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ON DOMINATION NUMBERS OF GRAPH BUNDLES

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On domination numbers of graph bundles

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Abstract

Let $\gamma(G)$ be the domination number of a graph G. It is shown that for any $k \geq 0$ there exists a Cartesian graph bundle $B \Box_\varphi F$ such that $\gamma(B \Box_\varphi F) = \gamma(B)\gamma(F) - 2k$. The domination numbers of Cartesian bundles of two cycles are determined exactly when the fibre graph is a triangle or a square. A statement similar to Vizing's conjecture on strong graph bundles is shown not to be true by proving the inequality $\gamma(B \boxtimes_\varphi F) \leq \gamma(B)\gamma(F)$ for strong graph bundles. Examples of graphs B and F with $\gamma(B \boxtimes_\varphi F) < \gamma(B)\gamma(F)$ are given.

Key words: graph bundle, dominating set, domination number, Cartesian product
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1 Introduction

In topology, bundles are objects which generalize both covering spaces and Cartesian products [9]. Analogously, graph bundles generalize the notion of covering graphs and graph products. Graph bundles can be defined with respect to arbitrary graph products [25]. (For a classification of all possible associative graph products, see [10] or [12].) Various problems on graph bundles were studied recently, including edge coloring [24], maximum genus [22], isomorphism classes [19], characteristic polynomials [20,26], chromatic numbers [15,16], and recognition problems [11,23,28–31,33].

A conjecture proposed by Vizing [27] has been a challenge for several authors (see, for example [1,17,4,12] and further references there). So far only partial

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solutions are known, which show that

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H)$$

holds for graphs G, H, which belong to certain classes of graphs. The best known general result is due to [4]

$$2\gamma(G\Box H) \geq \gamma(G)\gamma(H)$$

Because of the difficulty of the Vizing's conjecture, an interesting related problem is to determine the domination numbers of particular Cartesian products [13,14]. It is shown in [5] that even for subgraphs of $P_m \Box P_n$ this problem is NP-complete. Domination numbers of the Cartesian products of paths and cycles have been studied in a series of papers [2,6,8,13,14,17]. For complete grid graphs, i.e. graphs $P_k \Box P_n$, algorithms were given in [8] which for a fixed k compute $\gamma(P_k \Box P_n)$ in $O(n)$ time. An $O(\log n)$ algorithm was proposed in [18]. In fact, the domination number problem for $k \times n$ grids, where k is fixed, has a constant time solution [21,32]. It may be interesting to note that in [7] formulas are given for families $\{P_k \Box P_n \mid n \in N\}$ for k up to 19. For $k \geq 20$, it is conjectured [3] that

$$\gamma(P_k \Box P_n) = \left\lceil \frac{(k+2)(n+2)}{5} \right\rceil - 4$$

and the problem is still open. In [17] the domination numbers are determined for products of two cycles exactly if one factor is equal C_3, C_4 or C_5 and also for products $X = C_1 \Box C_2 \Box \cdots \Box C_m$.

Here we study domination numbers of graph bundles, in particular the Cartesian graph bundles over cycles. We show that for any $k \geq 0$ there exists a graph G such that the domination number of Cartesian graph bundle with fibre G over the base graph C_4 is equal to $\gamma(G)\gamma(C_4) - k$. This implies that the Vizing's conjecture can not be generalized to graph bundles.

2 Preliminaries

We will consider finite, undirected, connected graphs without loops and multiple edges. A set D of vertices of a simple graph G with a vertex set $V(G)$ is called *dominating* if every vertex $w \in V(G) - D$ is adjacent to some vertex $v \in D$. The *domination number* of a graph G, $\gamma(G)$ is the order of a smallest dominating set of G. A dominating set D with $|D| = \gamma(G)$ is called a *minimum dominating set*. The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, c)(b, d) \in E(G \Box H)$ whenever $c = d$ and

$ab \in E(G)$, or $a = b$ and $cd \in E(H)$. For $c \in (H)$ set $G_c = G\Box\{c\}$ and for $a \in V(G)$ set $H_a = \{a\}\Box H$. We call G_c a *layer* of G, and H_a a layer of H.

