(t,s) \in [0,1]^2$) covering the immersions $G_t \colon D \hookrightarrow X$. Such extension already exists for (t,s) in the bottom, top and left face of the parameter square $[0,1]^2$ where we respectively take dG_t, $\phi_t|_{TD}$ and ι_s (properties (i'), (ii') and (iii')). The homotopy lifting property provides an extension $\Theta_{t,s}$ for all $(t,s) \in [0,1]^2$, with the given boundary values on the bottom, top and left side of $[0,1]^2$. (See fig. 4; the front and back face belong to the homotopy $\widetilde{\theta}_{t,s}$

over $S = \partial D$, compare with fig. 3.) Over the right face $\{t = 1\}$ we thus obtain a homotopy $\Theta_{1,s} \in \mathrm{Hom}_{inj}(TD, TX|_{G_1(D)})$ $(s \in [0,1])$ satisfying

$$\Theta_{1,0} = dG_1 \colon TD \to TX|_{G_1(D)}, \quad \Theta_{1,1} = \phi_1|_{TD} \colon TD \to TX|_{G_1(D)}.$$

The homotopy $\Theta_{1,s}$ is fixed over S where it coincides with $\widetilde{\theta}_{1,s} = dG_1|_{TD|_S}$ by property (iv'). (On fig. 4, $\Theta_{1,s}$ appears on the right face of the cube, with bold vertical sides indicating that it is constant on $TD|_S$ where it equals $\ell_1 \oplus dg_1$.)

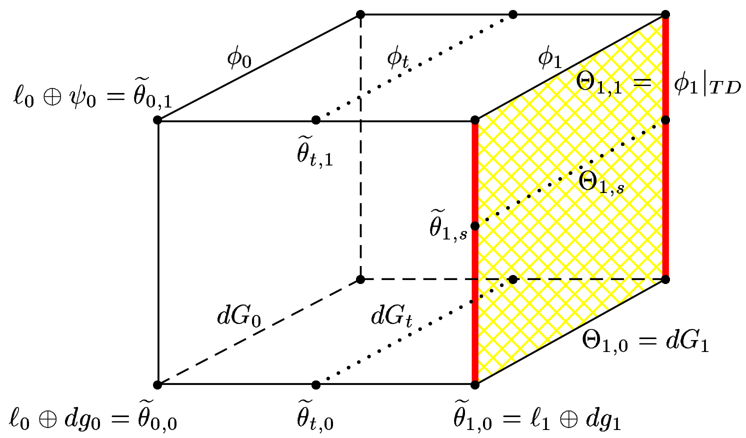

FIGURE 4. The homotopy $\Theta_{t,s}$

Since $\phi_1 \colon T\mathbb{C}^n|_D \to TX|_{G_1(D)}$ is a $\mathbb{C}$-linear vector bundle isomorphism, the h-principle for totally real immersions (see [43], [45] or [15]) provides a regular homotopy of immersions $G_t \colon D \to X\backslash W$ $(t \in [1,2])$, fixed near S, such that the immersion G_2 is J-real and its complexified differential $d^{\mathbb{C}}G_2$ is homotopic to ϕ_1 in the space of $\mathbb{C}$-linear maps $T\mathbb{C}^n|_D \to TX$ of maximal rank. If G_1 is an embedding then we can change it to a totally real embedding G_2 by an isotopy which is fixed near S. This follows from the fact that totally real embeddings also satisfy the h-principle (see [45]). For $k < n$ or $k = n > 2$ this can be seen from the results in [19], and for $k = n = 2$ it follows from the result of Eliashberg and Harlamov [14] on cancellations of complex points of real surfaces in complex surfaces (this is discussed in §7 below). Finally we reparametrize the family $\{G_t \colon t \in [0,2]\}$ back to the parameter interval $[0,1]$.

This proves the existence of a regular homotopy with the required properties. It remains to be seen that, unless $k = n = 2$, there also exists an *isotopy* $\{G_t\}$ with these properties. For $k < n$ this follows from the general position argument — a small perturbation of $\{g_t\}$ with fixed ends gives an isotopy which can be realized by an ambient diffeotopy, and we get an isotopy of embedded discs $G_t \colon D \hookrightarrow X\backslash W$ with $G_t|_S = g_t$. For $k = n = 1$ the conclusion of lemma 3.1 obviously holds for any attached 1-disc (segment).

In the remainder of the proof we consider the case $k = n > 2$. The following argument, up to the end of proof of lemma 3.1, was communicated to us by Y. Eliashberg (personal communications, June 2005).

A generic choice of the regular homotopy $g_t\colon S = S^{n-1} \to \Sigma$ insures that g_t is an embedding for all but finitely many parameter values $t \in [0,1]$ and has a simple (transverse) double point at each of the exceptional parameter values. We wish to change the Legendrian embedding g_1 by a regular homotopy of Legendrian immersions $g_t\colon S \hookrightarrow \Sigma$ $(t \in [1,2])$ to another Legendrian embedding g_2 so that the resulting regular homotopy $\{g_t\colon t \in [0,2]\}$ will have self-intersection index zero. More precisely, the map $\widetilde{g}\colon \widetilde{S} = S \times [0,2] \to \widetilde{\Sigma} = \Sigma \times [0,2]$, defined by $\widetilde{g}(x,t) = (g_t(x),t)$, is an immersion of the n-dimensional oriented manifold $\widetilde{S}$ into the $2n$-dimensional oriented manifold $\widetilde{\Sigma}$ such that the double points of $\widetilde{g}$ correspond to the double points of the regular homotopy $\{g_t\}$, and we define the index $i(\{g_t\})$ as the number of double points of $\widetilde{g}$ counted with the orientation signs. If this index equals zero then a foliated version of the Whitney trick allows us to find a deformation with fixed ends of $\{g_t\}$ to an isotopy of embeddings. This is done by connecting a chosen pair of double points $q_0, q_1 \in \widetilde{g}(\widetilde{S})$ of the opposite sign, lying over two different values $t_0 < t_1$ of the parameter, by a pair of curves $\lambda_j(t) = \widetilde{g}(c_j(t),t)$ $(t \in [t_0,t_1],\ j=1,2)$ which together bound an embedded Whitney disc $D^2 \subset \widetilde{\Sigma}$ such that $D^2 \cap (\Sigma_t \times \{t\})$ is an arc connecting $\lambda_1(t)$ to $\lambda_2(t)$ for every $t \in [t_0,t_1]$, and it degenerates to q_0 resp. q_1 over the endpoints t_0 resp. t_1. The rest of the procedure, removing this pair of double points by pulling $\widetilde{g}(\widetilde{S})$ across D^2, is standard [77]. Performing this operation finitely many times we remove all double points and change $\{g_t\}$ to an isotopy of embeddings. The rest of the proof can be completed exactly as before: we extend g_t to an isotopy of embedded discs $G_t\colon D \hookrightarrow X\backslash W$, with $G_t|_S = g_t$, covered by a homotopy of $\mathbb{C}$-linear isomorhisms $\phi_t\colon T\mathbb{C}^n|_D \to TX|_{G_t(D)}$, and observe that $\{dg_t\}$ still has the correct homotopy property so that the final embedding G_2 can be deformed (with fixed boundary) to a totally real embedding.

It remains to see that the index of $\{g_t\colon t \in [0,1]\}$ can be changed to an arbitrary number (in particular, to 0) by a small Legendrian deformation of g_1 in Σ. The deformation will be realized by a local Legendrian isotopy which introduces the correct number of double points. Set $L = L_1 := g_1(S) \subset \Sigma$, an embedded Legendrian sphere. Choose a point $a \in L \subset \Sigma$. In suitable local coordinates $(z,q,p) \in \mathbb{R}^{2n-1}$ on Σ, with a corresponding to $0 \in \mathbb{R}^{2n-1}$, the contact form is $\eta = dz - \sum_{j=1}^{n-1} p_j dq_j$, and L is given by the equations

$$\{(z,q,p) \in \mathbb{R}^{2n-1} : z^2 = q_1^3,\ p_1^2 = \frac{9}{4}q_1,\ p_2 = \cdots = p_{n-1} = 0\}.$$

(See section 2.4 in [10].) Let $\pi\colon \mathbb{R}^{2n-1} \to \mathbb{R}^{n-1}$ denote the projection $\pi(z,q,p) = q$. Choose a closed ball $\Delta \subset \mathbb{R}^{n-1}$ centered at a point $(q_1^0, 0, \ldots, 0)$ for a small $q_1^0 > 0$, of radius $q_1^0/2$. Let $\phi\colon \Delta \to \mathbb{R}$ be a smooth function equaling 0 near $\partial\Delta$. Set

$$h_t(q) = q_1^{3/2}\big(1 + (t-1)\phi(q)\big), \quad t \in [1,2].$$

Let L_t equal L outside $\pi^{-1}(\Delta)$ and equal

$$\Big\{(z,q,p)\colon z = h_t(q),\ p = \frac{\partial h_t}{\partial q}(q)\Big\} \cup \Big\{(z,q,p)\colon z = -h_t(q),\ p = -\frac{\partial h_t}{\partial q}(q)\Big\}$$

over Δ. (We choose ϕ with sufficiently small derivative to insure that we remain in the given coordinate patch; this can be done if $q_1^0 > 0$ is chosen sufficiently small.) Let $g_t\colon S \to \Sigma$ $(t \in [1,2])$ be the regular homotopy such that $g_t(S) = L_t$. The deformation is illustrated by fig. 5. The top diagrams show the projection onto the (z,q)-plane at three typical stages, with the cusp at $z=0$, $q=0$, and with a self-intersection point shown in the middle figure.

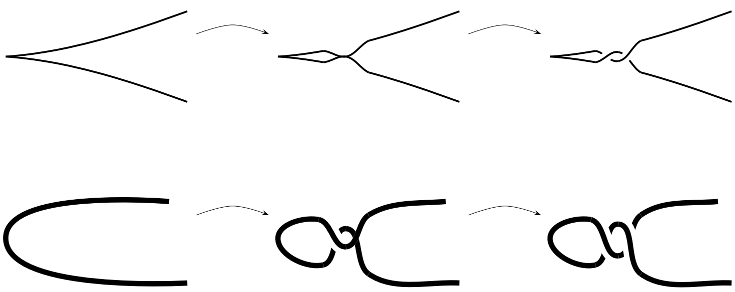

FIGURE 5. Changing the index of a regular homotopy by $+1$.

The index of $\{g_t\}_{t\in[1,2]}$ equals the intersection number between the manifolds

$$M_\pm = \Big\{(z,q,p,t)\in\mathbb{R}^{2n}\colon z=\pm h_t(q),\ p=\pm\frac{\partial h_t}{\partial q}(q),\ q\in\Delta,\ t\in[1,2]\Big\}.$$

The intersection points of M_+ and M_- are solutions of the equations

$$1+(t-1)\phi=0,\quad \frac{\partial\phi}{\partial q}=0,\qquad t\in[1,2].$$

This is precisely the set of all critical points of ϕ with the critical values belonging to $(-\infty,-1]$. By a generic choice of ϕ we can insure that -1 is not a critical value of ϕ. A computation shows that each point (q,t) satisfying the above equations adds ± 1 to the index $i(\{g_t\})$, with the sign equal to the sign of the determinant of the Hessian $Hess(\phi)$ at q; hence we get $+1$ at the critical points of even Morse index and -1 at the critical points of odd Morse index. Similarly, as we increase the constant c, the Euler characteristic of the sublevel set $\{\phi\le c\}$ increases by one at every critical point of ϕ of even Morse index, and it decreases by one at every critical point of odd Morse index. It follows that the index $i=i(\{g_t\}_{t\in[1,2]})$ equals the Euler number of the set $\{q\in\Delta\colon \phi(q)\le -1\}$. If $n>2$, this can be arranged to equal any preassigned integer by a suitable choice of ϕ, and hence we can arrange the index of $\{g_t\}_{t\in[0,2]}$ to equal zero. If $n=2$ then i can be arranged to be any nonnegative number since the set $\{\phi\le -1\}$ is a union of segments, but it can never be negative. (A similar argument had been used in [10], §2.4, for untwisting the normal framing when $n>2$.) $\square$

Remark 3.3. If $D=D^n$, $S=\partial D=S^{n-1}$ and $G\colon (D,S)\to (X\backslash W,\Sigma)$ is a totally real embedding which is normal to Σ and such that the attaching sphere $G(S)\subset\Sigma$ is Legendrian then we have a (unique) framing of the normal bundle $\beta\colon \nu\to TX|_{G(D)}$ satisfying the Cauchy-Riemann equations

$$\beta\circ J_{st}=J\circ dG \quad \text{on } TD. \tag{3.1}$$

Let us now omit the condition that G is totally real but keep the remaining conditions (which in particular imply that G is totally real near the boundary of D). Suppose that G admits a normal framing β which satisfies the CR equation (3.1) over the sphere S, i.e., $\beta = -J \circ dG \circ J_{st}$ on ν_x for every $x \in S$. Does it follow that G is isotopic to a totally real disc by an isotopy fixed near S? It turns out that the answer is affirmative in half of dimensions and negative in the remaining half, depending of the residue class of $n = \dim_{\mathbb{C}} X$ modulo 8. Here is the precise result.

