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DISTANCE-BALANCED GRAPHS

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Abstract

Distance-balanced graphs are introduced as graphs in which every edge uv has the following property: the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. Basic properties of these graphs are obtained. The new concept is connected with symmetry conditions in graphs and local operations on graphs are studied with respect to it. Distance-balanced Cartesian and lexicographic products of graphs are also characterized. Several open problems are posed along the way.

Key words: Graph distance; Distance-balanced graphs; Graph products; Connectivity

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1 Introduction

For an edge ab of a graph G let W_{ab}^G be the set of vertices closer to a than to b. That is,

$$W_{ab}^G = \{u \in G \mid d(u,a) < d(u,b)\}\,.$$

In addition, let ${}_aW_b^G$ be the set of vertices with equal distances to a and b;

$${}_aW_b^G = \{u \in G \mid d(u,a) = d(u,b)\}\,.$$

If the graph G is clear from the context, we write simply W_{ab} and ${}_aW_b$. Note that W_{ab}^G and W_{ba}^G form a partition of the vertex set of a connected, bipartite graph G.

The sets W_{ab} (and ${}_aW_b$) play an important role in metric graph theory. A classical result of this theory due to Djoković [3] asserts that a bipartite graph is a partial cube (a graph that admits an isometric embedding into a hypercube) if and only if for any edge ab of G the sets W_{ab} and W_{ba} are convex. Chepoi [2] generalized this result by proving that G isometrically embeds into a Hamming graph if and only if the sets W_{ab}, W_{ba}, $W_{ab} \cup {}_aW_b$, and $W_{ba} \cup {}_aW_b$ are convex for all edges ab of G. For more information on the research in this direction see [1].

We also note that the sets W_{ab} appear in chemical graph theory as well: The well-investigated Szeged index of a graph G is defined as $\sum_{ab\in E(G)} |W_{ab}| \cdot |W_{ba}|$, cf. [6, 7].

Here is our key definition. We call a graph G *distance-balanced*, if

$$|W_{ab}| = |W_{ba}|$$

holds for any edge ab of G.

Motivated by the results of [5], Handa considered distance-balanced partial cubes and proved that they are 3-connected, with the exception of cycles and the complete graph of order two. In general, any distance-balanced graph having at least two edges is 2-connected.

In this paper we do not restrict ourselves to the bipartite case but consider distance-balanced graphs in general. In the second section we consider several examples of distance-balanced graphs and derive some of their properties. In particular we characterize distance-balanced graphs of a given diameter and connect the concept with symmetry conditions in graphs. In the subsequent section we show that a removal of an edge always destroys the property of being distance-balanced. The same also holds in almost all the cases when an edge of a distance-balanced graph is subdivided. Then, in Section 4, we determine which Cartesian and lexicographic products are distance-balanced and show that this property is not preserved by direct and strong products of graphs. We conclude the paper suggesting several research topics and posing some open problems.

2 Examples and basic properties

We begin with some examples of distance-balanced graphs.

Note first that cycles and complete graphs are distance-balanced. In fact, complete graphs are the only distance-balanced chordal graphs. This follows easily by considering an edge $e = uv$ where u is the first vertex in a perfect elimination scheme and v is any neighbor of u that is not simplicial.

It is also easy to verify that hypercubes are distance-balanced. This time hypercubes are the only median distance-balanced graphs. (One way to prove this fact is to use Lemma 2.41 from [9].) On the other hand the variety of distance-balanced partial cubes is much richer, cf. Fig. 1, but a characterization of such graphs seems to be a difficult problem.

Our first result gives a slightly different view to the definition of distance-balanced graphs. For a vertex x of a connected graph G and $k \geq 0$ let $N_k(x) = \{y \in G \mid d(x,y) = k\}$ and $N_k[x] = \{y \in G \mid d(x,y) \leq k\}$. For $k = 1$ we shorten these to $N(x)$ and $N[x]$. Then we have:

Proposition 2.1 *A graph G of diameter d is distance-balanced if and only if*

$$|N[a] \setminus N[b]| + \sum_{k=2}^{d-1} |N_k(a) \setminus N_{k-1}(b)| = |N[b] \setminus N[a]| + \sum_{k=2}^{d-1} |N_k(b) \setminus N_{k-1}(a)|$$

holds for every edge $ab \in E(G)$.