Let B and F be graphs. A graph G is a *Cartesian graph bundle* with fibre F over the base graph B if there is a graph map $p : G \to B$ such that for each vertex $v \in V(B)$, $p^{-1}(v) \simeq F$, and for each edge $e \in E(B)$, $p^{-1}(e) \simeq K_2\Box F$. The triple (G, p, B) is called a *presentation* of G *as a Cartesian graph bundle*. We can also understand the Cartesian graph bundle as a graph, which is obtained from the base graph by replacing each of its vertices with a copy of the fibre graph and each of its edges by a matching between the copies of the fibre, corresponding to the endpoints of the edge. The edeges of the matching define an isomorphism between the copies of the fibre. Let $\varphi : E(B) \to Aut(F)$ be a mapping which assigns an automorphism of the graph F to any edge of B. The bundle G is obviously determined by F, B and φ, therefore we will write $G = B\Box_\varphi H$. It is well known [25] and easy to see that for any spanning tree T of B, there is a $\psi : E(B) \to Aut(F)$ such that $\psi(e) = id$ for $e \in T$ and $G = B\Box_\varphi H \sim B\Box_\psi H$. From this definition it follows that Cartesian graph bundles over paths and trees are exactly Cartesian product graphs and that we can represent Cartesian graph bundles over a cycle with a set of isomorphisms over the edges of the cycle, with at most one nonidentical isomorphism (if all isomorphisms are identities, the bundle is a product graph). Analogously we define *strong graph bundle* $G = B \boxtimes_\varphi H$ with fibre F over the base graph B by replacing the Cartesian product with the strong product in the definition.

In the rest of the paper we will denote Cartesian graph bundle over cycle C_n by $C_n\Box_\varphi F$, where φ gives the only possible nonidentity isomorphism over one edge of the cycle (usually it will be the edge $(n-1, 0)$). We can also get this automorphism of the fibre from any representation of Cartesian graph bundle over the cycle by the product (composition) of all automorphisms over the edges of the cycle (see Fig. 1).

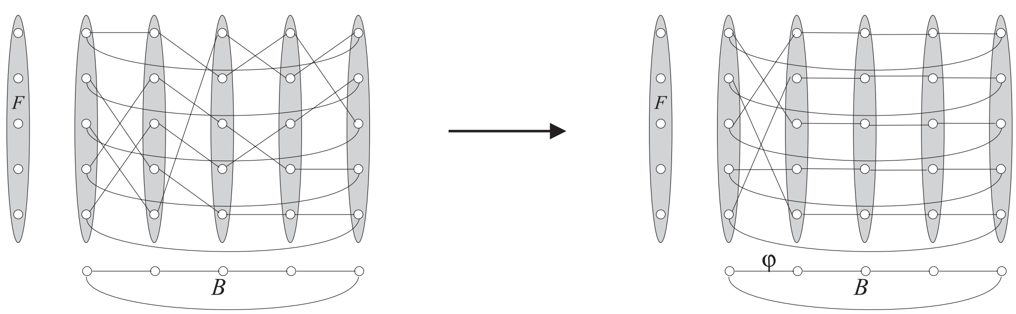

Fig. 1. Representation of Cartesian graph bundles over cycles

Throughout this paper the vertices of a path P_n or a cycle C_n will be denoted by $0, 1, \ldots, n-1$ and the automorphisms of cycles C_m over the edge $(n-1, 0)$ of C_n will be presented as follows:

- $\varphi = (+k)$ stands for rotation $\varphi(i) \equiv (i+k) \pmod{n}$ of cycle C_n and
- $\varphi = (\pm 2k)$ means reflection $\varphi(i) \equiv (-i+k) \pmod{n}$ over the vertex k of C_n,
- $\varphi = id$ denotes the identity.

3 Domination numbers of $C_n \Box_\varphi C_3$, $C_n \Box_\varphi C_4$ and $C_n \Box_\varphi C_5$

We continue by a direct derivation of $\gamma(C_n \Box_\varphi C_3)$. In the proof we will use Lemma 2.1. from [17]:

Lemma 1 *Let $n \geq 3$. Then there exists a minimum dominating set D of $C_m \Box_\varphi C_n$ such that for every $i \in V(C_n)$, $|(C_m)_i \cap D| \leq m-1$.*

Proposition 2 $\gamma(C_n \Box_\varphi C_3) = n - \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 1 \text{ ; } n = 4k \quad \wedge \quad \varphi = (+1) \\ 0 \text{ ; otherwise} \end{cases}$