Proposition 3.4. *Let W be an open, relatively compact domain with smooth strongly pseudoconvex boundary $\Sigma = \partial W$ in an almost complex manifold (X, J) of real dimension $2n$. Let $D = D^n$, $S = \partial D$, and let $G\colon (D, S) \to (X\backslash W, \Sigma)$ be a smooth embedding, normal to Σ, such that $G(S) \subset \Sigma$ is Legendrian. Assume that G admits a normal framing β satisfying (3.1) over S. If $n \in \{1, 3, 4, 5\}$ modulo 8 then G is isotopic to a totally real embedding by an isotopy which is fixed near S.*

Proof. We extend G to an embedding of a standard handle $H \subset \mathbb{C}^n$ such that $dG|_\nu = \beta$ over D. The difference on D between the almost complex structures $J' := G^*(J)$ and J_{st} (which agree over S) defines an element of the group

$$[S^n, GL^+_{2n}(\mathbb{R})/GL_n(\mathbb{C})] = \pi_n(SO(2n)/U(n))$$

(Milnor [61], p. 133). Using the long exact sequence of homotopy groups and the five lemma one sees that

$$\pi_n(SO(2n)/U(n)) = \pi_n(SO/U) = \pi_n(\Omega SO) = \pi_{n+1}(SO). \tag{3.2}$$

By the (real) Bott periodicity theorem this group equals $\mathbb{Z}$ if $n \in \{2, 6\}$ modulo 8, it equals $\mathbb{Z}_2$ if $n \in \{0, 7\}$ modulo 8, and it vanishes for the remaining values $n \in \{1, 3, 4, 5\}$ modulo 8. In the latter case we conclude that $J = J_1$ is homotopic along $G(D)$ to $J_0 := G_*(J_{st})$ by a homotopy which is fixed near $G(S) \subset \Sigma$. Since $G_0 = G$ is clearly J_0-real, Gromov's h-principle gives an isotopy of embedded discs $G_t\colon (D, S) \to (X\backslash W, \Sigma)$ which is fixed near S and such that G_t is J_t-real for every $t \in [0, 1]$. At $t = 1$ we obtain a J-real embedded disc G_1. □

4. A Hörmander-Wermer type approximation theorem

Let X be a complex manifold with an integrable complex structure J. We denote by $\mathcal{H}(X) = \mathcal{H}(X, J)$ the algebra of all holomorphic functions on X.

A compact set K in X is $\mathcal{H}(X)$-convex (or $\mathcal{H}(X, J)$-convex) if for every $p \in X\backslash K$ there exists $f \in \mathcal{H}(X)$ with $|f(p)| > \sup_{x \in K} |f(x)|$.

We say that a compact set K in X is *holomorphically convex* if there is an open Stein domain $\Omega \subset X$ containing K such that K is $\mathcal{H}(\Omega)$-convex. By the classical theory (Chapter 2 in [51]), holomorphic convexity of K is equivalent to the existence of a Stein neighborhood Ω of K and a continuous plurisubharmonic function $\rho \geq 0$ on Ω such that $\rho^{-1}(0) = K$ and ρ is strongly plurisubharmonic on $\Omega\backslash K$. We may take $\Omega = \{\rho < c_1\}$ for some $c_1 > 0$; for any $c \in (0, c_1)$ the sublevel set $\{\rho < c\} \subset\subset \Omega$ is then Stein and Runge in Ω (Sect. 4.3 in [51]).

A point p_0 in an immersed real k-dimensional submanifold $M \subset X$ is said to be a *special double point* if there is a holomorphic coordinate system

$$z = (x' + iy', z'')\colon U \to \widetilde{U} \subset \mathbb{C}^n = \mathbb{R}^k \oplus i\mathbb{R}^k \oplus \mathbb{C}^{n-k}$$

in a neighborhood $U \subset X$ of p_0 such that $z(p_0) = 0$ and

$$z(M \cap U) = \widetilde{U} \cap \big(\{(x' + i0', 0'')\colon x' \in \mathbb{R}^k\} \cup \{(0' + iy', 0'')\colon y' \in \mathbb{R}^k\}\big) \tag{4.1}$$

The following theorem will play a key role in the proof of our main results. It is far from the most general one with respect to the types of double points of M, but it will suffice for the present purposes.

Theorem 4.1. *Let X be a complex manifold. Let $K_0 \subset K = K_0 \cup M$ be compact sets in X such that M is a smoothly immersed, compact, totally real submanifold of X, possibly with boundary, all of whose nonembedding points are special double points (4.1). Assume that K_0 is holomorphically convex and it admits a compact relative neighborhood N in K which is also holomorphically convex (fig. 6).*

Given a continuous map $f\colon X \to Y$ to a complex manifold Y such that f is holomorphic in an open neighborhood of K_0, there exist open Stein domains $V_1 \supset V_2 \supset \cdots \supset \cap_j V_j = K$ and holomorphic maps $f_j\colon V_j \to Y$ $(j = 1, 2, \ldots)$ such that K_0 and K are $\mathcal{H}(V_j)$-convex for every j, and $f_j|_K \to f|_K$ uniformly as $j \to \infty$.

If M is an embedded submanifold and $f|_M\colon M \to Y$ is of class $\mathcal{C}^r$ then f_j can be chosen such that $f_j|_K \to f|_K$ uniformly and $f_j|_M \to f|_M$ in the $\mathcal{C}^r(M)$ topology as $j \to \infty$. The analogous results hold for a family of maps $f\colon X \times P \to Y$ parametrized by a compact Hausdorff space P, and in this case the neighborhoods $V_j \supset K$ can be chosen uniformly for all maps in the family.

The situation is shown on fig. 6: the initial map f is holomorphic on K_0 together with its shaded collar and is continuous elsewhere, and N equals K_0 together with the thick attaching arcs. Theorem 4.1 was proved by Hörmander and Wermer [52] in the special case when $X = \mathbb{C}^n$, $Y = \mathbb{C}$ and M has no double points.

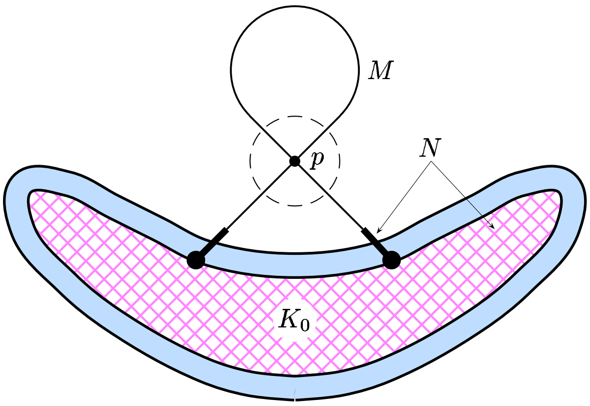

FIGURE 6. A kinky disc M attached to K_0

Proof. Suppose first that M is a smoothly embedded, totally real submanifold. By theorem 3.1 in [24] the hypotheses in theorem 4.1 imply that the set $K = K_0 \cup M$ is holomorphically convex. We approximate f uniformly on M by a smooth map without changing its values near K_0 (where it is holomorphic), and we denote the new map again by f. Theorem 3.2 in [24] then gives Stein domains $V_j \supset K$ and holomorphic maps $f_j\colon V_j \to Y$ satisfying the conclusion of theorem 4.1. Each V_j is a sublevel set of a nonnegative continuous plurisubharmonic function in a neighborhood of K in X which vanishes precisely on K and is strongly plurisubharmonic in the complement of K_0. Hence K is $\mathcal{H}(V_j)$-convex, and by adding a sufficiently small nonnegative term which vanishes on K_0 and is positive on $M\backslash K_0$ we see that K_0 is also $\mathcal{H}(V_j)$-convex. Although V_j depends on the rate of approximation of f by f_j, it can be chosen uniform with respect to maps in a compact family.

Suppose now that M is an immersed totally real submanifold with special double points $p_1,\ldots,p_m$ (by compactness of M there are at most finitely many of them). By slightly enlarging K_0 if necessary we may assume that all of these points lie in $M\backslash K_0$ (the points in the interior of K_0 will not matter). Let $B_j \subset X\backslash K_0$ be a small open neighborhood of p_j in X which is mapped onto a small ball around $0 \in \mathbb{C}^n$ by a local coordinate map (4.1). By a uniformly small change of f we make it smooth on M and constantly equal to $f(p_j)$ on a neighborhood of $\overline{B}_j \cap M$; the latter change can be made small by choosing the balls B_j as small as necessary. We extend f to the constant (holomorphic) map $x \to f(p_j)$ on a neighborhood of $\overline{B}_j$ in X. Note that $M \cap \partial B_j$ is a union of two disjoint Legendrian spheres in ∂B_j. We can now apply the same argument as in the embedded case, with K_0 replaced by $K_0' = K_0 \cup (\cup_{j=1}^m \overline{B}_j)$ and M replaced by the embedded totally real submanifold $M' = M\backslash \cup_{j=1}^m B_j$. The set $K' := K_0' \cup M' = K \cup (\cup_{j=1}^m \overline{B}_j)$ is holomorphically convex by theorem 3.1 in [24]. (The addition of balls $\overline{B}_j$ does not affect this property since the union of the round ball in $\mathbb{C}^n$ with $\mathbb{R}^n \cup i\mathbb{R}^n$ is holomorphically convex [10], [28].) This gives a uniform approximation of f by holomorphic maps in small open neighborhoods of K' in X; if $f|_M$ is smooth then the approximation can be made in the $\mathcal{C}^r(M)$ topology away from the double points of M.

An extension to a family of maps $f\colon X \times P \to Y$, with P a compact Hausdorff space, is obtained by covering the graph of the family in $X \times Y$ (after an initial smoothing of the maps $f(\cdot\,,p)\colon X \to Y$ on the totally real submanifold M) by finitely many Stein neighborhoods in $X \times Y$, using these to approximate f by local (in P) families of holomorphic maps, and patching these families by a continuous partition of unity in the parameter $p \in P$. The latter is possible since we can introduce a complex linear structure on the fibers of the projection $X \times Y \to X$ within a small Stein neighborhood of each individual graph. The details in a very similar context can be found in [29] (the proof of Theorem 4.2, pp. 138-139). $\square$

5. Extending a holomorphic map across a handle

The following lemma is central to the proof of our main results. Its geometric part (without holomorphic maps) is due to Eliasherg [10].