Proof. For $k \geq 1$, let $D_k(ab) = \{u \in G \mid d(u,a) = k, d(u,b) = k+1\}$. Then W_{ab} can be written as

$$W_{ab} = \{a\} \cup \bigcup_{k=1}^{d-1} D_k(ab)\,.$$

Note that for any $k \geq 1$, $D_k(ab) = (N_k(a) \setminus N_k[b])$ and observe that $N(a) \setminus N[b] = N[a] \setminus N[b]$. For $k \geq 2$ we can further compute $D_k(ab) = (N_k(a) \setminus N_k[b]) = [N_k(a) \setminus N_{k-1}(b)] \setminus [N_k(a) \cap N_k(b)]$. Therefore,

$$W_{ab} = \{a\} \cup (N[a] \setminus N[b]) \cup \bigcup_{k=2}^{d-1} \Big([N_k(a) \setminus N_{k-1}(b)] \setminus [N_k(a) \cap N_k(b)] \Big).$$

and

$$W_{ba} = \{b\} \cup (N[b] \setminus N[a]) \cup \bigcup_{k=2}^{d-1} \Big([N_k(b) \setminus N_{k-1}(a)] \setminus [N_k(a) \cap N_k(b)] \Big).$$

Since $N_k(a) \cap N_k(b)$ is a subset of both $N_k(a) \setminus N_{k-1}(b)$ and $N_k(b) \setminus N_{k-1}(a)$, we have $|W_{ab}| = |W_{ba}|$ if and only if

$$|N[a] \setminus N[b]| + \sum_{k=2}^{d-1} |N_k(a) \setminus N_{k-1}(b)| = |N[b] \setminus N[a]| + \sum_{k=2}^{d-1} |N_k(b) \setminus N_{k-1}(a)|$$

which completes the argument. $\square$

Corollary 2.2 *Let G be a regular graph of diameter d. Then G is distance-balanced if and only if*

$$\sum_{k=2}^{d-1} |N_k(a) \setminus N_{k-1}(b)| = \sum_{k=2}^{d-1} |N_k(b) \setminus N_{k-1}(a)|$$

holds for every edge $ab \in E(G)$.

The following result yields many distance-balanced graphs. In particular it implies that the Petersen graph is such.

Corollary 2.3 *Let G be a graph of diameter two. Then G is distance-balanced if and only if G is regular.*

The study of distance-balanced graphs was initiated because of the importance of the sets W_{ab} and ${}_aW_b$ in metric graph theory. However, there is also a strong link to symmetry conditions in graphs. The following proposition suggests this connection.

Proposition 2.4 *Let G be a graph. If for any edge ab of G there exists an automorphism φ of G such that $\varphi(a) = b$ and $\varphi(b) = a$, then G is distance-balanced.*

Proof. Let $ab \in E(G)$ and let φ be an automorphism of G for which $\varphi(a) = b$ and $\varphi(b) = a$. Let $x \in W_{ab}$ and let $d(a,x) = k$. Then $d(b,x) = k+1$. Since automorphisms preserve distances, $k = d(a,x) = d(\varphi(a), \varphi(x)) = d(b, \varphi(x))$ and $k+1 = d(b,x) = d(\varphi(b), \varphi(x)) = d(a, \varphi(x))$. It follows that $\varphi(x) \in W_{ba}$. Likewise, if $y \in W_{ba}$ then $\varphi(y) \in W_{ab}$. Hence $|W_{ab}| = |W_{ba}|$. $\square$

Circulant graphs and all vertex-transitive, generalized Petersen graphs have automorphisms as described in Proposition 2.4 [4, 10]. We know of no graph that is vertex-transitive and not distance-balanced. However, there are many more generalized Petersen graphs that are distance-balanced but not vertex-transitive. For example, $P(7,3)$ is such a graph. Evidence generated by computer algorithm suggests that there are distance-balanced generalized Petersen graphs $P(n,k)$ for almost all n. On the other hand, we suspect the following is true.

