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STERN POLYNOMIALS

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Stern polynomials*

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Abstract

Stern polynomials $B_k(t)$, $k \geq 0$, $t \in \mathbb{R}$, are introduced in the following way: $B_0(t) = 0$, $B_1(t) = 1$, $B_{2n}(t) = tB_n(t)$, and $B_{2n+1}(t) = B_{n+1}(t) + B_n(t)$. It is shown that $B_n(t)$ has a simple explicit representation in terms of the hyperbinary representations of $n-1$ and that $B'_{2n-1}(0)$ equals the number of 1's in the standard Gray code for $n-1$. It is also proved that the degree of $B_n(t)$ equals the difference between the length and the weight of the non-adjacent form of n.

Key words: Stern (diatomic) sequence, Stern polynomials, hyperbinary representation, standard Gray code, non-adjacent form

1 Introduction

Stern sequence [17] or, as it is often called, *Stern diatomic series* $b(n)$ is defined recursively by

$$
\begin{aligned}
&b(0) = 0,\ b(1) = 1,\\
&b(2n) = b(n),\ n \geq 1,\\
&b(2n+1) = b(n) + b(n+1),\ n \geq 1.
\end{aligned}
$$

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The sequence thus starts as

$$0,\ 1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5,\ 2,\ 5,\ 3,\ 4,\ 1,\ 5,\ 4,\ 7,\ 3,\ \ldots$$

and can, for instance, be obtained as a one-dimensional extract of the so-called Stern-Brocot array. This sequence is `A002487` in Sloane's online database of integer sequences [16].

Stern sequence has been studied in several different fields of mathematics, as a sample of references we propose [9, 11, 12, 15] and a comprehensive survey [18]. The sequence also appears in a very general theory of k-regular sequences due to Allouche and Shallit [1, 2].

A nice application of the Stern sequence is given in [4], where Calkin and Wilf prove that the sequence defined by the quotients $b(n)/b(n+1)$, $n \geq 1$, encounters every positive rational exactly once. The Calkin and Wilf encoding of positive rationals hence begins as:

$$\frac{1}{1},\ \frac{1}{2},\ \frac{2}{1},\ \frac{1}{3},\ \frac{3}{2},\ \frac{2}{3},\ \frac{3}{1},\ \frac{1}{4},\ \frac{4}{3},\ \frac{3}{5},\ \frac{5}{2},\ \frac{2}{5},\ \frac{5}{3},\ \frac{3}{4},\ \frac{4}{1},\ \frac{1}{5},\ \frac{5}{4},\ \frac{4}{7},\ \frac{7}{3},\ \ldots$$

Motivated by the definition of the Stern sequence and the above application we now introduce *Stern polynomials* $B_k(t)$, $k \geq 0$, $t \in \mathbb{R}$, in the following way.

$$\begin{aligned} &B_0(t) = 0,\ B_1(t) = 1, \\ &B_{2n}(t) = tB_n(t),\ n \geq 1, \\ &B_{2n+1}(t) = B_{n+1}(t) + B_n(t),\ n \geq 1. \end{aligned}$$

The first few of them are: $B_0(t) = 0$, $B_1(t) = 1$, $B_2(t) = t$, $B_3(t) = t + 1$, $B_4(t) = t^2$, $B_5(t) = 2t + 1$, $B_6(t) = t^2 + t$, $B_7(t) = t^2 + t + 1$, and $B_8(t) = t^3$, see also Table 1. Note that

$$B_n(1) = b(n),\ n \geq 0\,. \tag{1}$$

It is also interesting to observe that the sequence of natural numbers can be encoded as $B_n(2) = n$.