Lemma 5.1. *Let (X,J) be an almost complex manifold of real dimension $2n$. Let $W \subset\subset X$ be a smoothly bounded domain such that J is integrable in a neighborhood of $\overline{W}$, the manifold (W,J) is Stein, and $\Sigma = \partial W$ is strongly J-pseudoconvex. Let $D = D^k$ and $S = S^{k-1} = \partial D$ $(1 \le k \le n)$. Let $G\colon (D,S) \to (X\backslash W, \Sigma)$ be a smooth J-real embedding which is normal to Σ and such that $G|_S\colon S \to \Sigma$ is Legendrian. Assume that Y is a complex manifold and $f\colon X \to Y$ is a continuous map which is J-holomorphic in an open neighborhood of $\overline{W}$. Let d_Y be a distance function on Y induced by a smooth Riemannian metric.*

After a small smooth perturbation of G there exist an integrable complex structure $\widetilde{J}$ in an open neighborhood $U \subset X$ of $K := \overline{W} \cup G(D)$, a homotopy J_t $(t \in [0,1])$ of almost complex structures on X which is fixed on a neighborhood of $\overline{W}$, with

$J_0 = J$ and $J_1 = \widetilde J$, and for every $\epsilon > 0$ there exist a smoothly bounded strongly $\widetilde J$-pseudoconvex Stein domain $\widetilde W$ and a map $\widetilde f\colon X \to Y$ satisfying the following:

(i) *$K \subset \widetilde W \subset U$, $\widetilde W$ is a handlebody with core K, and $\overline W$ is $\mathcal{H}(\widetilde W, \widetilde J)$-convex,*
(ii) *$\widetilde f|_{\widetilde W}\colon \widetilde W \to Y$ is $\widetilde J$-holomorphic, and*
(iii) *there is a a homotopy $f_t\colon X \to Y$ $(t \in [0,1])$, with $f_0 = f$ and $f_1 = \widetilde f$, such that for each $t \in [0,1]$ the map f_t is J-holomorphic on a neighborhood of $\overline W$ and satisfies $\sup_{x\in W} d(f(x), f_t(x)) < \epsilon$.*

If in addition f is covered by a complex vector bundle map $\iota\colon (TX, J) \to TY$ which is of maximal rank on every fiber and such that $df = \iota$ on a neighborhood of $\overline W$ then we can choose $\widetilde f$ to be of maximal rank at every point of $\widetilde W$ and such that $d\widetilde f$ is homotopic to ι through complex vector bundle maps $\iota_t\colon (TX, J_t) \to TY$ of pointwise maximal rank.

If Σ is real analytic and the almost complex structure J is integrable in a neighborhood of K then the above conclusions hold with $J = \widetilde J$.

Proof. After a small enlargement of W we may assume that ∂W is real analytic and strongly J-pseudoconvex, and the k-disc $M := G(D) \subset X$ is attached to this new domain along the Legendrian $(k-1)$-sphere $G(S) \subset \partial W$.

By Lemma 2.5.1. in [10] (which uses Gray's theorem on approximation of Legendrian embeddings by real analytic Legendrian embeddings) we can approximate G by a map which is real analytic in a neighborhood of $S = \partial D$, normal to Σ, and such that the attaching sphere $G(S) \subset \Sigma$ is Legendrian in $\Sigma = \partial W$. (The almost complex structure J is assumed integrable only in a neighborhood of $\overline W$.)

We first consider the case $k = n$. For every $x \in D$ let $A_x\colon T_x\mathbb{C}^n \to T_{G(x)}X$ denote the (unique) complex linear map which agrees with dG_x on T_xD. (We use the standard structure J_{st} on $\mathbb{C}^n$ and the structure J on X.) We extend G to a smooth diffeomorphisms $\widetilde G$ from a standard handle $H \subset \mathbb{C}^n$ (2.1) onto a neighborhood $\widetilde H = \widetilde G(H)$ of M in X such that $d\widetilde G_x = A_x$ for each $x \in D$. Near the sphere $S = \partial D$ we take $\widetilde G$ to be the complexification of G, hence biholomorphic.

Let $W' \subset X$ be a slightly larger domain containing W. Let $\widetilde J$ denote the complex structure on $W_1 := W' \cup \widetilde H$ which equals J on W' and equals $\widetilde G_*(J_{st})$ (the push forward of the standard structure J_{st} on $\mathbb{C}^n$ by $\widetilde G$) on the handle $\widetilde H$. By choosing $W' \supset W$ and $H \supset D$ sufficiently small we insure that these two complex structures coincide on $W' \cap \widetilde H$ (since $\widetilde G$ maps a neighborhood of $S \subset \mathbb{C}^n$ biholomorphically onto a neighborhood of $G(S) \subset X$). Notice that $J = \widetilde J$ at every point of $M = G(D)$ by our choice of $\widetilde G$. By the construction there clearly is a homotopy of almost complex structures from J to $\widetilde J$ satisfying (i); using this homotopy we can patch J and $\widetilde J$ outside of a larger neighborhood of K.

If J is integrable near $\overline W \cup M$ then we can choose G to be real analytic by lemma 3.1, and in this case $\widetilde G$ may be simply be taken as the complexification of G. Hence the desired conclusion holds with $J = \widetilde J$.

Our next goal is to (approximately) extend $f\colon X \to Y$ to a holomorphic map across the handle. By the assumption f is J-holomorphic in a neighborhood of $\overline W$ in X, and we may assume by approximation that it is smooth on X. Since $\widetilde J = J$ near $\overline W$, f is also $\widetilde J$-holomorphic near $\overline W$. We wish to apply theorem 4.1 in the complex manifold $(W_1, \widetilde{J})$, with the compact sets $K_0 = \overline{W}$ and $K = K_0 \cup M$, in order to obtain a $\widetilde{J}$-holomorphic map $\widetilde{f}\colon V \to Y$ in an open neighborhood $V \supset K$ such that $\widetilde{f}|_K$ approximates $f|_K$ as close as desired. In order to do so, we must verify that $\overline{W}$ has a small compact, holomorphically convex relative neighborhood N in K. It is well known (see e.g. Lemma 1 in [68]) that the problem is local near the attaching sphere $G(S) = \partial M \subset \Sigma$. Thus, taking a closed tubular neighborhood $T \subset X$ of $M = G(D)$, it suffices to show that the set $\widetilde{G}^{-1}(T \cap K) \subset \mathbb{C}^n$ is holomorphically convex for a suitable choice of T. The latter set is the union of the closed ball $D \subset \mathbb{R}^n \subset \mathbb{C}^n$ with a piece of a strongly pseudoconvex domain which essentially looks like the quadric Q_λ (2.2). In fact, by a small outward bumping of $\Sigma = \partial W$ (from the side of W) which is localized in a tubular neighborhood of the circle $G(S)$ (keeping Σ and its tangent bundle fixed on $G(S)$) we can reduce to the situation when $\widetilde{G}^{-1}(T \cap K) = \widetilde{G}^{-1}(T) \cap (Q_\lambda \cup D)$ and $\widetilde{G}^{-1}(T)$ is a compact convex set in $\mathbb{C}^n$. The holomorphic polynomial $h(z) = z_1^2 + \ldots + z_n^2$ on $\mathbb{C}^n$ maps the disc D to the segment $[0,1] \subset \mathbb{R} \subset \mathbb{C}$, it maps the sphere $S = \partial D$ to the point 1, and $\Re h > 1$ on the set $Q_\lambda \backslash S$ (compare with [68] and the proof of Lemma 6.6 in [23]). Thus h separates the polynomially convex sets $Q_\lambda \cap \widetilde{G}^{-1}(T)$ and D, and hence their union is polynomially convex by Lemma 29.21 in [72].

Hence theorem 4.1 applies and gives a $\widetilde{J}$-holomorphic map $\widetilde{f}$ in a neighborhood of K which approximates f uniformly on K. A homotopy from f to $\widetilde{f}$ with the required properties clearly exists near K provided that the approximation is sufficiently close, and it is then used to patch $\widetilde{f}$ with f outside of a larger neighborhood of K.

Remaining in the case $k = n$ for the moment, we also consider the situation when f is of maximal complex rank in a neighborhood of $\overline{W}$ and is covered by a complex vector bundle map $\iota\colon (TX, J) \to (TY, J_Y)$ of fiberwise maximal rank, with $\iota = df$ near $\overline{W}$. In this case we must show that $\widetilde{f}$ can also be chosen of maximal rank on M, and hence in a neighborhood of K provided that the approximation is sufficiently close. To this end we first deform f (without changing it near $\overline{W}$) such that for every $x \in M$ its differential $df_x\colon T_xM \to T_{f(x)}Y$ is of maximal complex rank (equal to $\min\{n, \dim Y\}$), and the map $x \to df_x$ $(x \in M)$ is homotopic to $x \to \iota|_{T_xM}$ by a homotopy of vector bundle maps of pointwise maximal complex rank which is fixed near ∂M. This is a straightforward application of Gromov's h-principle, the main point being that the pertinent differential relation is *ample* on any totally real submanifold. For the details see Lemma 6.4 in [23] or Lemma 4.3 in [24] (page 1931). Applying theorem 4.1 to this new map f we obtain a $\widetilde{J}$-holomorphic map $\widetilde{f}$ in a neighborhood of K which approximates f uniformly on $\overline{W}$ and in the $\mathcal{C}^1$-topology on M. If the approximation is sufficiently close than the latter property ($\mathcal{C}^1$-approximation on M) insures that $\widetilde{f}$ is of maximal rank at every point of K. The existence of a homotopy from ι to $d\widetilde{f}$ with the required properties follows from the construction (see [23]).

To complete the proof of lemma 5.1 (still in the case $k = n$) it remains to find a $\widetilde{J}$-convex Stein domain $\widetilde{W} \supset K$ contained in V (so that $\widetilde{f}$ will be holomorphic in $\widetilde{W}$) and satisfying the other required properties; see fig. 7. Assuming as we may that Σ has been standardized along $G(S)$ as described above, this is an immediate application of lemma 2.1 (due to Eliasberg [10]) — one takes $\widetilde{W} = \widetilde{G}(\mathrm{Int}L)$ where

$L \subset \mathbb{C}^n$ is a standard strongly pseudoconvex handlebody around $Q_\lambda \cup D$ as in the cited lemma. Eliashberg also showed how to extend a J-convex defining function ρ for W to a $\widetilde{J}$-convex defining function $\widetilde{\rho}$ for $\widetilde{W}$ with precisely one additional Morse critical point of index $\dim_{\mathbb{R}} D$ which may be placed at the center of the attached handle. It follows in particular that W and $\widetilde{W}$ are two sublevel sets of the same $\widetilde{J}$-convex exhaustion function and hence W is Runge in $\widetilde{W}$. This also follows from our earlier argument on polynomial convexity of $Q_\lambda \cup D$.

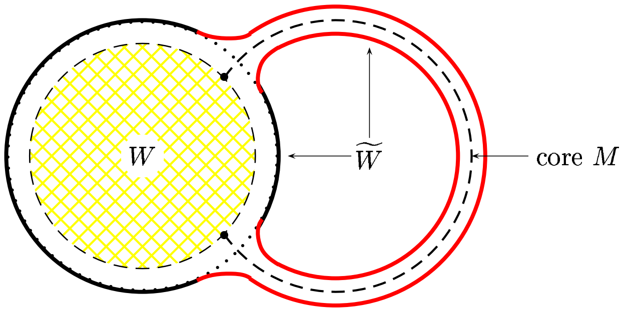

FIGURE 7. The handlebody $\widetilde{W}$ around $K = \overline{W} \cup M$

This completes the proof of lemma 5.1 for $k = n$. When $1 \leq k < n$, we apply the same proof with a totally real n-disc M' obtained by thickening $M = G(D)$ in the missing $n-k$ real directions. To find such M' we choose a $\mathbb{C}$-linear normal framing $\beta \colon \nu'' \to TX|_M$ over $G \colon D \hookrightarrow X$ such that $d^{\mathbb{C}}G \oplus \beta \colon T^{\mathbb{C}}D \oplus \nu'' = T\mathbb{C}^n|_D \to TX|_{G(D)}$ is a $\mathbb{C}$-linear isomorphism. Furthermore, we may choose β to map $\nu''|_S$ into the contact subbundle $\xi|_{G(S)}$. Let rD^{n-k} denote a closed ball of radius $r > 0$ in the real subspace $\{0\}^k \oplus \{i0\}^k \oplus \mathbb{R}^{n-k} \oplus \{i0\}^{n-k} \subset \mathbb{C}^n$. For a small $r > 0$ we can extend G to a smooth J-real embedding $(1+r)D^k \times rD^{n-k} \hookrightarrow X$, still denoted G, which is real analytic near $S^{k-1} \times rD^{n-k}$ and maps the latter manifold to a Legendrian submanifold of Σ, and such that dG equals β'' in the directions tangent to rD^{n-k} at every point of D^k. Taking $M' = G(D^k \times rD^{n-k})$ and $K' = \overline{W} \cup M'$ reduces the proof to the case $k = n$. (The fact that $\partial M'$ is not entirely contained in ∂W does not cause any complication in the above proof for $k = n$.) $\square$

6. Existence theorems for holomorphic maps

In this section we state and prove our main results when $\dim_{\mathbb{R}} X \neq 4$; the case $\dim_{\mathbb{R}} X = 4$ will be considered in §7.