Conjecture 2.5 *For any $k \geq 2$ there exists an n_0 such that $P(n,k)$ is not distance-balanced for every $n \geq n_0$.*

3 Local operations

In this section we consider local operations on graphs and show that they typically destroy the property of being distance-balanced.

Proposition 3.1 *Let G be a distance-balanced graph that has at least two edges and let $e \in E(G)$. Then $G - e$ is not distance-balanced.*

Proof. Let $e = ab$. Let $H = G - e$ and let P be a shortest a,b-path in H. (P exists because G is distance-balanced and hence 2-connected, cf. [8, Lemma 2.1].) Let x be the neighbor of a on P and consider the edge ax of H. Let $y \in W_{xa}^G$. Since e does not lie on any shortest y,x-path in G (for otherwise we would have $y \in W_{ax}^G$), we have $d_H(y,x) = d_G(y,x)$. It follows that

$$d_H(y,a) \geq d_G(y,a) > d_G(y,x) = d_H(y,x)\,,$$

and hence $y \in W_{xa}^H$. Therefore $W_{xa}^G \subseteq W_{xa}^H$. Consider now the vertex b and note that $b \in W_{ax}^G \cup {}_aW_x^G$. Since x lies on a shortest a,b-path, we have $b \in W_{xa}^H$ and hence $|W_{xa}^H| \geq |W_{xa}^G| + 1$. We now have

$$|W_{xa}^H| \geq |W_{xa}^G| + 1 = |W_{ax}^G| + 1 \geq |W_{ax}^H| + 1\,,$$

where the last inequality holds since a shortest u,x-path in G that traverses the edge e contains a shortest u,a-path (of shorter length). We conclude that H is not distance-balanced. □

For a graph G and an edge e of G let G_e be the graph obtained from G by subdividing e. An arbitrary cycle shows that both G and G_e can be distance-balanced. However, this cannot happen if G is 3-connected as our next result asserts.

Theorem 3.2 *Let G be a 3-connected, distance-balanced graph and let e be any edge of G. Then G_e is not distance-balanced.*

Proof. Let $e = ab$, $H = G_e$, and let x be the vertex added in the subdivision of e. Suppose first that $|{}_aW_b^G| > 1$. Let $y \in V(H)$, $y \neq x$. Since $N(x) = \{a,b\}$, we can reach x only from a or b and hence $d_H(y,x) = \min\{d_G(y,a), d_G(y,b)\} + 1$. Consider now the edge ax of H. For $y \in W_{ab}^G \cup {}_aW_b^G$ we have $d_H(y,a) = d_G(y,a)$ (since e does not lie on any shortest y,a-path in G) and $d_H(y,x) = d_G(y,a) + 1$. It follows that $y \in W_{ax}^H$. This in particular implies that W_{xa}^H can (besides x) contain only the vertices from W_{ba}^G and hence $|W_{xa}^H| \leq |W_{ba}^G| + 1 = |W_{ab}^G| + 1 < |W_{ab}^G \cup {}_aW_b^G| \leq |W_{ax}^H|$. We conclude that H is not distance-balanced if $|{}_aW_b^G| > 1$.

Assume next that $|{}_aW_b^G| = 1$. Let ${}_aW_b^G = \{y\}$. Since $\{a,y\}$ does not separate W_{ab}^G and W_{ba}^G in G, there exists an edge $uv \neq ab$ in G with $u \in W_{ab}^G$ and $v \in W_{ba}^G$. By the definition of W_{ab}^G, $u \neq a$ and similarly $v \neq b$. Then $d_G(a,u) = d_G(b,v)$ by [8, Lemma 2.2] and it follows that $v \in {}_aW_x^H$. Since $y \in W_{ax}^H$ and $W_{ab}^G \subseteq W_{ax}^H$, this implies that $|W_{xa}^H| \le |W_{ba}^G| < |W_{ax}^H|$, and hence H is not distance-balanced in this case.