Several well-known sequences of polynomials are defined in a way similar to the one in which we define the Stern polynomials. For instance, the Fibonacci polynomials $F_n(t)$ are introduced by $F_0(t) = 0$, $F_1(t) = 1$, and $F_n(t) = tF_{n-1}(t) + F_{n-2}(t)$, [19, 20]; see also [7, 22] for recent results on these polynomials. Another such class is formed by the Lucas polynomials $L_n(t)$ defined with $L_0(t) = 2$, $L_1(t) = t$, and $L_n(t) = tL_{n-1}(t) + L_{n-2}(t)$, see [19, 21]. Analogously to (1), the Fibonacci numbers and the Lucas numbers are obtained as $F_n(1)$ and $L_n(1)$, respectively. It is interesting to add that Lucas and Fibonacci polynomials have several applications, even in mathematical physics [5].

The main purpose of this paper is to introduce Stern polynomials and to demonstrate that they have many appealing properties. We begin by showing that the polynomial $B_n(t)$ has a simple explicit representation in terms of the hyperbinary representations of n. More precisely,

$$B_n(t) = \sum_{\ell \geq 0} \left| \begin{matrix} n-1 \\ \ell \end{matrix} \right| t^{\ell},$$

where we use the symbol $\left| \begin{smallmatrix} m \\ k \end{smallmatrix} \right|$ to denote the number of hyperbinary representations of m containing exactly k digits 1. Then we prove that $B'_{2n-1}(0)$, $n \geq 1$, equals the number of 1's in the standard Gray code for $n-1$. We conclude the paper by proving that the degree of $B_n(t)$, $\deg(B_n(t))$, equals the difference between the length and the weight of the non-adjacent form of n.

2 Explicit representation of Stern polynomials

A *hyperbinary representation* of a non-negative integer n is a representation of n as a sum of powers of 2, each power being used at most twice. We will employ the notation $(a_1 a_2 \ldots a_m)_{[2]}$ to describe the hyperbinary representation $\sum_{i=1}^{m} a_i 2^{m-i}$, $a_i \in \{0, 1, 2\}$. Let $\mathcal{H}(n)$ denote the set of all hyperbinary representations $(a_1 a_2 \ldots a_m)_{[2]}$ of n, where any two representations of the same integer differing only in zeros on the left-hand side are identified. For instance, $(1)_{[2]}$ is the same representation of 1 as $(01)_{[2]}$. It is well-known, see [4, 15], that $b(n)$ counts the number of hyperbinary representations of $n-1$.

Theorem 1. *For any $n \in \mathbb{N}$, $b(n) = |\mathcal{H}(n-1)|$.*

The proof idea for Theorem 1 is that the recursive formulas are established by noting that when $n - 1 = (a_1 a_2 \ldots a_m)_{[2]}$ is odd, then a_m must be 1, and if $n-1$ is even, a_m may be 0 or 2, but not 1. This theorem can be extended to the Stern polynomials in the following way.

Theorem 2. *For any $n \in \mathbb{N}$,*

$$B_n(t) = \sum_{(a_1 a_2 \ldots a_m)_{[2]} \in \mathcal{H}(n-1)} t^{|\{i \,|\, a_i = 1\}|}.$$

Proof. The assertion is easily verified to be true for small n. Let n be even, say $2k$, and consider an arbitrary hyperbinary representation $n - 1 = (a_1 a_2 \ldots a_m)_{[2]}$. Since $n-1$ is odd, $a_m = 1$ by the observation before the theorem. As $k - 1 = (a_1 a_2 \ldots a_{m-1})_{[2]}$, the polynomial that counts the number of 1's in the representation $(a_1 a_2 \ldots a_m)_{[2]}$ is obtained from the polynomial for $(a_1 a_2 \ldots a_{m-1})_{[2]}$ by multiplication by t. As $B_{2k}(t) = tB_k(t)$, the theorem holds for even n by induction.