Let P be a compact Hausdorff space and X, Y smooth manifolds. A *P-map* from X to Y is a continuous map $f \colon X \times P \to Y$. If X and Y are complex manifolds then such f is said to be a *holomorphic P-map* if $f_p = f(\cdot\,, p) \colon X \to Y$ is holomorphic for every fixed $p \in P$. The following is one of our main results; for $\dim_{\mathbb{R}} X = 4$ see also theorem 7.1.

Theorem 6.1. *Let (X, J) be a smooth almost complex manifold of real dimension $2n$ which is exhausted by a Morse function $\rho \colon X \to \mathbb{R}$ without critical points of index*

$> n$. *Assume that for some* $c \in \mathbb{R}$, J *is integrable in* $X_c = \{x \in X : \rho(x) < c\}$ *and* ρ *is strongly* J*-plurisubharmonic in* X_c. *Let* Y *be a complex manifold with a distance function* d_Y *induced by a Riemannian metric. Let* P *be a compact Hausdorff space and* $f: X \times P \to Y$ *be a* P*-map which is* J*-holomorphic in* X_c.

If $n \neq 2$, *or if* $n = 2$ *and* ρ *has no critical points of index* > 1 *in* $X \backslash W$ *then for every compact set* $K \subset X_c$ *and every* $\epsilon > 0$ *there exist a Stein structure* $\widetilde{J}$ *on* X *and a homotopy of* P*-maps* $f^t: X \times P \to Y$ $(t \in [0,1])$ *satisfying the following:*

(i) $f^0 = f$,
(ii) *the* P*-map* $f^1 = \widetilde{f}$ *is* $\widetilde{J}$*-holomorphic on* X.
(iii) *there is a homotopy* J_t *of almost complex structures on* X *which is fixed in a neighborhood of* K *such that* $J_0 = J$ *and* $J_1 = \widetilde{J}$; *if* J *is integrable then the homotopy* J_t *can be chosen to consist of integrable structures, and*
(iv) *for every* $t \in [0,1]$ *the* P*-map* f^t *is* J*-holomorphic in a neighborhood of* K *and satisfies* $\sup\{d_Y(f^t(x,p), f(x,p)) : x \in K,\ p \in P\} < \epsilon$.

The restriction to compact families of maps is essential — the following example shows that maps in a noncompact family in general cannot be simultaneously made holomorphic with respect to any complex structure on the source manifold.

Example 6.2. Let $X = A_r = \{z \in \mathbb{C}\colon 1/r < |z| < r\}$, and let $Y = A_R$ for another $R > 1$. We have $[X,Y] = \mathbb{Z}$, and a homotopy class represented by $k \in \mathbb{Z}$ admits a holomorphic representative if and only if $r^{|k|} \leq R$; in this case a representative is $z \to z^k$. Since every complex structure on an annulus is biholomorphic to A_r for some $r > 1$, we see that at most finitely many homotopy classes of maps between any pair of holomorphic annuli contain a holomorphic map. The conclusion of theorem 6.1 can be obtained by a radial dilation, decreasing the value of $r > 1$ to another value satisfying $r^k \leq R$, which amounts to a homotopic change of the complex structure. This allows us to simultaneously deform any compact family of maps $X \to Y$ to holomorphic maps, but it is impossible to do it for a sequence of maps belonging to infinitely many different homotopy classes. The problem disappears in the limit as $R \to +\infty$ when Y becomes the complex Lie group $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and the Oka-Grauert principle applies [39], [64].

The same phenomenon appears whenever the fundamental group $\pi_1(Y)$ contains an element $[\alpha]$ of infinite order such that the minimal Kobayashi length l_N of loops in Y representing the class $N[\alpha] \in \pi_1(Y)$ tends to $+\infty$ as $N \to +\infty$. A homotopically nontrivial loop γ in X with positive Kobayashi length $K_X(\gamma)$ can be mapped to the class $N[\alpha]$ by a holomorphic map $X \to Y$ only if $l_N \leq K_X(\gamma)$, and this is possible for at most finitely many $N \in \mathbb{N}$.

Proof. We shall write the proof in the nonparametric case since the parameters do not present any additional complication.

Fix a compact set $K \subset X_c$ and choose a regular value $c_0 \in \mathbb{R}$ of ρ such that $K \subset X_{c_0} \subset\subset X_c$. Hence the structure $J = J_0$ is integrable in a neighborhood of $\overline{X}_{c_0}$ and the map $f: X \to Y$ is J_0-holomorphic in a neighborhood of $\overline{X}_0$.

Let $p_1, p_2, \ldots$ be the critical points of ρ in $\{x \in X : \rho(x) > c_0\}$, ordered so that $\rho(p_j) < \rho(p_{j+1})$ for every j. Choose numbers c_j satisfying $c_{-1} = -\infty < c_0 < \rho(p_1) < c_1 < \rho(p_2) < c_2 < \ldots$. Let k_j denote the Morse index of p_j, so $k_j \leq n$. For each $j = 0, 1, \ldots$ we set $X_j = \{x \in X : \rho(x) < c_j\}$, $\Sigma_j = \partial X_j = \{x : \rho(x) = c_j\}$, and $A_j = \{x \in X : c_{j-1} \le \rho(x) < c_j\}$. We shall inductively construct a sequence of almost complex structures J_j on X and a sequence of maps $f_j : X \to Y$ satisfying the following for $j = 0, 1, 2, \ldots$:

(i) J_j is integrable in a neighborhood of $\overline{X}_j$, and the manifold (X_j, J_j) is Stein with strongly pseudoconvex boundary,
(ii) $J_j = J_{j-1}$ in a neighborhood of $\overline{X}_{j-1}$,
(iii) $\overline{X}_{j-1}$ is $\mathcal{H}(X_j, J_j)$-convex,
(iv) f_j is J_j-holomorphic in a neighborhood of $\overline{X}_j$,
(v) $\sup_{x \in X_{j-1}} d(f_j(x), f_{j-1}(x)) < \epsilon\, 2^{-j-1}$, and
(vi) there is a homotopy from f_{j-1} to f_j which is J_j-holomorphic and uniformly close to f_{j-1} in a neighborhood of $\overline{X}_{j-1}$ (satisfying the estimate in (v)).

These conditions clearly hold for $j = 0$, and in this case (ii), (iii) and (v) are vacuous. Assume inductively that the above hold for $j-1$. By Morse theory [61] X_j is diffeomorphic to a handlebody contained in X_j which is obtained by attaching to X_{j-1} an embedded disc $M_j \subset X \backslash X_{j-1}$ of dimension k_j and smoothly thickening the union $\overline{X}_{j-1} \cup M_j$. ($M_j$ may be taken as the unstable manifold of p_j for the gradient flow of ρ with respect to some metric on X.) Applying lemmas 3.1 and 5.1 with $W = X_{j-1}$, $J = J_{j-1}$ and $f = f_{j-1}$ gives a Stein structure $\widetilde{J}$ on a handlebody $\widetilde{W}$ which is isotopic to X_j and satisfies $\overline{X}_{j-1} \subset \widetilde{W} \subset X_j$. We also get a map $\widetilde{f} : X \to Y$, homotopic to f_{j-1}, which is $\widetilde{J}$-holomorphic on $\widetilde{W}$ and approximates f_{j-1} uniformly on X_{j-1}. (If $k_j = 0$, a new connected component of the sublevel set $\{\rho < c\}$ appears at p_j when c passes the value $\rho(p_j)$, and it is trivial to find $\widetilde{f}$ and $\widetilde{J}$ with these properties.) There is a smooth diffeotopy $h_t : X \to X$ ($t \in [0,1]$) which is fixed in a neighborhood of $\overline{X}_{j-1}$ such that h_0 is the identity map on X and $h = h_1$ satisfies $h(X_j) = \widetilde{W}$. Taking $J_j = h^*(\widetilde{J})$ and $f_j = \widetilde{f} \circ h$ completes the inductive step. (The homotopy from f_{j-1} to f_j is obtained by composing the homotopy from f_{j-1} to $\widetilde{f}$ by the map h.) Hence the induction may proceed.

By properties (i) and (ii) there is a unique integrable complex structure $\widetilde{J}$ on X which agrees with J_j on X_j. By the construction $\widetilde{J}$ is homotopic to the initial structure $J = J_0$ since at the j-th stage of the construction, the structure J_j was chosen homotopic to the previous structure J_{j-1} by a homotopy which is fixed on a neighborhood of $\overline{X}_{j-1}$. The complex manifold $(X, \widetilde{J})$ is exhausted by the increasing sequence of Stein domain X_j, and the Runge property (iii) implies that it is itself Stein. Properties (iv) and (v) insure that the sequence $f_j : X \to Y$ converges uniformly on compacts in X to the $\widetilde{J}$-holomorphic map $\widetilde{f} = \lim_{j\to\infty} f_j : X \to Y$ satisfying $\sup_{x \in X_0} d(\widetilde{f}(x), f_0(x)) < \epsilon$. By (vi) the homotopies from f_{j-1} to f_j also converge, uniformly on compacts in X, and give a homotopy from the initial map f_0 to $\widetilde{f}$. Denoting $\widetilde{f}$ by f_1 and the homotopy by f_t ($t \in [0,1]$) completes the proof.

If the initial structure J on X is integrable then all steps can be made within the class of integrable structures. However, assuming that (X, J) is Stein does not help very much, the main reason being that the size of the set on which the holomorphic approximation in theorem 4.1 can be defined in general depends on the previous step of the construction. □

The reader familiar with Eliashberg's paper [10] will notice that the geometry in the above proof essentially comes from Theorem 1.3.1 in [10]. An inessential difference is that we built everything on X, and not on an abstract new manifold.

Proof of theorem 1.2. This requires only minor modifications of the proof of theorem 6.1, and the underlying geometric part is similar to the proof of Theorem 1.3.6 in [10]. The main difference is that we do not change the given integrable structure J during the construction at the cost of remaining on subsets of X which are only diffeomorphic to sublevel sets of ρ, and not equal to them as before. We will in fact obtain a stronger version with approximation, similar to theorem 6.1.