Finally, assume that ${}_aW_b^G = \emptyset$. Since G is 2-connected, there is $uv \in E(G)$ such that $uv \neq ab$, $u \in W_{ab}^G$ and $v \in W_{ba}^G$. As above we note that $u \neq a$ and $v \neq b$. For each such edge uv we obtain as above that $v \in {}_aW_x^H$. The fact that G is 3-connected implies that there must be at least one additional vertex like v. Since $W_{ab}^G \subseteq W_{ax}^H$, it follows that H is not distance-balanced. □

We observe in the above that G need not be 3-connected if it has an edge ab with $|{}_aW_b| > 1$. The connectivity assumption is also not needed in case G is bipartite.

Proposition 3.3 *Let G be a bipartite, distance-balanced graph that has at least two edges and let $e \in E(G)$. Then G_e is distance-balanced if and only if G is a cycle.*

Proof. Let $e = ab$. Let H and x be as in the proof of Theorem 3.2. Consider the edge ax of H. Then $W_{ab}^G \subseteq W_{ax}^H$. Let uv be an edge where $u \in W_{ab}^G$ and $v \in W_{ba}^G$. Then $d_G(a,u) = d_G(b,v)$ by [8, Lemma 2.2] and hence $d_H(a,v) = d_H(x,v) = d_G(a,u)+1$. Since $|H| = 2|W_{ab}^G|+1$ and $v \in {}_aW_x^H$ it follows that $|W_{ax}^H| = |W_{xa}^H| = |W_{ab}^G|$. This in particular implies that the set of edges between W_{ab}^G and W_{ba}^G contains exactly two edges. By [8, Lemma 2.3.(ii)] this is only possible if G is a cycle. □

4 Distance-balanced product graphs

In this section we study the conditions under which the standard graph products produce a distance-balanced graph. All of the graph products constructed from two graphs G and H have vertex set $V(G) \times V(H)$. Let (a,u) and (b,v) be two vertices in $V(G) \times V(H)$. They are adjacent in the *Cartesian product* $G \,\Box\, H$ if they are equal in one coordinate and adjacent in the other and are adjacent in the *direct product* $G \times H$ if they are adjacent in both coordinates. The edge set of the *strong product* $G \boxtimes H$ is $E(G \,\Box\, H) \cup E(G \times H)$. These vertices are adjacent in the *lexicographic product* $G \circ H$ if $ab \in E(G)$ or if $a = b$ and $uv \in E(H)$. See [9] for a more complete treatment of graph products.

Proposition 4.1 *Let G and H be connected graphs. Then $G \,\Box\, H$ is distance-balanced if and only if both G and H are distance-balanced.*

Proof. Set $X = G \,\Box\, H$. Assume X is distance-balanced and let e be an edge of X. We may assume without loss of generality that $e \in G^u$, so that $e = (a,u)(b,u)$ for $ab \in E(G)$. Then the sets $W_{ab} \times V(H)$, $W_{ba} \times V(H)$, and ${}_aW_b \times V(H)$ form a partition of $V(X)$. For $(x,y) \in W_{ab} \times V(H)$ we have

$$d_X((x,y),(a,u)) = d_G(x,a) + d_H(y,u) < d_G(x,b) + d_H(y,u) = d_X((x,y),(b,u))\,.$$

Hence $(x,y) \in W_{(a,u)(b,u)}$. For $(x,y) \in W_{ba} \times V(H)$ (resp. $(x,y) \in {}_aW_b \times V(H)$) we similarly get $(x,y) \in W_{(b,u)(a,u)}$ (resp. $(x,y) \in {}_{(a,u)}W_{(b,u)}$). It follows that $W_{(a,u)(b,u)} = W_{ab} \times V(H)$ and $W_{(b,u)(a,u)} = W_{ba} \times V(H)$. Since $G \square H$ is distance-balanced, we have $|W_{(a,u)(b,u)}| = |W_{(b,u)(a,u)}|$ and hence $|W_{ab}| = |W_{ba}|$. We conclude that G is distance-balanced.