Suppose next that n is odd, say $2k+1$. Then $n - 1 = (a_1 a_2 \ldots a_m)_{[2]}$ is even and a_m must be 0 or 2. Hence no multiplication by t based on counting the number of 1's

n	0	1	2	3	4	5	6	7	8	9
$\mathcal{H}(n-1)$		$(0)_{[2]}$	$(1)_{[2]}$	$(10)_{[2]}$ $(2)_{[2]}$	$(11)_{[2]}$	$(100)_{[2]}$ $(12)_{[2]}$ $(20)_{[2]}$	$(101)_{[2]}$ $(21)_{[2]}$	$(110)_{[2]}$ $(102)_{[2]}$ $(22)_{[2]}$	$(111)_{[2]}$	$(1000)_{[2]}$ $(112)_{[2]}$ $(120)_{[2]}$ $(200)_{[2]}$
$B_n(t)$	0	1	t	$t+1$	t^2	$2t+1$	t^2+t	t^2+t+1	t^3	t^2+2t+1

n	10	11	12	13	14	15	16
$\mathcal{H}(n-1)$	$(1001)_{[2]}$ $(121)_{[2]}$ $(201)_{[2]}$	$(1010)_{[2]}$ $(1002)_{[2]}$ $(122)_{[2]}$ $(210)_{[2]}$ $(202)_{[2]}$	$(1011)_{[2]}$ $(211)_{[2]}$	$(1100)_{[2]}$ $(1012)_{[2]}$ $(1020)_{[2]}$ $(212)_{[2]}$ $(220)_{[2]}$	$(1101)_{[2]}$ $(1021)_{[2]}$ $(221)_{[2]}$	$(1110)_{[2]}$ $(1102)_{[2]}$ $(1022)_{[2]}$ $(222)_{[2]}$	$(1111)_{[2]}$
$B_n(t)$	$2t^2+t$	t^2+3t+1	t^3+t^2	$2t^2+2t+1$	t^3+t^2+t	t^3+t^2+t+1	t^4

Table 1: Polynomials $B_n(t)$ obtained from hyperbinary representations of $n-1$

in $(a_1a_2\dots a_{m-1})_{[2]}$ appears. Now, if $a_m = 0$ then $(a_1a_2\dots a_{m-1})_{[2]}$ is a hyperbinary representation of k, and if $a_m = 2$, then $(a_1a_2\dots a_{m-1})_{[2]}$ is a hyperbinary representation of $k-1$. Applying the recursive formula $B_{2k+1}(t) = B_{k+1}(t)+B_k(t)$ the argument is complete, again by induction. □

Theorem 2 is illustrated in Table 1 for $n \le 16$.

Recall that by the symbol $\left|{m \atop k}\right|$ we denote the number of hyperbinary representations of m containing exactly k digits 1. Then Theorem 2 may be rewritten as

Corollary 3.

$$B_n(t) = \sum_{\ell\ge 0} \left|{n-1 \atop \ell}\right| t^\ell . \qquad \square$$

We close this section by the following property of the Stern polynomials.

Proposition 4. *For any $m \ge 0$ and any $k \ge 1$,*

$$B_{2^{k-1}(2m+1)}(t) = \frac{1}{t}\Big(B_{2^k m}(t) + B_{2^k(m+1)}(t)\Big) .$$

Proof. Compute as follows:

$$\begin{aligned}\frac{1}{t}\Big(B_{2^k m}(t) + B_{2^k(m+1)}(t)\Big) &= \frac{1}{t}\Big(t^k B_m(t) + t^k B_{m+1}(t)\Big)\\ &= t^{k-1}\Big(B_m(t) + B_{m+1}(t)\Big)\\ &= t^{k-1}B_{2m+1}(t)\\ &= B_{2^{k-1}(2m+1)}(t) .\end{aligned}$$

□

3 Stern polynomials and standard Gray code

The *standard Gray code* of n is defined as the binary representation of $g(n)$, where $g : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by

$$g(0) = 0, \quad g(2^p + j) = 2^p + g(2^p - 1 - j) \quad \text{for} \quad 0 \leq j < 2^p \,, \tag{2}$$

see [6]. This looks like a complicated definition but the construction of the standard Gray code is simple. The first two words of the code are 0 and 1. Suppose that for some $k \geq 1$, the first 2^k words are already known, where every word is written using k digits. Then the next 2^k words of the code are obtained by attaching 1 on the left of each of the first 2^k words in the reverse order. See Table 2 where this construction is indicated for $k = 4$. For some algorithmic aspects on the standard Gray code we refer to [10].