We begin with the same assumptions and notation as in the proof of theorem 6.1. Thus, $W_0 = X_0$ is a sublevel set of a Morse exhaustion function $\rho\colon X \to \mathbb{R}$ which has no critical points of index $> n$ in $X \backslash W_0$, ρ is strongly plurisubharmonic in a neighborhood of $\overline{W}_0$, and the initial map $f_0 = f\colon X \to Y$ is holomorphic in a neighborbood of $\overline{W}_0$. Let $X_j = \{\rho < c_j\}$ where the constants c_j are chosen as in the proof of theorem 6.1, so ρ has a unique critical point p_j in $X_j \backslash X_{j-1}$. Choose $\epsilon > 0$ and let d_Y denote a distance function on the manifold Y. Assuming that $n \neq 2$ (the case $n = 2$ will be treated in §7 below) we inductively construct an increasing sequence of relatively compact, strongly pseudoconvex domains $W_1 \subset W_2 \subset \cdots$ in X with real analytic boundaries, a sequence of maps $f_j\colon X \to Y$, and a sequence of diffeomorphisms $h_j\colon X \to X$ such that the following hold for all $j = 1, 2, \ldots$:

(i) $\overline{W}_{j-1}$ is $\mathcal{H}(W_j)$-convex,
(ii) f_j is holomorphic in a neighborhood of $\overline{W}_j$ and is homotopic to f_{j-1} by a homotopy $f_{j,t}\colon X \to Y$ $(t \in [0,1])$ such that each $f_{j,t}$ is holomorphic near $\overline{W}_{j-1}$ and satisfies $\sup_{x \in W_{j-1}} d_Y\big(f_{j,t}(x), f_{j-1}(x)\big) < \epsilon 2^{-j}$,
(iii) $h_j(X_j) = W_j$, and
(iv) $h_j = g_j \circ h_{j-1}$ where $g_j\colon X \to X$ is a diffeomorphism of X which is diffeotopic to id_X by a diffeotopy which is fixed in a neighborhood of $\overline{W}_{j-1}$. (In particular, h_j agrees with h_{j-1} near $\overline{W}_{j-1}$.)

Granted such sequences, it is easily verified that the limit map

$$\widetilde{f} = \lim_{j\to\infty} f_j\colon \Omega = \cup_j W_j \to Y$$

and the limit diffeomorphism $h = \lim_{j\to\infty} h_j\colon X \to \Omega$ then satisfy the conclusion of theorem 1.2.

To prove the inductive step, we begin by attaching to $W_{j-1} = h_{j-1}(X_{j-1})$ the disc $M_j := h_{j-1}(D_j)$, where $D_j \subset X_j \backslash X_{j-1}$ (with $\partial D_j \subset \partial X_{j-1}$) is the unstable disc for ρ at the unique critical point $p_j \subset X_j \backslash X_{j-1}$ of ρ in this region. By lemma 3.1 we can isotope M_j to a totally real, real analytic disc in X attached to ∂W_{j-1} along a Legendrian sphere. Applying lemma 5.1 with the integrable structure J we find the next map $f_j\colon X \to Y$ which is holomorphic in a thin handlebody $W_j \supset \overline{W}_{j-1} \cup M_j$. The next diffeomorphism h_j with the stated properties is then furnished by the Morse theory. This concludes the proof of theorem 1.2 in a stronger version than stated.

With slightly more care one can insure that $\partial\Omega$ is smoothly bounded and strongly pseudoconvex, but in general we cannot choose such Ω to be relatively compact in X unless X has finite topology (i.e., there is an exhaustion function $\rho\colon X \to \mathbb{R}$ with at most finitely many critical points).

In the remainder of this section we discuss the existence of holomorphic maps of maximal rank (immersions resp. submersions). Let X and Y be complex manifolds. A necessary condition for a continuous map $f\colon X \to Y$ to be homotopic to a holomorphic map of maximal rank is that f is covered by a complex vector bundle map $\iota\colon TX \to TY$, i.e., such that for every $x \in X$ the map $\iota_x\colon T_xX \to T_{f(x)}Y$ is $\mathbb{C}$-linear and of maximal rank. If X is a Stein manifold, this condition is known to be sufficient in the following cases:

(i) $\dim X = 1$ and $Y = \mathbb{C}$ (Gunning and Narasimhan [47]);
(ii) $Y = \mathbb{C}^q$ with $q > \dim X$ (Eliashberg and Gromov [12], [45]);
(iii) $Y = \mathbb{C}^q$ with $q < \dim X$ (Forstnerič [23]);
(iv) $n = \dim X \geq \dim Y$ and Y satisfies a Runge approximation property for holomorphic submersion $\mathbb{C}^n \to Y$ on compact convex sets in $\mathbb{C}^n$ (the Property $\mathrm{S_n}$ in [24]).

By an obvious modification of the proof of theorem 6.1, using the part of lemma 5.1 for maps of maximal rank, one obtains the following result which in particular implies theorem 1.3.

Theorem 6.3. *Let (X, J) be a smooth almost complex manifold of real dimension $2n$, exhausted by a Morse function $\rho\colon X \to \mathbb{R}$ without critical points of index $> n$. Let $f\colon X \to Y$ be a continuous map to a complex manifold Y, and let $\iota\colon TX \to TY$ be a complex vector bundle map covering f such that $\iota_x\colon T_xX \to T_{f(x)}Y$ is of maximal rank* $\min\{\dim X, \dim Y\}$ *for every $x \in X$.*

If $n \neq 2$, or if $n = 2$ and ρ has no critical points of index > 1, there is a homotopy (J_t, f_t, ι_t) $(t \in [0,1])$ where J_t is an almost complex structure on X, $f_t\colon X \to Y$ is a continuous map, and $\iota_t\colon TX \to TY$ is a J_t-complex linear vector bundle map of pointwise maximal rank covering f_t, such that the following hold:

(i) *$J_0 = J$, $f_0 = f$, $\iota_0 = \iota$,*
(ii) *(X, J_1) is a Stein manifold,*
(iii) *the map $f_1\colon X \to Y$ is J_1-holomorphic and of maximal rank (an immersion resp. a submersion), and $df_1 = \iota_1$.*

If in addition J is integrable Stein on $X_c = \{\rho < c\}$ for some $c \in \mathbb{R}$, f is holomorphic on X_c and $\iota = df$ on X_c then for every compact set $K \subset X_c$ the homotopy J_t may be chosen fixed near K, the map f_t may be chosen holomorphic near K and uniformly close to $f = f_0$ on K, and ι_t may be chosen to satisfy $\iota_t = df_t$ near K.

The analogous result holds for a family of maps parametrized by a compact Hausdorff space (compare with theorem 6.1).

7. The four dimensional case

In this section we consider the case when the source manifold X is of real dimension four. We first state and prove the following, more precise version of theorem 1.1 in this case. The notion of a (holomorphic) P-map has been defined at the beginning of §6.

Theorem 7.1. *Let X be a smooth, oriented 4-manifold, exhausted by a Morse function $\rho\colon X \to \mathbb{R}$ without critical points of index > 2. Assume that for some $c \in \mathbb{R}$ there is an integrable complex structure J on $X_c = \{x \in X\colon \rho(x) < c\}$ such that $\rho|_{X_c}$ is strongly J-plurisubharmonic. Let Y be a complex manifold with* *a distance function d_Y induced by a Riemannian metric, P a compact Hausdorff space and $f\colon X\times P\to Y$ a P-map which is J-holomorphic in X_c.*

Given a compact set $K\subset X_c$ and $\epsilon>0$, there are a Stein surface (X',J'), an orientation preserving homeomorphism $h\colon X\to X'$ which is biholomorphic in a neighborhood of K, and a holomorphic P-map $f'\colon X'\times P\to Y$ such that the P-map $\widetilde f\colon X\times P\to Y$ defined by $\widetilde f(x,p)=f'(h(x),p)$ is homotopic to f and satisfies

$$\sup\left\{d_Y\big(f(x,p),\widetilde f(x,p)\big): x\in K,\ p\in P\right\}<\epsilon.$$

Remark 7.2. Unlike in the case $n>2$, we do not need to assume that the almost complex structure J in theorem 7.1 is defined on all of X since the obstruction to extending it across an attached handle only appears for handles of index >2. However, if J is already given on all of X, one can choose (X',J') such that the homotopy class of almost complex structures on X determined by $h^*(J')$ equals the class of J. Gompf showed that this notion makes sense under orientation preserving homeomorphisms (p. 645 in [35]).

Proof. For the geometric part of the proof we follow Gompf's construction of exotic Stein structures [35], but with a modification which better suits our purpose of finding a holomorphic map in the given homotopy class. Subsequently we will show how the construction can be carried out inside a given complex surface as in [36], thereby proving theorem 1.2 (for $\dim_{\mathbb R} X=4$) and corollary 1.6.

The proof of theorem 6.1 applies without any changes when attaching handles of index zero or one, but the difficulty arises when attaching 2-handles because lemma 3.1 fails to give an embedded core 2-disc attached along a Legendrian curve. As shown by Gompf [35], [36], the obstruction can be removed by using *Casson handles* at the cost of changing the underlying smooth structure on X. We begin by reviewing the necessary background material, referring to Gompf [35] for a more complete discussion of the topological part.

Let W be a relatively compact, smoothly bounded domain in X such that J is defined on a neighborhood of $\overline{W}$, and $\Sigma=\partial W$ is strongly pseudoconvex with respect to this structure. Let $G\colon D=D^2\to X\backslash W$ be an embedded 2-disc attached along the circle $G(S)\subset\Sigma=\partial W$. Let $g=G|_S\colon S\hookrightarrow\Sigma$. The restriction of the contact subbundle $\xi=T\Sigma\cap J(T\Sigma)$ to the circle $C:=g(S)$ is a trivial bundle (since every oriented two-plane bundle over a circle is trivial). The Legendrization theorem (see §3) gives an isotopy of G to another embedding, still denoted G, which is normal to ∂W along the Legendrian boundary circle $C=g(S)\subset\Sigma$. (The boundary circle remains in Σ during this isotopy.) We denote by $M=G(D)$ the resulting embedded 2-disc in $X\backslash W$, with $\partial M=C$.

Let $\nu_C\subset T\Sigma|_C$ denote the normal bundle of C in Σ. It is spanned by the pair of vector fields $(Jw,J\tau)$ where w is normal to Σ in X, with $Jw\in T\Sigma$, and τ is tangent to the circle C. This pair of fields determines what is called the *canonical framing*, or the *Thurston-Bennequin framing* TB, of the normal bundle ν_C. (Such framing is only defined for Legendrian knots or links.)

Let $\beta\colon\nu\to TX|_M$ denote a normal framing over M (a trivialization of the normal bundle of M in X), chosen such that $\beta(\nu|_S)=\nu_C$. We thus have two framings of ν_C, namely β (which extends to the disc M) and the TB framing. Since ν_C is a trivial 2-plane bundle over the circle C, any two framings differ up to homotopy by a map $C \to SO(2) = S^1$, hence by an integer. We can thus write $[\beta] = TB + k$; the integer $k = k([\beta]) \in \mathbb{Z}$ will be called the *framing index* of β.

In the model case when $M = D^2 \subset \mathbb{C}^2$ is the core of a standard handle in $\mathbb{C}^2$ attached to a quadric domain $Q_\lambda \subset \mathbb{C}^2$ (2.2) we easily see that

$$[\beta] = TB - 1. \tag{7.1}$$

Indeed, the tangent field τ to $S = \partial D^2$ rotates once in the positive (counterclockwise) direction as we trace S in the positive direction. Since the complex structure operator J_{st} on $T_z\mathbb{C}^2$ is an orientation reversing map of $\mathbb{R}^2$ onto $i\mathbb{R}^2$, the vector field $J\tau$ (which determines TB) rotates once in the clockwise direction, hence β is obtained from the TB framing by one left (negative) twist, explaining (7.1).

When the normal framing β satisfies (7.1) then J extends to an integrable complex structure in a neighborhood of $\overline{W} \cup M$ in X such that the core disc M is J-real (this is precisely as in [10]). In this case lemma 5.1 in §5 applies and yields a holomorphic map in a neighborhood of $\overline{W} \cup M$ which approximates the previous map uniformly on W. If this ideal situation occurs for all 2-handles in $X\backslash W$ then the construction of a Stein structure on X and a holomorphic map $X \to Y$ can be completed exactly as in §6.