Conversely, let G be distance-balanced and $ab \in E(G)$. Then $|W_{ab} \times V(H)| = |W_{ba} \times V(H)|$ and hence $|W_{(a,u)(b,u)}| = |W_{(b,u)(a,u)}|$. The same argument applies for edges from the fibers ${}^aH$, and so $G \square H$ is distance-balanced. $\square$

Theorem 4.2 *Let G and H be connected graphs. Then $G \circ H$ is distance-balanced if and only if G is distance-balanced and H is regular.*

Proof. Assume that G is distance-balanced and H is regular. Let $e \in E(G \circ H)$. Consider first the case where $e = (a,x)(a,y)$. Then $xy \in E(H)$. Let (u,v) be a vertex not incident with e. Note that for $u \neq a$, $d_{G \circ H}((a,x),(u,v)) = d_G(a,u) = d_{G \circ H}((a,y),(u,v))$. That is, every vertex of ${}^uH$ belongs to ${}_{(a,x)}W_{(a,y)}$. On the other hand, for $u = a$ the distance in $G \circ H$ between (a,x) and (a,v) is two unless $xv \in E(H)$ in which case this distance is one. Similarly, $d_{G \circ H}((a,y),(a,v)) = 1$ if $yv \in E(H)$, and $d_{G \circ H}((a,y),(a,v)) = 2$ if $yv \notin E(H)$. It follows that in this case $W_{(a,x)(a,y)} = \{(a,v) \mid xv \in E(H) \text{ and } yv \notin E(H)\}$, and $W_{(a,y)(a,x)} = \{(a,v) \mid yv \in E(H) \text{ and } xv \notin E(H)\}$. Since H is regular we see that $|W_{(a,x)(a,y)}| = |W_{(a,y)(a,x)}|$. Now assume that $e = (a,x)(b,y)$ where $a \neq b$. Then $ab \in E(G)$. It follows from the edge structure of $G \circ H$ that $(u,v) \in W_{(a,x)(b,y)}$ if and only if either $u \notin \{a,b\}$ and $d_G(a,u) < d_G(b,u)$, or $u = b$ and $d_H(v,y) \geq 2$. In a similar way we see that $(u,v) \in W_{(b,y)(a,x)}$ if and only if either $u \notin \{a,b\}$ and $d_G(b,u) < d_G(a,u)$, or $u = a$ and $d_H(v,x) \geq 2$. Since G is distance-balanced $|W_{ab}| = |W_{ba}|$ and because H is regular it follows that $|\{v \mid d_H(v,y) \geq 2\}| = |\{v \mid d_H(v,x) \geq 2\}|$. Therefore, $|W_{(a,x)(b,y)}| = |W_{(b,y)(a,x)}|$ in this case as well, and thus $G \circ H$ is distance-balanced.

Conversely, assume that $G \circ H$ is distance-balanced. Let ab be an arbitrary edge of G. By following the above argument for an edge of the form $(a,x)(a,y)$ we see that $|W_{(a,x)(a,y)}| = |W_{(a,y)(a,x)}|$ implies that any two adjacent vertices of H have the same degree. Since H is connected this implies that H is r-regular for some r. For an edge $e = (a,x)(b,y)$ where $a \neq b$ it follows again from earlier analysis that

$$\begin{aligned} |W_{(a,x)(b,y)}| &= |\{(u,v) \mid u \notin \{a,b\}\ , d_G(a,u) < d_G(b,u)\}| \\ &\quad + |\{(b,v) \mid d_H(v,y) \geq 2\}| \\ &= (|W_{ab}| - 1) \cdot |V(H)| + (|V(H)| - (r+1)) \end{aligned}$$

and that

$$\begin{aligned} |W_{(b,y)(a,x)}| &= |\{(u,v) \mid u \notin \{a,b\}\ , d_G(b,u) < d_G(a,u)\}| \\ &\quad + |\{(a,v) \mid d_H(v,x) \geq 2\}| \\ &= (|W_{ba}| - 1) \cdot |V(H)| + (|V(H)| - (r+1)). \end{aligned}$$

Now, since $G \circ H$ is distance-balanced the two above equations imply that $|W_{ab}| = |W_{ba}|$ and hence G is distance-balanced. $\square$

The other two standard graph products, the direct and the strong ones, do not preserve the property of being distance-balanced. Let H be the graph from Fig. 1 that was obtained by Handa in [8]. H is an example of a distance-balanced partial cube; its embedding into the 5-cube Q_5 is also shown on the figure.

