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Abstract

Using a newly introduced operation on graphs and its counterpart on graph drawings, we prove the conjecture of Jendrol' and Ščerbová from 1982 about the crossing number of the Cartesian product $K_{1,m} \Box P_n$. Our approach is applicable to the capped Cartesian products of P_n with any graph containing a dominating vertex.

Keywords: crossing number, Cartesian product, path.

1 Introduction

Determination of crossing numbers of graphs (see [7] for the folklore definitions) is a notorious problem in Graph Theory, as in general it is quite easy to find a drawing of sufficiently "nice" graph in which the number of crossings can hardly be decreased, but it is very difficult to prove that such a drawing indeed has the smallest possible number of crossings. Thus it is not surprising that the exact crossing number is known only for few graph families, and that the arguments often strongly depend on the structure of the graphs considered.

The crossing number problem was for some time studied for Cartesian product (cf. [1]) of paths, cycles and stars with various other graphs. In this case the problem seems more tractable, mostly due to the richness of patterns that the Cartesian product of graphs possesses. Usually, the optimality is established in the way that the crossing number of a small graph is obtained exploiting its structure, and then an inductive proof is deduced with the help of the following contradiction: if the larger graph would have fewer crossings than stated, we would be able to modify its drawing to a drawing of a smaller graph with too few crossings. Often also some insight into the structure of the optimal drawing of the larger graph is needed in the sense that if it possesses a certain structure, then some lower bound on the number of crossings follows.

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Several results were obtained using this and similar techniques. To our knowledge, the first result of this kind is the proof of Ringeisen and Beineke [8] that $\mathrm{cr}(C_3 \Box C_n) = n$. In [1] they extend this result to $G \Box C_n$ where G is any graph on four vertices, except the 3-star $S_3 = K_{1,3}$. This is later established by Jendrol' and Ščerbová [4], who obtain $\mathrm{cr}(S_3 \Box C_n)$ and $\mathrm{cr}(S_3 \Box P_n)$ for all values $n \geq 3$. In their paper they conjecture that

$$\mathrm{cr}(S_m \Box P_n) = (n-1)\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \tag{1}$$

for an arbitrary star $S_m = K_{1,m}$ and path P_n of length $n \geq 1$. This conjecture is proved by Klešč [6] for $m = 4$, along with the explicit formula for $S_4 \Box C_n$. An overview of other results about crossing numbers of some graphs and graph families obtained as Cartesian products is given in [7]. The most recent result on the topic was the proof of the conjecture about crossing number of product of cycles $C_m \Box C_n$ for $n \geq m(m-1)$ by Glebsky and Salazar [3].

In this contribution we prove the conjecture (1). The approach is seemingly new: instead of exploiting the structure of drawings of the graphs in question, we propose an operation on drawings of graphs and describe a sufficient combinatorial condition that this operation preserves the optimality of the drawings. This operation was initially developed for constructing crossing critical graphs and applied to settle a question of Salazar [9] about existence of infinite families of crossing critical graphs with prescribed average degree [2], and it was observed afterwards that it solves this quite unrelated conjecture.

2 Zip product

For $i = 1, 2$ let G_i be a graph and let $v_i \in V(G_i)$ be its vertex of degree d. Let $N_i = N_{G_i}(v_i)$ be the set of neighboring vertices of v_i and let $\sigma : N_1 \to N_2$ be a bijection. We call σ a *zip function* of the graphs G_1 and G_2 at vertices v_1 and v_2. The *zip product* of the graphs G_1 and G_2 according to σ is defined to be the graph $G_1 \odot_\sigma G_2$, obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ after adding the edge $u\sigma(u)$ for every $u \in N_1$.