Suppose now that $k = [\beta] - TB \neq -1$ for some 2-handle. A basic fact from the theory of Legendrian knots [2], [11] is that for any Legendrian knot K there is a $\mathcal{C}^0$-small isotopy (preserving the knot type but changing its Legendrian knot type) which adds any number of left (negative) twists to the TB framing. (One adds small spirals to K.) Since the homotopy class of the β framing is preserved under an isotopy of C in Σ, we see that $k = [\beta] - TB$ can be increased by any number of units. If $k < -1$, it is therefore possible to add spirals to the boundary circle and obtain an isotopic embedding $(D, S) \hookrightarrow (X\backslash W, \Sigma)$ satisfying (7.1), thereby reducing the problem to the previous case.

The problem is more difficult when $k \geq 0$ since it is in general impossible to add right twists to the TB framing or, equivalently, to decrease the framing index k. This is only possible in a contact structure which is *overtwisted*, in the sense that it contains a topologically unknotted Legendrian knot K with the Thurston-Bennequin index $tb(K) = 0$; adding such knot to a Legendrian knot adds a positive twist to the TB framing which makes it possible to decrease the difference $k = [\beta] - TB$ and hence reach $k = -1$. However, Eliashberg proved [11] that contact structures arising as boundaries of strongly pseudoconvex Stein manifolds are never overtwisted (they are *tight*). A 2-handle for which we cannot find an isotopy of the boundary circle to a Legendrian knot so that 7.1 holds will be called in the sequel *a wrongly attached handle*.

In [38] Gompf showed how one can circumvent the problem by replacing each wrongly attached 2-handle by a *Casson handle* which is only homeomorphic, but not diffeomorphic to the standard 2-handle $D^2 \times D^2$. In such case the new domain can be chosen to admit a Stein structure and is homeomorphic (but not diffeomorphic) to the original manifold. In the following two paragraph we summarize Gompf's construction for future reference.

Let Z be a 4-manifold that is the result of a 2-handle h with the core disc M attached to a compact Stein surface $\overline{W}$ with strongly pseudoconvex boundary $\Sigma = \partial W$. We first isotope $C = \partial M \subset \Sigma$ to a Legendrian knot in Σ. Since the TB invariant can be increased by a non-Legendrian isotopy by an arbitrary integer, while the homotopy class $[\beta]$ does not change by the isotopy, we can assume that the framing coefficient $[\beta] - TB$ is odd, $[\beta] - TB = -1 + 2k$ for some $k \in \mathbb{Z}$. If $k < 0$ we can further isotope the boundary of h to get (7.1) and we are done. If not, we remove the 2-handle h and reattach it to W along C using the framing $TB - 1$, meaning that we add $2k$ negative twists to the framing of h. Lets us call this new handle h', and let Z' be the new manifold obtained by attaching h' to $\overline{W}$ in this way. As mentioned above, we can now extend the complex (Stein) structure from W across the 2-handle h' to a Stein structure on Z'. If in Z the 2-handle h gives a homology class in $H_2(Z, \mathbb{Z})$ with self-intersection m, the self-intersection of h' in the homology of Z' is $m - 2k$. In order to compensate this error we make k positive *self-plumbings* to the handle $h' \subset Z'$ to get a yet new manifold Z_1. (A self-plumbing is done by locating two discs M_1, M_2 in the core M of h, trivializing the normal bundles over M_1 and M_2 to get $M_1 \times M_1'$ and $M_2 \times M_2'$ in Z', and identifying them by using the map $M_1 \times M_1' \to M_2 \times M_2'$ that interchanges the factors: $(z, w) \mapsto (w, z)$ in local coordinates. For a negative plumbing we would use $(z, w) \to (\overline{w}, z)$. We can do the self-plumbing so that the manifold Z_1 is still endowed with a Stein structure: locally at a double point, the manifold can be made to look as a tubular Stein neighborhood of $\mathbb{R}^2 \cup i\mathbb{R}^2 \subset \mathbb{C}^2$; see Theorem 1.3.5. in [10] or Theorem 2.2 in [21].) Even though Z_1 is not even homotopically equivalent to Z, we do have $H^2(Z, \mathbb{Z})$ isomorphic to $H^2(Z_1, \mathbb{Z})$ by an isomorphism preserving the self-intersection form. This follows from the fact that every positive self-plumbing of a handle h' introduces a positive transverse double point to its core, thus raising the self-intersection number of h' by 2.

In the next step we show that Z can be reconstructed back from the manifold Z_1 by attaching additional 2-cells to Z_1. The group $H_1(Z_1, \mathbb{Z})$ differs from $H_1(Z, \mathbb{Z})$ by $\mathbb{Z}^k$, with new homology classes represented by one loop in Z_1 for each performed plumbing, the only requirement being that the loop passes once through the plumbed double point. Attaching a 2-handle to Z_1 along each such loop cancels the extra homology and moreover, if the framing is correct, reconstructs the original manifold Z. The details can be found in [35], but will also be obvious from the construction below. The problem now is that the framing of the 2-handles we need to attach to Z_1 in order to reconstruct Z may not be correct, in the sense that the attaching circles of the cores cannot be isotoped to Legendrian knots satisfying (7.1). To correct this one repeats the above steps, adding kinks to each of these wrongly attached handles, thus beginning the *Casson tower* procedure. In this way one gets an increasing sequence of Stein manifolds $X_1 \subset X_2 \subset X_3 \subset \cdots$ with strongly pseudoconvex boundaries, each of them Runge in the next one. The limit manifold $\cup_j X_j$ is then also Stein, and by Freedman's result on Casson handles [32] it is homeomorphic to the original manifold X.

Here is a somewhat different explanation of the above procedure which is better suited to our purpose; its key advantage is that we remain inside X during the entire construction.

Let us take an immersed totally real sphere $S^2 \to \mathbb{C}^2$ with a positive double point at $0 \in \mathbb{C}^2$ and no other double points; an explicit Lagrangian example is due to Weinstein [74]:

$$F(x, y, z) = \big(x(1 + 2iz), y(1 + 2iz)\big) \in \mathbb{C}^2 \tag{7.2}$$

where $(x, y, z) \in \mathbb{R}^3$, $x^2 + y^2 + z^2 = 1$. Let us think of S^2 as the union of two closed discs $D_0 \cup D_\infty$, glued along their boundary circles $S^1 = D_0 \cap D_\infty$ and chosen such that $F(D_\infty) \subset \mathbb{C}^2$ is embedded while $F(D_0)$ contains the positive double point at the origin. The oriented normal bundle ν of F is isomorphic to $\overline{TS^2}$, the tangent bundle of S^2 with the reversed orientation. Indeed, $T\mathbb{C}^2|_{F(S^2)} = F_*(TS^2) \oplus \nu$, and the complex structure J_{st} gives an orientation reversing isomorphism of the first onto the second summand. By a small modification of F we may assume that the double point at 0 (the *center of the kink*) is special (4.1) which means that, in a suitable local holomorphic coordinate system, we have the union $\mathbb{R}^2 \cup i\mathbb{R}^2$. We shall take $K = F(D_0) \subset \mathbb{C}^2$ as our model kink which will be used to correct the framing coefficient of wrongly attached handles.

Since TS has Euler number $\chi(TS) = 2$, the normal bundle ν of the Weinstein sphere in $\mathbb{C}^2$ has $\chi(\nu) = -2$. Hence a copy of this model kink K, glued into a 2-disc M attached to ∂W along a Legendian knot as in fig. (8), will reduce the framing coefficient of M by 2 units. This is seen very explicitly as follows. Thinking of S^2 as the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with $D_0 = \{z \in \mathbb{C}\colon |z| \le 1\}$ and $D_\infty = \{w \in \mathbb{C} \cup \{\infty\}\colon |w| \ge 1\}$, the real and imaginary part of the complex vector field $\frac{\partial}{\partial w}$ provide a reference framing for $TS|_{D_\infty}$. From $\frac{\partial}{\partial w} = -z^2 \frac{\partial}{\partial z}$ we see that $\frac{\partial}{\partial w}$ makes two left (negative) twists when compared to the framing $\frac{\partial}{\partial z}$ for $TS^2|_{D_0}$ as we trace the circle $S = \partial D_0 = \{|z| = 1\}$ in the positive direction. Conversely, $\frac{\partial}{\partial z}$ makes two right (positive) twists in comparison to $\frac{\partial}{\partial w}$. Considering these framings on the immersed sphere $F(S^2) \subset \mathbb{C}^2$ and applying J_{st} we obtain framings of the normal bundle ν over the respective discs. Due to the reversal of the orientation under J_{st} we see that the framing for $\nu|_{D_0}$ makes two left (negative) twists when compared to the framing for $\nu|_{D_\infty}$, which explains $\chi(\nu) = -2$.

With F as in (7.2), let $\Delta = \{F(0, y, z)\colon y \ge 0,\ y^2 + z^2 \le 1\} \subset \mathbb{C}^2$. This 2-disc is embedded in $\mathbb{C}^2$, except along the side $\{y = 0\}$ which gets pinched to $0 \in \mathbb{C}^2$. Note that $\partial\Delta \subset F(S^2)$, and the union $F(S^2) \cup \Delta$ has a tubular neighborhood diffeomorphic to $S^2 \times \mathbb{R}^2$, a tubular neighborhood of an embedded sphere.

In order to make a self-intersection at a point p in a core disc M of the handle h in our 4-manifold X, we replace a small disc in M around p by a copy of our model kink K. (See fig. 8; we removed the small dotted disc and smoothly attached along its boundary the kinky disc shown on the right.) We have seen that this surgery reduces the relative Euler number over the immersed disc M by 2 for each kink added. Adding k kinks on M inside X and then taking a tubular neighborhood has the same effect as first removing the handle h from X, reattaching it with a framing of the boundary reduced by $2k$, and then performing k self-plumbings on h (as was done by Gompf in [35]). In this way we can see that the manifold Z_1, constructed above when discussing Gompf's proof, can be seen as a submanifold of the original manifold X, changed only by a surgery in a small coordinate neighborhood of the kinked points on the core disc of the handle h. We can also explicitly see the trivializing 2-cell Δ that needs to be added to each of the kinks in order to reconstruct the desired manifold.

In the next stage of the construction every such disc Δ will also have to receive a kink in order to correct its framing coefficient. This begins the *Casson tower* procedure which will converge to a Casson handle in place of the original removed disc in M.

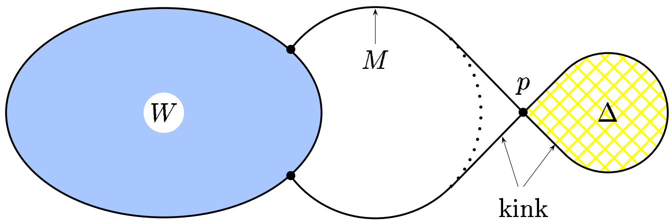

FIGURE 8. A kinky disc M with a trivializing 2-cell Δ

We are now ready to complete the proof of theorem 7.1. Assume that our 4-manifold X is constructed by successively attaching handles $h_1, h_2, h_3, \ldots$ of index ≤ 2, beginning with the compact domain $\overline{W} \subset X$ with smooth boundary $\Sigma = \partial W$. By assumption we also have an integrable complex structure J in a neighborhood of $\overline{W}$ such that W is Stein and its boundary Σ is strongly pseudoconvex. Let $M_1, M_2, \ldots$ be the cores of the handles $h_1, h_2, \ldots$, chosen such that their union is a smoothly embedded CW complex inside X. Since we have not assumed that our handlebody is finite, we can not ask for the ordering of the handles with regards to their indices. However, due to local compactness we can, and will, ask that when h_j with core M_j is attached, all handles whose core discs intersect the boundary ∂M_j have already been attached. Note also that we can assume that ∂M_j consists only of the core discs of handles of lower indices. We can now proceed with the induction as in the proof of 6.1, but with the following modifications:

(1) When a 2-handle is attached with a wrong framing, we insert the right number of kinks to its core disc (inside X) in order to change the framing coefficient to -1, thereby insuring that we can extend J to a Stein structure in a tubular strongly pseudoconvex neighborhood of the immersed disc. (The disc is totally real in this structure, with a special double point (4.1) at each kink.)