Proof. The automorphism group of C_3 consists of identity, two rotations and three reflections of cycle C_3. Using the proper enumeration, we can consider only identity, rotation $\varphi_1 = (+1) = (012)$ (rotation of cycle by one vertex $i \mapsto i+1 \pmod 3$) and reflection $\varphi_2 = (02)$ over a vertex 1 of the cycle C_3. For convenience, let the automorphism of C_3 be defined over the edge $(n-1, 0)$ in cycle C_n. Let D consist of vertices $(i, 1); i \equiv 0 \pmod 4$, and vertices $(i, 0), (i, 2); i \equiv 2 \pmod 4$. If $\varphi = (+1)$ and $n \equiv 0 \pmod 2$ then add the vertex $(n-1, 0)$ to the set D. It is straightforward to check that D is a dominating set of $C_n \Box_\varphi C_3$ and that $|D| = n - \left\lfloor \frac{n}{4} \right\rfloor$ (+1 if $\varphi = (+1)$ and $n \equiv 0 \pmod 4$)). Now we have to show that $\gamma(C_n \Box_\varphi C_3) \geq n - \left\lfloor \frac{n}{4} \right\rfloor$ (+1 if $n \equiv 0 \pmod 4$)). We start in the same way as in the proof of Lemma 2.3 [17]. Let $n = 4k + t$, $k \geq 1$, $3 \geq t \geq 0$, and let D be a minimum dominating set which satisfies Lemma 1. Let s be the number of C_3-layers which contain no vertex of D. Then, since no two empty layers are adjacent,

$$k + 1 \leq s \leq \left\lfloor \tfrac{n}{2} \right\rfloor = 2k + \left\lfloor \tfrac{t}{2} \right\rfloor.$$

As every empty C_3-layer is dominated by exactly two other layers, there are at least $\lceil s/2 \rceil$ C_3-layers with precisely two vertices from D. Hence

$$|D| \geq 2\left\lceil \tfrac{s}{2} \right\rceil + \left(n - \left\lceil \tfrac{s}{2} \right\rceil - s\right) = n - \left(n - \left\lceil \tfrac{s}{2} \right\rceil\right)$$

The argument $s - \lceil s/2 \rceil$ is maximal when $s = 2k + \lfloor t/2 \rfloor$, therefore

$$|D| \geq n - \left(2k + \left\lfloor \tfrac{t}{2} \right\rfloor - \left\lceil \tfrac{2k + \lfloor t/2 \rfloor}{2} \right\rceil\right) = 3k + t - \left\lfloor \tfrac{t}{2} \right\rfloor + \left\lceil \tfrac{\lfloor t/2 \rfloor}{2} \right\rceil = n - \left\lfloor \tfrac{n}{4} \right\rfloor.$$

Suppose that there exists a dominating set with $|D| = n - k$ also in the case, when $\varphi = (+1)$ and $n = 4k$ ($s = 2k$). Then each empty C_3-layer is dominated by one layer with one vertex from D and one layer with two vertices from D. Since no two nonempty layers are adjacent ($s = 2k$), no two vertices from adjacent layers, which lie in D, have the same second coordinate. Therefore, the empty layer over an endpoint of the edge $(n-1, 0)$ in base graph, can not be dominated by three vertices (with different second coordinate) from adjacent layers. Hence $|D| > n - k$. □

Proposition 3 $\gamma(C_n \Box_\varphi C_4) = n$

Proof. Exactly the same arguments can be used as in the proof of Theorem 2.5 in [17]. □

Lemma 4 $\gamma(C_n \Box_\varphi C_5) \leq n + 2$

Proof. Clearly, $\gamma(C_n \Box_\varphi C_5) \leq \gamma(P_n \Box C_5)$ and $\gamma(P_n \Box C_5) \leq n + 2$, which can be verified by a straightforward construction. □

Lemma 5 $\gamma(C_n \Box_\varphi C_5) = n$, *whenever* $\varphi = (+k)$ *where* $k \equiv (2n) \pmod 5$.