n	$g(n)$	$g(n)_{(2)}$	
0	0	0 0 [0 0 0]	0=x(1)
1	1	0 0 [0 0 1]	1=x(2)
2	3	0 0 [0 1 1]	2=x(3)
3	2	0 0 [0 1 0]	1=x(4)
4	6	0 0 [1 1 0]	2=x(5)
5	7	0 0 [1 1 1]	3=x(6)
6	5	0 0 [1 0 1]	2=x(7)
7	4	0 0 [1 0 0]	1=x(8)
8	12	0 [1] 1 0 0	2=x(9)
9	13	0 [1] 1 0 1	3=x(10)
10	15	0 [1] 1 1 1	4=x(11)
11	14	0 [1] 1 1 0	3=x(12)
12	10	0 [1] 0 1 0	2=x(13)
13	11	0 [1] 0 1 1	3=x(14)
14	9	0 [1] 0 0 1	2=x(15)
15	8	0 [1] 0 0 0	1=x(16)
16	24	1 1 0 0 0	2=x(17)
17	25	1 1 0 0 1	3=x(18)

Table 2: Standard Gray code $g(n)_{(2)}$ of n and the number of 1's in it

In this section we show that certain coefficients of the Stern polynomials are closely related to this Gray code. More precisely, let $x(n)$ be the coefficient at t^1 of the polynomial $B_{2n-1}(t)$, that is,

$$x(n) = B'_{2n-1}(0). \tag{3}$$

The first few values of this sequence are shown in Table 2.

To establish the connection between the sequence $x(n)$ and the standard Gray code, we need the following auxiliary sequence. For $n \geq 0$ let $y(n)$ be the coefficient at t^1 of the polynomial $B_{2n}(t)$, that is,

$$y(n) = B'_{2n}(0). \tag{4}$$

Lemma 5. *For any $n \geq 0$, $y(n) = 0$, if n is even, and $y(n) = 1$, if n is odd.*

Proof. It is easily seen that the lemma holds for small n. Then, using $B'_{2n}(t) = tB'_n(t) + B_n(t)$, we get $y(2k) = B'_{4k}(0) = B_{2k}(0) = 0 \cdot B_k(0) = 0$ and $y(2k+1) = B'_{4k+2}(0) = B_{2k+1}(0) = B_k(0) + B_{k+1}(0) = 1$. Here $\{B_k(0), B_{k+1}(0)\} = \{0, 1\}$ holds by the induction assumption. □

Lemma 6. *For any $\ell \geq 0$,*

$$\begin{aligned} x(4\ell) &= x(2\ell), \\ x(4\ell+1) &= x(2\ell+1), \\ x(4\ell+2) &= x(2\ell+1)+1, \\ x(4\ell+3) &= x(2\ell+2)+1. \end{aligned}$$

Proof. Applying Lemma 5 and having in mind that $B'_{2n+1}(t) = B'_n(t) + B'_{n+1}(t)$ we compute as follows:

$$\begin{aligned} x(4\ell) &= B'_{8\ell-1}(0) = B'_{4\ell-1}(0) + B'_{4\ell}(0) = x(2\ell) + y(2\ell) = x(2\ell), \\ x(4\ell+1) &= B'_{8\ell+1}(0) = B'_{4\ell}(0) + B'_{4\ell+1}(0) = y(2\ell) + x(2\ell+1) = x(2\ell+1), \\ x(4\ell+2) &= B'_{8\ell+3}(0) = B'_{4\ell+1}(0) + B'_{4\ell+2}(0) = x(2\ell+1) + y(2\ell+1) \\ &= x(2\ell+1)+1, \qquad \text{and} \\ x(4\ell+3) &= B'_{8\ell+5}(0) = B'_{4\ell+2}(0) + B'_{4\ell+3}(0) = y(2\ell+1) + x(2\ell+2) \\ &= x(2\ell+2)+1. \end{aligned}$$

□

We now prove that the sequence $x(n)$ satisfies the following recursive formula:

Lemma 7. $x(1) = 0,\ x(2^k+i) = x(2^k-i+1)+1,$ *for* $k \geq 0$ *and* $0 < i \leq 2^k$.