(2) Each time before proceeding to the next handle h_{j+1}, we perform one more step on each of the kinked discs appearing in the sequence before. More precisely, we add a new kinked disc which cancels the superfluous loop at the self-intersection point introduced in the previous step. (Of course this new kinked disc introduces a new superfluous loop which will have to be cancelled in the following step.)

The first condition is essential since we need to build a manifold that is Stein, and the second condition insures that each handle is properly worked upon, thereby producing a Casson tower at every place where a kink was made in the initial 2-disc. At every step we also apply lemma 5.1 to approximate the given map, which has already been made holomorphic in a tubular strongly pseudoconvex neighborhood of our partial (finite) subcomplex, by a map holomorphic in a tubular neighborhood of the previous domain with all core discs that have been added at the given step.

The proof can now be concluded similarly as in the proof of theorem 6.1. We construct an increasing sequence of Stein domains $X_1 \subset X_2 \subset \cdots$ inside the original smooth 4-manifold X, each of them Runge in the next one, together with a sequence of maps $f_j\colon X \to Y$ $(j = 1, 2, \ldots)$ such that f_j is holomorphic on X_j, it approximates f_{j-1} uniformly on X_{j-1}, and is homotopic to f_{j-1} by a homotopy which is holomorphic and uniformly close to f_{j-1} on X_{j-1}. The limit manifold $X' = \cup_j X_j$ is Stein with respect to the limit complex structure and, by the construction, is homeomorphic to X. (It is even diffeomorphic to X if no Casson handles were used in the construction.) A small ambient topological deformation moves the initial CW complex (made of the cores of the attached handles) into X'; see Gompf [36] for more details. By construction the limit map $f' = \lim_{j\to\infty} f_j\colon X' \to Y$ is holomorphic, and the map $f' \circ h\colon X \to Y$ is homotopic to f.

The same proof applies to any smoothly embedded 2-complex M inside X, and after a small ambient topological deformation we find a new embedding $M' \subset X$ with a Stein thickening $X' \subset X$ such that a given continuous map $M \to Y$ has a holomorphic representative $X' \to Y$. □

Remark 7.3. We note here that whenever a handle is wrongly attached, we cannot expect the process to be finite. One can actually check that in a model case of a kink, the disc Δ needed to be added to reconstruct the original manifold requires exactly one positive kink to be able to extend the Stein structure across it's neighborhood.

Remark 7.4. The complex structure J in the above proof was built by a stepwise extension across handles. If one already has an almost complex structure J defined on all of X, the Stein structure constructed by this process is in general not homotopic to J (compare with proposition 3.4). Below we shall give a different construction which will produce a Stein structure homotopic to the given initial complex structure on X. With a bit more work as in [35] one can also show that every relative cohomology class in $H^2(X, W; \mathbb{Z})$ which reduces mod 2 to the second Stiefel-Whitney class $w_2(TX) \in H^2(X, W; \mathbb{Z}_2)$ can be obtained as a reduction of a cohomology class of $c_1(X, J)$ for an appropriate J in theorem 7.1.

It remains to prove the case $n = 2$ of theorem 1.2 and corollary 1.6. We assume that (X, J) is a complex surface with an integrable (not necessarily Stein) structure J, and with a correct handlebody structure if we are proving theorem 1.2. By a modification of the proof given above we shall construct an increasing sequence of domains $X_1 \subset X_2 \subset \ldots$ in X and J-holomorphic maps $f_j\colon X_j \to Y$ such that each X_j is a Stein domain in X with respect to the given structure J, and the other properties are the same as before. The proof which we shall give is similar to the construction in Gompf's recent paper [39], the difference being that we do everything by using the special kink K introduced above.

We recall a few fact about the complex points of immersed real surfaces in complex surfaces. Let M be a compact real surface, smoothly immersed in a complex surface X. A generic such M has finitely many complex points $p \in M$ where the tangent space T_pM is a complex line in T_pX. Locally near such point p the surface M is given in suitable local holomorphic coordinates (z, w), with $z(p) = w(p) = 0$, by an equation $w = |z|^2 + \lambda(z^2 + \bar{z}^2) + o(|z|^2)$ for a unique $\lambda \geq 0$ (Bishop [3]). The point p is *elliptic* if $\lambda < 1/2$ and *hyperbolic* if $\lambda > 1/2$; the degenerate case $\lambda = 1/2$ does not arise for a generic M. At an elliptic point M has a nontrivial envelope of holomorphy consisting of a family of small analytic discs (the so called Bishop discs, [3]). On the other hand, near a hyperbolic point M is locally holomorphically convex [31] and it admits a basis of tubular Stein neighborhoods [69].

If M is oriented, one further divides its complex points into positive and negative ones, depending on whether the standard complex line orientation of T_pM agrees or disagrees with the given orientation of M. Let $e_\pm(M)$ resp. $h_\pm(M)$ indicate the numbers of elliptic resp. hyperbolic points in each orientation class. The *Lai indices* $I_\pm(M) = e_\pm(M) - h_\pm(M)$ are invariant under a regular homotopy of the immersion $M \to X$ and are given by the formula

$$2\, I_\pm(M) = \chi(M) \pm \langle c_1(X), [M] \rangle + [M]^2 - 2\, d(M) \tag{7.3}$$

where $\chi(M)$ is the Euler number of M, $[M] \in H_2(X;\mathbb{Z})$ is the homology class of M in X, $c_1(X) = c_1(TX)$ is the first Chern class of the tangent bundle TX, and $d(M)$ is the algebraic number of self-intersection points of M, counted with oriented intersection indices (see [6], [3], [56], [14], [73], [62], [21]).

A basic fact, proved by Eliashberg and Harlamov [14], is that one can cancel a pair of an elliptic and a hyperbolic point in the same orientation class by a $\mathcal{C}^0$-small isotopy supported in a neighborhood of a suitable chosen embedded arc in M connecting these two points. (See [20] and [62] for more details.) If $I_\pm(M) \le 0$ then we can deform M to an immersed surface with special double points which is totally real (if $I_\pm(M) = 0$) or has only *special hyperbolic points*, given in local holomorphic coordinates by $w = z^2 + \bar z^2$ (Theorem 1.2 in [21], p. 82). By Theorem 2.2 in [21] this new M admits a basis of smoothly bounded Stein neighborhoods diffeomorphic to $M \times \mathbb{R}^2$, given as sublevel sets of a smooth plurisubharmonic function $\tau \ge 0$ which vanishes precisely on M and has no critical points in a deleted neighborhood of M. (Locally at a special hyperbolic point $w = z^2 + \bar z^2$ we can take $\tau(z,w) = |w - z^2 - \bar z|^2$. It was shown in [69] how to find *strongly pseudoconvex* tubular neighborhoods of any surface with only hyperbolic complex points.)

By adding a positive kink to M we do not change its homology class $[M]$ and its Euler number $\chi(M)$, but we increase $d(M)$ by one. Hence we see from (7.3) that a positive kink reduces each of the numbers $I_\pm(M)$ by 1. This can also be seen directly: if in local coordinates $(z,w) = (x+iy, u+iv)$ the double point is given by $R^2 \cup i\mathbb{R}^2 = \{(x+iu)(y+iv) = 0\}$ (with the orientation reversed on the second summand), the perturbation $(x+iu)(y+iv) = 2\epsilon^2$ for $\epsilon > 0$ is a smooth surface with two hyperbolic complex points $p_1 = \epsilon(1+i, -1+i)$, $p_2 = -p_1$. Adding a sufficient number of positive kinks and performing a $\mathcal{C}^0$-small regular homotopy as above we find an immersed surface M with a basis of strongly pseudoconvex tubular Stein neighborhoods. For this argument we do not need to know the precise formula (7.3), it suffices to know that a positive kink reduces each of the numbers $I_\pm$ by one so that eventually they become nonpositive.

The same argument holds for a compact surface $M \subset X \backslash W$ with boundary attached to $\Sigma = \partial W$ along a Legendrian curve — after kinking it enough times we can find an isotopy, fixed near ∂M, to a new surface with only special hyperbolic points and special double points. (If M is a disc, it can be seen that $I_+(M) + I_-(M) = [\beta] - TB + 1$, but this will not be needed in the proof.)

We also note that a surface M is locally holomorphically convex near a hyperbolic complex point $p \in M$, and every continuous function on M near p can be approximated uniformly on M by functions holomorphic in a neighborhood of p in X. If M only has hyperbolic complex points then we can approximate each continuous function on M uniformly by holomorphic functions in a Stein neighborhood of M. It is now easily seen that theorem 4.1 still holds in the presence of hyperbolic points (with uniform approximation on M). The existence of Stein neighborhoods of the graph of f in $X \times Y$ (needed in the proof of theorem 4.1) is easily insured by chosing f to be constant in a small neighborhood of each hyperbolic point.

We can now complete the proof of theorem 1.2 when $\dim_{\mathbb{R}} X = 4$. Assume that X is obtained from a strongly J-pseudoconvex domain $\overline{W} \subset X$ by successively attaching handles $h_1, h_2, \ldots$ with core discs $M_1, M_2, \ldots$, where the ordering of these handles satisfies the same condition as in the proof of theorem 7.1. (Of course we may begin with $W = \emptyset$.) We shall use the same notation as in the proof of theorem 1.2 for $\dim_{\mathbb{R}} X \neq 4$ (see §6), beginning with $W_0 = W$ and $f_0 = f$. In the inductive step we have a smoothly bounded, strongly pseudoconvex domain $W_j \subset X$ and a map $f_j \colon X \to Y$ which is J-holomorphic in a neighborhod of $\overline{W}_j$. (The set W_j is a tubular neighborhood of the union of $\overline{W}$ with the cores of handles attached in the earlier steps; since these cores may have received kinks, W_j does not have the correct homeomorphic type, but this will be corrected in the limit by the Casson handles resulting from the construction.) We now attach to $\overline{W}_j$ the next handle in the sequence. As before, attaching a 1-handle does not pose a problem. For a 2-handle h we first make sure that the boundary of its core M is a Legendrian curve in ∂W_j, and then, if needed, we add enough positive kinks to algebraically cancel off all elliptic points on the core disc M. We denote this new immersed disc by M'. After a $\mathcal{C}^0$-small regular homotopy fixing the boundary $\partial M'$ we can assume that $\overline{W}_j \cup M'$ has a basis of tubular, strongly pseudoconvex, Stein neighborhoods in X. We also add to $\overline{W}_j$ a new trivializing kinky disc at each of the kinks from the earlier stages of the construction, making sure that the conditions (1) and (2) in the proof of theorem 7.1 are satisfied. These additional kinky discs $\Delta_1, \ldots, \Delta_k$ can be chosen such that $L_j = \overline{W}_j \cup M' \cup (\cup_j \Delta_j)$ has a basis of tubular, strongly pseudoconvex, Stein neighborhoods in X. By lemma 5.1 we approximate f_j uniformly on L_j by a map $f_{j+1} \colon X \to Y$ which is holomorphic in a neighborhood of L_j. Choosing a strongly pseudoconvex tubular neighborhood $W_{j+1} \supset L_j$ contained in the domain of holomorphicity of f_{j+1} completes the induction step. In the limit we obtain a holomorphic map $f' \colon \Omega \to Y$ on the Stein domain $\Omega = \cup_j W_j \subset X$ with the stated properties.

Corollary 1.6 is proved by applying the same construction to an embedded CW 2-complex in X.