Proof. Since each vertex of any dominating set in $(C_n \Box_\varphi C_5)$ covers exactly five vertices, we can dominate at most $5n = |V(C_n \Box_\varphi C_5)|$ vertices with n vertices. Therefore

$$\gamma(C_n \Box_\varphi C_5) \geq n.$$

It is easy to see that a dominating set of $(C_n \Box_\varphi C_5)$ which has two vertices in the same fibre is of size more than n (because two vertices in the same fibre together dominate at most 9 vertices). Hence we now consider only dominating sets with exactly one element per fibre. The pattern of dominating vertices in $(C_n \Box_\varphi C_5)$ repeats after each five fibres, because the vertices of dominating set in this pattern can be uniquely enumerated: $(i, 2i \pmod 5)$. If we want the pattern to continue over the edge $(n-1, 0)$ of Cartesian graph bundle $(C_n \Box_\varphi C_5)$ the equality

$$2n \equiv k \pmod 5$$

must hold, otherwise at least one vertex is not dominated. □

By constructing dominating sets in all possible cases we obtain the following exact domination numbers. The proof is straightforward and omitted.

Proposition 6
(1) $\gamma(C_n \Box_\varphi C_5) \leq n + 1$, *for* $\varphi = (+k)$ *where* $k + 1 \not\equiv (2n) \pmod 5$.
(2) $\gamma(C_n \Box_\varphi C_5) = n + 1$, *for any reflection* $\varphi = (\pm k)$.

Remark. We believe that $\gamma(C_n \Box_\varphi C_5) = n+2$ for $\varphi = (+k)$ where $k+1 \equiv (2n) \pmod 5$. In this case, the proof is likely to be more difficult.

Remark. It seems that the domination numbers of $\gamma(C_3 \Box_\varphi C_n)$, $\gamma(C_4 \Box_\varphi C_n)$, and $\gamma(C_5 \Box_\varphi C_n)$ are more difficult than the examples solved above.

4 On a statement similar to Vizing's conjecture

Since graph bundles generalize product graphs, it is natural question, whether an analog to the Vizing's conjecture holds for graph bundles. In the following we will construct an infinite class of Cartesian graph bundles for which the difference between their domination numbers and the products of the domination numbers of their bases and fibres can be arbitrary large. Let G_l be the graph, obtained from the complete graph K_l by adding a new neighbor to each vertex of the complete graph (see Fig. 2). We denote the vertices of

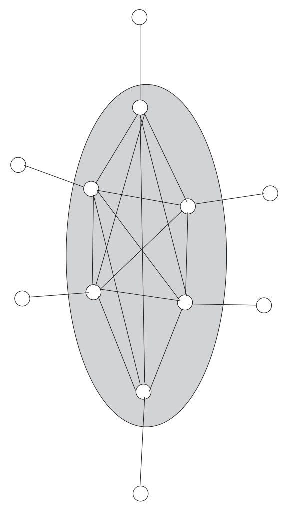

Fig. 2. Graph G_6

the complete subgraph K_l by $1, 2, \ldots, l$ and the other vertices of graph G_l by $-1, -2, \ldots, -l$ such that all pairs of vertices $(i, -i)$ are adjacent. The domination number of the graph G_l is equal l ($\gamma(G_l) = l$), since the vertex set of subgraph K_l is a dominating set of G_l and there are no common neighbor of any two vertices from $G_l \setminus K_l$. Note also that $\gamma(C_4) = 2$.

Proposition 7 *For any $k \geq 1$ there exists an isomorphism φ of G_{4+3k} such that*

$$\gamma(C_4 \Box_\varphi G_{4+3k}) = \gamma(G_{4+3k})\gamma(C_4) - 2k.$$

Proof. Take the isomorphism φ of G_{4+3k} over the edge $(2,3) \in E(C_4)$ as follows:

$$\varphi(i) = \begin{cases} i & ; i = 1,2,3,4 \\ i+1 & ; i = 3j+2, 3j+3; j = 1,\dots,k \\ i-2 & ; i = 3j+4; j = 1,\dots,k \\ -|\varphi(-i)| & ; i < 0 \end{cases}.$$

Let D consist of vertices $(j-1,-j)$, $(j-1,(j+1) \pmod 4 + 1)$; $j = 1,2,3,4$ and $(0,-(3i+4)), (1,-(3i+3)), (2,-(3i+2)), (3,-(3i+2))$; $i = 1,\dots,k$ (see Fig. 3). It is straightforward to check that D is a dominating set of $C_4\square_\varphi G_{4+3k}$