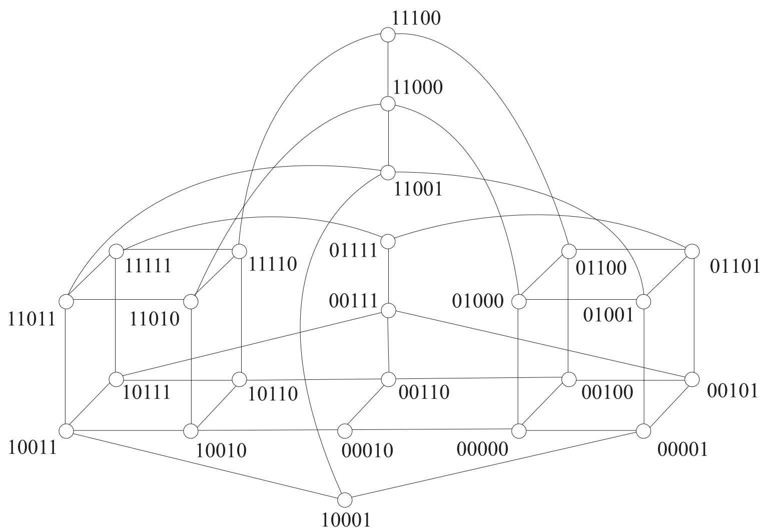

Figure 1: Non-regular bipartite distance-balanced graph H

For the strong product, consider the product $H \boxtimes C_4$. Let w_1, w_2, w_3, w_4 be the vertices of C_4 with natural adjacencies and let $u = 00010$, $v = 00110$ be vertices of H. Consider the edge $(u, w_2)(v, w_3)$ in $H \boxtimes C_4$. Then $|(W_{uv} \times \{w_4\}) \cap W_{(u,w_2)(v,w_3)}| = 9$, while $|(W_{vu} \times \{w_1\}) \cap W_{(v,w_3)(u,w_2)}| = 8$. Since $W_{(u,w_2)(v,w_3)}$ and $W_{(v,w_3)(u,w_2)}$ are balanced elsewhere, we conclude that $H \boxtimes C_4$ is not distance-balanced.

For the direct product consider $H \times H$ and let $x = 00000$ and $u = 00010$ be vertices of H. A tedious but straightforward computation shows that in the connected component of $H \times H$ containing the edge $(x, x)(u, u)$ we have $|W_{(x,x)(u,u)}| = 128$ but $|W_{(u,u)(x,x)}| = 160$, so $H \times H$ is not distance-balanced.

5 Concluding remarks

Note that in a k-regular, triangle free graph G of diameter two, we have $|W_{ab}| = k$ for all $ab \in E(G)$. However, it is possible for G to be regular of diameter two such that there are edges uv and ab in G such that $|W_{uv}|$ and $|W_{ab}|$ are not the same. For example, "expand" one of the vertices of a $K_{3,3}$ to a triangle and the resulting graph F (keep it 3-regular) is distance-balanced, but the range of values of $|W_{uv}|$

considering all edges uv is $\{2,3\}$. One can ask - must this range be an interval of integers for a distance-balanced graph? How many different values could be in this range for a distance-balanced graph?

By using Proposition 4.1 the answer to the first question above is no, and the answer to the second one is that graphs can be constructed having arbitrarily many different positive integers in the range. For example, the Cartesian product of the graph F (the one from modifying $K_{3,3}$) and the 5-cycle has values 10, 15 and 16. The Cartesian product of this graph and the 7-cycle has values 70, 105, 112, and 120. This can be continued.

Let G be a graph and let $b(G)$ be the smallest number of edges that can be added to G such that the obtained graph is distance-balanced. Since complete graphs are distance-balanced, $b(G)$ is a well-defined graph invariant. In general it seems that the computation of $b(G)$ is quite hard, but it might be interesting to obtain it in some special cases.

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