Remark 7.5. By a small addition to the above argument it is actually possible to change every real surface M, attached to a strongly pseudoconvex domain $W \subset X$ along a Legendrian curve, to a totally real immersed surface all of whose double points are positive kinks. Indeed, by an isotopy of $\partial M \hookrightarrow \partial W$ to another Legendrian link one can insure that $I_+(M) - I_-(M) = 0$. (This difference equals the *rotation number* of the canonical contact extension of the Legendrian link $\partial M \subset \partial W$ which can be made equal to any given number by isotopies; see [10] and [35].) If $I_+(M) = I_-(M) > 0$, we add this many positive kinks and reduce $I_\pm(M)$ to 0, so our disc becomes totally real after a small isotopy. If $I_+(M) = I_-(M) < 0$, we can increase them to zero by adding left spirals to ∂M (see [35]), again making M totally real.

Acknowledgements. We sincerely thank Yakov Eliashberg who patiently answered our questions regarding his work [10], and whose help was essential to us in understanding the handle attachment lemma for handles of maximal allowed index.

The critical case $k = n$ in the proof of lemma 3.1 resulted from these communications. We also thank Robert Gompf for helpful communication regarding his works [35], [36] on Stein surfaces. We thank Petar Pavešić and Sašo Strle for their help with the homotopy theory. Finally, we thank Irena Majcen who introduced us to drawing pictures with *pstricks*.

References

1. ANDREOTTI, A., FRAENKEL, T., The Lefshetz theorem on hyperplane sections. *Ann. of Math.*, 69 (1959), 713–717.
2. ARNOL'D, V., Lagrangian and Legendrian cobordisms. *Funct. Anal. Appl.*, 14 (1980), 167–177.
3. BISHOP, E., Differentiable manifolds in complex Euclidean spaces. *Duke Math. J.*, 32 (1965), 1–21.
4. CARTAN, H., Espaces fibrés analytiques. *Symposium Internat. de topologia algebraica*, Mexico, 97–121 (1958). (Also in H. Cartan, *Oeuvres*, Vol. 2, Springer-Verlag, New York, 1979.)
5. ČERNE, M., FORSTNERIČ, F., Embedding some bordered Riemann surfaces in the affine plane. *Math. Res. Lett.*, 9 (2002), 683–696.
6. CHERN, S. S., SPANIER, E., A theorem on orientable surfaces in four-dimensional space. *Comm. Math. Helv.*, 25 (1951), 205–209.
7. COSTANTINO, F., Stein domains and branched shadows of 4-manifolds. Preprint, 2005.
8. DUCHAMP, T., The classification of Legendre immersions. Preprint, 1984.
9. EELLS, J. Jr., SAMPSON, J. H., Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86 (1964) 109–160.
10. ELIASHBERG, Y., Topological characterization of Stein manifolds of dimension > 2. *Internat. J. Math.*, 1 (1990), 29–46.
11. ———, Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98 (1989), 623–637.
12. ELIASHBERG, Y., GROMOV, M., Nonsingular maps of Stein manifolds. *Func. Anal. Appl.*, 5 (1971), 82–83.
13. ———, Embeddings of Stein manifolds. *Ann. of Math.*, 136 (1992), 123–135.
14. ELIASHBERG, Y., HARLAMOV, V., Some remarks on the number of complex points of a real surface in a complex one. *Proc. LITC–82*, 143–148.
15. ELIASHBERG, Y., MISHACHEV, N., *Introduction to the h-principle.* Graduate Studies in Math., 48. Amer. Math. Soc., Providence, RI, 2002.
16. FORNÆSS WOLD, E., Proper holomorphic embeddings of finitely and infinitely connected subsets of $\mathbb{C}$ into $\mathbb{C}^2$. *Math. Z.*, to appear.
17. ———, Embedding Riemann surfaces properly into $\mathbb{C}^2$. Preprint, 2005.
18. FORSTER, O., RAMSPOTT, K. J., Analytische Modulgarben und Endromisbündel. *Invent. Math.*, 2 (1966), 145–170.
19. FORSTNERIČ, F., On totally real embeddings into $\mathbb{C}^n$. *Expo. Math.*, 4 (1986), 243–255.
20. ———, Complex tangents of real surfaces in complex surfaces. *Duke Math. J.*, 67 (1992), 353–376.
21. ———, Stein domains in complex surfaces, *J. Geom. Anal.*, 13 (2003), 77–94.
22. ———, The Oka principle for sections of subelliptic submersions. *Math. Z.*, 241 (2002), 527–551.
23. ———, Noncritical holomorphic functions on Stein manifolds. *Acta Math.*, 191 (2003), 143–189.
24. ———, Holomorphic submersions from Stein manifolds. *Ann. Inst. Fourier*, 54 (2004), 1913–1942.
25. ———, Runge approximation on convex sets implies Oka's property. *Ann. of Math.*, 163 (2006). [arXiv: math.CV/0402278]
26. ———, Extending holomorphic mappings from subvarieties in Stein manifolds. *Ann. Inst. Fourier*, 55 (2005), 733-751.
27. ———, Holomorphic flexibility of complex manifolds. *Amer. J. Math.*, to appear. [arXiv: math.CV/0401439]

28. FORSTNERIČ, F., KOZAK, J., Strongly pseudoconvex handlebodies. *J. Korean Math. Soc.*, 40 (2003), 727-745.
29. FORSTNERIČ, F., PREZELJ, J., Oka's principle for holomorphic fiber bundles with sprays. *Math. Ann.*, 317 (2000), 117-154.
30. ——— , Oka's principle for holomorphic submersions with sprays. *Math. Ann.*, 322 (2002), 633-666.
31. FORSTNERIČ, F., STOUT, E. L., A new class of polynomially convex sets in $\mathbb{C}^2$. *Ark. Mat.*, 29 (1991), 51–62.
32. FREEDMAN, M., The topology of four-dimensional manifolds. *J. Diff. Geom.*, 17 (1982), 357–453.
33. FREEDMAN, M. H., QUINN, F., *Topology of 4-manifolds.* Princeton Mathematical Series, 39. Princeton Univ. Press, Princeton, NJ, 1990.
34. GLOBEVNIK, J., STENSØNES, B., Holomorphic embeddings of planar domains into $\mathbb{C}^2$. *Math. Ann.*, 303 (1995), 579–597.
35. GOMPF, R. E., Handlebody construction of Stein surfaces. *Ann. of Math.*, (2) 148 (1998), 619–693.
36. ——— , Stein surfaces as open subsets of $\mathbb{C}^2$. Preprint, 2005.
37. GOMPF, R. E., STIPSICZ, A. I., *4-manifolds and Kirby calculus.* Graduate Studies in Math., 20. Amer. Math. Soc., Providence, RI, 1999.
38. GRAUERT, H., Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen. *Math. Ann.*, 133 (1957), 139–159.
39. ——— , Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. *Math. Ann.*, 133 (1957), 450–472.
40. ——— , Analytische Faserungen über holomorph-vollständigen Räumen. *Math. Ann.*, 135 (1958), 263–273.
41. ——— , On Levi's problem. *Ann. of Math.*, 68 (1958), 460–472.
42. GROMOV, M., Stable maps of foliations into manifolds. *Izv. Akad. Nauk, S.S.S.R.*, 33 (1969), 707–734.
43. ——— , Convex integration of differential relations, I. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.*, 37 (1973), 329–343. English translation in *Math. USSR–Izv.*, 7 (1973), 329–343.
44. ——— , Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82 (1985), 307–347.
45. ——— , *Partial Differential Relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 9. Springer, Berlin–New York, 1986.
46. ——— , Oka's principle for holomorphic sections of elliptic bundles. *J. Amer. Math. Soc.*, 2 (1989), 851-897.
47. GUNNING, R. C., NARASIMHAN, R., Immersion of open Riemann surfaces. *Math. Ann.*, 174 (1967), 103–108.
48. GUNNING, R. C., ROSSI, H., *Analytic functions of several complex variables.* Prentice–Hall, Englewood Cliffs, 1965.
49. HENKIN, G., LEITERER, J., The Oka-Grauert principle without induction over the basis dimension. *Math. Ann.*, 311 (1998), 71–93.
50. HIRSCH, M., Immersions of manifolds. *Trans. Amer. Math. Soc.*, 93 (1959), 242–276.
51. HÖRMANDER, L., *An Introduction to Complex Analysis in Several Variables.* Third ed. North Holland, Amsterdam, 1990.
52. HÖRMANDER, L., WERMER, J., Uniform approximations on compact sets in $\mathbb{C}^n$. *Math. Scand.*, 23 (1968), 5–21.
53. KOBAYASHI, S., Intrinsic distances, measures and geometric function theory. *Bull. Amer. Math. Soc.* 82 (1976), 357–416.
54. KODAIRA, K., Holomorphic mappings of polydiscs into compact complex manifolds. *J. Diff. Geom.*, 6 (1971/72), 33–46.
55. KRONHEIMER, P. B., MROWKA, T. S., Monopoles and contact structures. *Invent. Math.* 130 (1997), 209–255.
56. LAI, H. F., Characteristic classes of real manifolds immersed in complex manifolds. *Trans. Amer. Math. Soc.*, 172 (1972), 1–33.
57. LÁRUSSON, F., Model structures and the Oka principle. *J. Pure Appl. Algebra*, 192 (2004), 203–223.

58. ——— , Mapping cylinders and the Oka principle. Preprint, 2004. [http://www.math.uwo.ca/ larusson/papers/]
59. LEFSHETZ, S., *L'analysis Situs et la Géometrie Algébrique.* Paris, 1924.
60. LISCA, P., MATIĆ, G., Tight contact structures and Seiberg-Witten invariants. *Invent. Math.*, 129 (1997), 509–525.
61. MILNOR, J., *Morse Theory.* Ann. of Math. Studies, 51. Princeton Univ. Press, Princeton, 1963.
62. NEMIROVSKI, S., Complex analysis and differential topology on complex surfaces. *Uspekhi Mat. Nauk*, 54, 4 (1999), 47–74. English transl. in *Russian Math. Surveys*, 54, 4 (1999), 729–752.
63. NEWLANDER, A., NIRENBERG, L., Complex analytic coordinates in almost complex manifolds. *Ann. of Math.*, (2) 65 (1957), 391–404.
64. OKA, K., Sur les fonctions des plusieurs variables. III: Deuxième problème de Cousin. *J. Sc. Hiroshima Univ.*, 9 (1939), 7–19.
65. OZBAGCI, B., STIPSICZ, A. I., *Surgery on Contact 3-Manifolds and Stein Surfaces.* Springer-Verlag, 2005.
66. OZSVÁTH, P., SZABO, Z., The symplectic Thom conjecture. *Annals of Math.*, 151 (2000), 93–124.
67. PHILLIPS, A., Submersions of open manifolds. *Topology*, 6 (1967), 170–206.
68. ROSAY, J.-P., A counterexample related to Hartog's phenomenon (a question by E. Chirka). *Michigan Math. J.*, 45 (1998), 529–535.
69. SLAPAR, M., On Stein neighborhood basis of real surfaces. *Math. Z.*, 247 (2004), 863–879.
70. SMALE, S., The classification of immersions of spheres in Euclidean spaces. *Ann. of Math.* (2), 69 (1959), 327–344.
71. STEIN, K., Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem. *Math. Ann.*, 123 (1951), 201–222.
72. STOUT, E. L., *The Theory of Uniform Algebras.* Bogden&Quigley, Inc., Tarrytown-on-Hudson, New York, 1971.
73. WEBSTER, S., Minimal surfaces in a Kähler surface. *J. Diff. Geom.*, 20 (1984), 463–470.
74. WEINSTEIN, A., *Lectures on Symplectic Manifolds.* Reg. Conf. Ser. Math., 29, Amer. Math. Soc., Providence, 1977.
75. ——— , Contact surgery and symplectic handlebodies. *Hokkaido Math. J.*, 20 (1991), 241–251.
76. WHITEHEAD, G. W., *Elements of Homotopy Theory.* Graduate Texts in Math., 61, Springer-Verlag, 1978.
77. WHITNEY, H., The self-intersections of a smooth manifold in a $2n$-space. *Ann. of Math.*, 45 (1944), 220–246.

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: franc.forstneric@fmf.uni-lj.si, marko.slapar@fmf.uni-lj.si