Let D_i be a drawing of a graph G_i. The drawing imposes a cyclic ordering of the edges incident with v_i, which can be extended to its neighborhood N_i. Let the bijection $\pi_i : N_i \to \{1, \ldots, d\}$ be one of the corresponding labelings. We define $\sigma : N_1 \to N_2$, $\sigma = \pi_2^{-1}\pi_1$, to be the *zip function* of the drawings D_1 and D_2 at vertices v_1 and v_2. The *zip product* of D_1 and D_2 according to σ is the drawing $D_1 \odot_\sigma D_2$, obtained from D_1 by adding a mirrored copy of D_2 that has v_2 on the infinite face disjointly into some face of D_1 that contains the vertex v_1, removing the vertices v_1 and v_2 together with small disks around them from the drawings, and then joining the edges according to the function σ. As σ respects the ordering of the edges around v_1 and v_2, the edges between D_1 and D_2 may be drawn without introducing any new crossings. Clearly $D_1 \odot_\sigma D_2$ is a drawing of $G_1 \odot_\sigma G_2$, which implies the following.

Lemma 1 *For $i = 1, 2$ let D_i be an optimal drawing of G_i, $v_i \in V(G_i)$ be a vertex of degree d and let σ be a zip function of D_1 and D_2 according to v_1 and v_2. Then* $\mathrm{cr}(G_1 \odot_\sigma G_2) \leq \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$.

In the sequel of the section we present a sufficient condition for equality in Lemma 1. Let $S_n = K_{1,n}$ be a star graph with n vertices of degree 1 (called the *leaves* of the star) and one vertex of degree n (the *center*). Let G be a graph and $S \subseteq V(G)$, $|S| = n$. We say that S is

k*-star-connected* in G if there exist k disjoint sets $F_1, \ldots, F_k \subseteq E(G)$, such that either $G[F_i]$ is a subdivision of S_n with S being the leaves, or $G[F_i]$ is a subdivision of S_{n-1} with all its leaves and the center belonging to S. We call each F_i a *star-set* of S.

Lemma 2 *Let G_1 and G_2 be disjoint graphs, $v_i \in V(G_i)$, $\deg(v_i) = d$, and let the neighborhood N_i of v_i be 2-star-connected in $G_i - v_i$, $i = 1, 2$. Then $\operatorname{cr}(G_1 \odot_\sigma G_2) \geq \operatorname{cr}(G_1) + \operatorname{cr}(G_2)$ for any bijection $\sigma : N_1 \to N_2$.*

Proof. Let $F = \{u\sigma(u) \mid u \in N_{G_1}(v_1)\}$ be the set of edges joining $G_1 - v_1$ and $G_2 - v_2$ in $G := G_1 \odot_\sigma G_2$. For $i, j = 1, 2$, let $F_{ij} \subseteq E_i := E(G_i - v_i)$ be disjoint star-sets of N_i. Let G_{1j} be the subgraph of G, generated with edges E_1, F_{2j} and F. We define similarly the graph G_{2j}. Clearly G_{ij} is isomorphic to a subdivision of G_i, thus $\operatorname{cr}(G_{ij}) = \operatorname{cr}(G_i)$.

Let D be an optimal drawing of G, and let D_{ij} be the corresponding subdrawing of G_{ij}. We have $\operatorname{cr}(G_1) \leq \operatorname{cr}_D(E_1, E_1) + \operatorname{cr}_D(E_1, F) + \operatorname{cr}_D(E_1, F_{2j})$, thus also $\operatorname{cr}(G_1) \leq \operatorname{cr}_D(E_1, E_1) + \operatorname{cr}_D(E_1, F) + \frac{1}{2}\operatorname{cr}_D(E_1, F_{21} \cup F_{22})$.