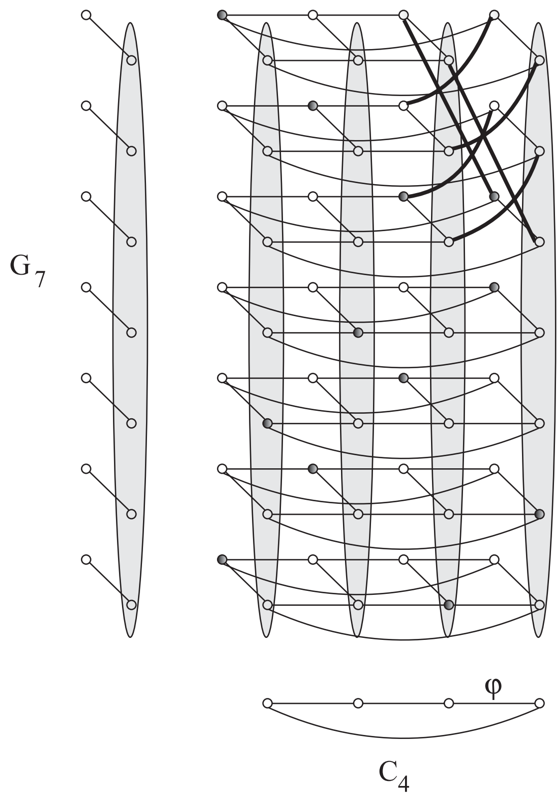

Fig. 3. The domination set of $C_4\square_\varphi G_7$

and that $|D| = 8+4k$.

Next we show that $\gamma(C_4\square_\varphi G_{4+3k}) \geq 8+4k$. Let us denote the vertex set of subgraph $C_4\square_\varphi K_{4+3k}$ in graph $C_4\square_\varphi G_{4+3k}$ by S and $V(C_4\square_\varphi G_{4+3k}) \setminus S$ by T. Each vertex in T dominates only one vertex in S, but for domination of all vertices from S only four vertices from S suffice. If there are less than four vertices in the dominating set from S, then at least $6k+4$ dominating vertices from T is needed ($3k+4$ vertices in T from the G_{4+3k}-layer which is not dominated by vertices from S and at least $3k$ vertices from the nonadjacent G_{4+3k}-layer) and therefore $|D| \geq 6k+8 > 4k+8$. On the other hand, if $|D \cap S| \geq 4$, in any minimal dominating set there are at least four vertices from the set S. Each vertex from T dominates exactly three vertices from T

and four vertices from S dominate four vertices in S, therefore for dominating vertex set T we need at least $\lceil \frac{|T|-4}{3} \rceil = \lceil \frac{4(4+3k)-4}{3} \rceil = 4+4k$ vertices from the set T. Hence, any dominating set of graph $C_4 \Box_{\varphi} G_{4+3k}$ has at least $8+4k$ vertices. $\Box$

5 A remark on dominating strong bundles

It is straightforward to check that Cartesian product of dominating sets of graphs B and F is dominating set of $B \boxtimes_{\varphi} F$, hence $\gamma(B \boxtimes_{\varphi} F) \leq \gamma(B)\gamma(F)$.

Furthermore, let $B \simeq F \simeq C_4$ and $\varphi = (+2)$ (Fig. 4). It is easy to check that

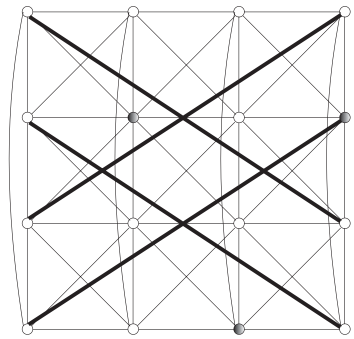

Fig. 4. A dominating set of $C_4 \boxtimes_{\varphi} C_4$

set $D = \{(1,2),(2,0),(2,2)\}$ is dominating set of $C_4 \boxtimes_{\varphi} C_4$. Since each C_4-layer must be adjacent to at least two vertices from dominating set, $|D| \leq 3$, therefore $\gamma(C_4 \boxtimes_{\varphi} C_4) = 3 < 4 = \gamma(C_4)\gamma(C_4)$.

Hence we have

Proposition 8 $\gamma(B \boxtimes_{\varphi} F) \leq \gamma(B)\gamma(F)$

and there exist graphs B and F with

$$\gamma(B \boxtimes_{\varphi} F) < \gamma(B)\gamma(F).$$

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