Proof. For $k = 0, 1, 2$ we easily check that $x(2) = x(1)+1$, $x(3) = x(2)+1$, $x(4) = x(1)+1$, $x(5) = x(4)+1$, $x(6) = x(3)+1$, $x(7) = x(2)+1$, $x(8) = x(1)+1$. Suppose

$k \geq 2$. Then, using Lemma 6, we inductively get (for appropriate values of ℓ):

$$\begin{aligned}
x(2^k + 4\ell) &= x(2^{k-1} + 2\ell) = x(2^{k-1} - 2\ell + 1) + 1 = x(2^k - 4\ell + 1) + 1, \\
x(2^k + 4\ell + 1) &= x(2^{k-1} + 2\ell + 1) = x(2^{k-1} - 2\ell) + 1 = x(2^k - 4\ell) + 1, \\
x(2^k + 4\ell + 2) &= x(2^{k-1} + 2\ell + 1) + 1 = x(2^{k-1} - 2\ell) + 2 \\
&= x(2^k - 4\ell - 1) + 1, \\
x(2^k + 4\ell + 3) &= x(2^{k-1} + 2\ell + 2) + 1 = x(2^{k-1} - 2\ell - 1) + 2 \\
&= x(2^k - 4\ell - 2) + 1.
\end{aligned}$$

□

Theorem 8. *For any $n \geq 1$, $x(n)$ equals the number of 1's in the standard Gray code for $n - 1$.*

Proof. Compare Lemma 7 with (2) and keep in mind that 2^p adds an additional digit 1 to the binary representation of the second summand of (2). (Beware of the 1-shift!) □

Corollary 9. *The number of 1's in the standard Gray code for n is the same as the number of hyperbinary representations of $2n$ containing exactly one digit 1.*

Proof. By Theorem 8, the number of 1's in the standard Gray code for n equals $x(n+1)$, which by definition equals to the coefficient at t^1 of the polynomial $B_{2n+1}(t)$, i.e. to $\left|{2n \atop 1}\right|$, by Corollary 3. By definition the symbol equals to the number of hyperbinary representations of $2n$ containing exactly one digit 1. □

The sequence $x(n)$ is `A005811` from [16]. It appears under the name of Kuczma's sequence in [2], where it is proved that $x(n)$ is 2-regular.

4 Stern polynomials and the NAF

A *signed bit representation* of a positive integer is a base 2 representation of the integer in which digits -1, 0, and 1 are allowed. A signed bit representation $n = \sum s_i 2^i = s_m \dots s_0$ is called *non-adjacent form*, NAF for short, if $s_m \neq 0$ and if $s_i \neq 0$ implies $s_{i-1} = 0$ for $i \geq 1$. It is well-known that every positive integer has a unique NAF, see [14]. The second column of Table 3 gives the NAFs for positive integers up to 17, where $\bar{1}$ stands for -1.

NAF proved to be very useful in computer science, especially for fast computations and in coding theory, see [3, 8, 13]. This is related to the following well-known remarkable fact. Among all signed bit representations of an integer n, the NAF minimizes the weight, that is, the number of non-zero digits of a representation. The weight of the NAF of n is denoted by $w(n)$. Let in addition the length $\ell(n)$ of the NAF of n be the number of digits in the NAF, that is, if $s_m \dots s_0$ is the NAF of n, then $\ell(n) = m + 1$. Here is the main result of this section.

n		$z(n)$
1	1	0
2	10	1
3	$10\bar{1}$	1
4	100	2
5	101	1
6	$10\bar{1}0$	2
7	$100\bar{1}$	2
8	1000	3
9	1001	2
10	1010	2
11	$10\bar{1}0\bar{1}$	2
12	$10\bar{1}00$	3
13	$10\bar{1}01$	2
14	$100\bar{1}0$	3
15	$1000\bar{1}$	3
16	10000	4
17	10001	3

Table 3: The NAFs of n and $z(n)$—the number of 0's in it

Theorem 10. *For any $n \geq 1$,*

$$w(n) = \ell(n) - \deg(B_n(t)) .$$

In the rest of the paper we prove Theorem 10. To shorten the notation set $z(n) = \deg(B_n(t))$.