A similar inequality holds for $\operatorname{cr}(G_2)$. Summing up we obtain:

$$\begin{aligned}
\operatorname{cr}(G_1) + \operatorname{cr}(G_2) &\leq \operatorname{cr}_D(E_1, E_1) + \operatorname{cr}_D(E_2, E_2) + \\
&\quad \tfrac{1}{2}\left(\operatorname{cr}_D(E_1, F_{21} \cup F_{22}) + \operatorname{cr}_D(E_2, F_{11} \cup F_{12})\right) + \\
&\quad \operatorname{cr}_D(E_1 \cup E_2, F) \\
&\leq \operatorname{cr}_D(E_1, E_1 \cup F) + \operatorname{cr}_D(E_2, E_2 \cup F) + \operatorname{cr}_D(E_1, E_2) \\
&= \operatorname{cr}(D) - \operatorname{cr}_D(F, F) \leq \operatorname{cr}(G).
\end{aligned}$$

Note that $F_{ij} \subseteq E_i$ are disjoint for $i, j = 1, 2$, thus a crossing of any $e \in E_1$ with some $f \in E_2$ can be counted at most twice in the sum of $\operatorname{cr}_D(E_i, F_{3-i,j})$. This justifies the second inequality. □

3 The crossing number of $S_m \Box P_n$

With $\tilde{G}$ we denote a double suspension of G, that is the graph, obtained from G by adding two vertices v_1 and v_2 and the edges $v_i v$ for $i = 1, 2$ and each $v \in V(G)$. We call a vertex $v \in V(G)$ a *dominating vertex* of G, if it is adjacent to all other vertices in G. With P_n we denote the path of length n, its vertices are $0, \ldots, n$. With $G \widehat{\Box} P_n$ we denote the *capped Cartesian product* of G and P_n, i.e. the graph, obtained from $G \Box P_n$ by adding two new vertices v_0 and v_n and connecting v_0 with all the vertices of $G \Box \{0\}$ and v_n with all the vertices of $G \Box \{n\}$ in $G \Box P_n$.

Theorem 3 *Let G be a graph with a dominating vertex. Then for $n \geq 0$, $\operatorname{cr}(G \widehat{\Box} P_n) = (n+1)\operatorname{cr}(\tilde{G})$.*

Proof. Let $G_n = G \widehat{\Box} P_n$. Then $G_0 = \tilde{G}$ and $G_n = G_{n-1} \odot_{\iota_n} \tilde{G}$, where $\iota_n : N_{n-1} \to N_1$ maps a vertex $v \in N_{n-1} = N_{G_{n-1}}(v_{n-1})$ to its counterpart in $N_1 = N_{\tilde{G}}(v_1)$. The proof of this observation is merely technical and left to the reader. As G has a dominating vertex, the vertices $v_1, v_2 \in V(\tilde{G})$ have 2-star connected neighborhoods in $\tilde{G}$; the same holds for

$v_0, v_n \in G_n$. Lemma 2 implies $\mathrm{cr}(G_n) \geq (n+1)\,\mathrm{cr}(\tilde{G})$. A drawing D_n of G_n with its crossing number reaching this bound can be obtained from any optimal drawing D of $\tilde{G}$:

For $i = 1, 2$ let π_i be the circular labeling of vertices of $V(G)$ around v_i in D. Then in the mirror drawing D' the vertex v_i has π_i^{-1} as the circular labeling of its neighborhood. Set $D_0 = D$ and for odd n, let $D_n = D_{n-1} \odot_\iota D$ using a vertex with labeling π_1 in both drawings, and for even n let $D_n = D_{n-1} \odot_\iota D'$ using a vertex with labeling π_2^{-1}. Vertices with such labelings exist, as we mirror the second drawing in the zip product. □

Corollary 4 *For $m, n \geq 1$ we have* $\mathrm{cr}(S_m \Box P_n) = (n-1) \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$.

Proof. The double suspension $\tilde{S}_m$ has $K_{3,m}$ as a subgraph. Kleitman [5] proved that $\mathrm{cr}(K_{3,m}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$ and we leave to the reader to augment the drawing of $K_{3,m}$ presented in [10] to the drawing of $\tilde{S}_m$ without introducing any new crossings.

We may assume $n \geq 2$. Then Theorem 3 implies that $\mathrm{cr}(S_m \widehat{\Box} P_{n-2}) = (n-1) \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$. The claim follows, as $S_m \Box P_n$ is a subdivision of $S_m \widehat{\Box} P_{n-2}$. □

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