For the proof it suffices to show that $z(n)$ equals the number of zero digits in the NAF of n. From the definition of $z(n)$ it follows that $z(0)$ is not defined and that $z(1) = 0$, $z(2n) = z(n) + 1$, $z(2n+1) = \max\{z(n+1), z(n)\}$, for $n \geq 1$. The first few values of this sequence are presented in Table 3.

Lemma 11. *For any $n > 1$, $z(n)+1 \geq \max\{z(n-1), z(n+1)\}$. Also, $z(1)+1 \geq z(2)$.*

Proof. The claim obviously holds for small n. For the induction step we have

$$\begin{aligned} z(2n+1)+1 &= \max\{z(n+1), z(n)\} + 1 \\ &= \max\{z(n+1)+1, z(n)+1\} \\ &= \max\{z(2n), z(2n+2)\} . \end{aligned}$$

For the even case we proceed as follows. From $z(n+1)+1 \geq \max\{z(n), z(n+2)\}$ it follows that

$$\begin{aligned} z(n+1)+1 &\geq \max\{z(n), z(n+1), z(n+2)\} \\ &= \max\{\max\{z(n), z(n+1)\}, \max\{z(n+1), z(n+2)\}\}, \end{aligned}$$

and thus $z(2n+2) \geq \max\{z(2n+1), z(2n+3)\}$. It follows that

$$z(2n+2)+1 > \max\{z(2n+1), z(2n+3)\}\,. \qquad \square$$

Proposition 12. *The sequence $z(n)$, $n \geq 1$, is defined recursively as follows: $z(1) = 0$, $z(2n) = z(n)+1$, $z(4n-1) = z(n)+1$, $z(4n+1) = z(n)+1$, for $n \geq 1$.*

Proof. Using Lemma 11, we get

$$\begin{aligned} z(4n-1) &= \max\{z(2n-1), z(2n)\} \\ &= \max\{\max\{z(n-1), z(n)\}, z(n)+1\} \\ &= z(n)+1\,, \end{aligned}$$

as well as

$$\begin{aligned} z(4n+1) &= \max\{z(2n), z(2n+1)\} \\ &= \max\{z(n)+1, \max\{z(n), z(n+1)\},\} \\ &= z(n)+1\,. \end{aligned} \qquad \square$$

Proposition 12 and the definition of the sequence $z(n)$ immediately imply the following recursive form.

Corollary 13. *The sequence $z(n)$, $n \geq 1$, is defined recursively as follows: $z(1) = 0$, $z(2n) = z(n)+1$, $z(4n+1) = z(2n)$, $z(4n+3) = z(2n+2)$, for $n \geq 1$.* $\square$

From Corollary 13 it follows that $z(m)$, $m \geq 1$, counts the number of zero digits in the NAF of m. In other words, if $m = \sum s_i 2^i$ is the NAF of m, then the digit s_0 is 0 if and only if m is an even number, in which case the remaining digits coincide with the digits of $m/2$. (This corresponds to $z(2n) = z(n)+1$.) In addition, the digit s_0 of $m = 4n+1$ is 1, and the remaining digits coincide with the digits of $2n$. (This corresponds to $z(4n+1) = z(2n)$.) Finally, the digit s_0 of $m = 4n+3$ is -1, and the remaining digits coincide with the digits of $2n+2$. (This corresponds to $z(4n+3) = z(2n+2)$.) The proof of Theorem 10 is complete.

The sequence $z(n)$, $n \geq 1$, is the sequence `A057526` from [16].

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