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Preprint series, Vol. 44 (2006), 1006

APPROXIMATION OF HOLOMORPHIC MAPPINGS ON STRONGLY PSEUDOCONVEX DOMAINS

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ISSN 1318-4865

July 7, 2006

Ljubljana, July 7, 2006

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Abstract. Let $D \Subset S$ be a strongly pseudoconvex domain in a Stein manifold S, and let Y be a complex manifold. We prove that the set $\mathcal{A}(D,Y)$ of all continuous maps $\bar D \to Y$ which are holomorphic in D is a complex Banach manifold, and every $f \in \mathcal{A}(D,Y)$ can be approximated uniformly on $\bar D$ by maps holomorphic in open neighborhoods of $\bar D$ in S. We obtain analogous results for maps of class $\mathcal{A}^r(D)$ ($r \in \mathbb{N}$) and global approximation theorems for sections of certain holomorphic fiber bundles on strongly pseudoconvex domains which extend smoothly to the boundary.

1. Introduction

Let D be a relatively compact, strongly pseudoconvex domain in a Stein manifold S, and let Y be a complex manifold or a (reduced, paracompact) complex space. For $r \in \mathbb{Z}_+$ we denote by $\mathcal{C}^r(\bar D, Y)$ the set of all $\mathcal{C}^r$ maps $\bar D \to Y$ and by $\mathcal{A}^r(D,Y)$ the set of all $f \in \mathcal{C}^r(\bar D, Y)$ which are holomorphic in D. Set $\mathcal{A}(D,Y) = \mathcal{A}^0(D,Y)$, $\mathcal{C}^r(\bar D, \mathbb{C}) = \mathcal{C}^r(\bar D)$, $\mathcal{A}^r(D,\mathbb{C}) = \mathcal{A}^r(D)$.

In this paper we use the technique of holomorphic sprays to investigate the structure of the space $\mathcal{A}^r(D,Y)$ and to obtain approximation theorems for holomorphic maps. A *holomorphic spray* is a family of maps in a given space of maps (such as $\mathcal{A}(D,Y)$), depending holomorphically on a parameter t in an open set $P \subset \mathbb{C}^N$; the domination condition means submersivity with respect to the parameter $t \in P$ at $t = 0$ (Def. 2.1).

Theorem 1.1. *Assume that D is a relatively compact, strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell \geq 2$) in a Stein manifold. For each $r \in \{0,1,\ldots,\ell\}$ and for every complex manifold Y the space $\mathcal{A}^r(D,Y)$ is a complex Banach manifold.*

Theorem 1.1 follows from Corollary 4.4 to the effect that the set of all maps in $\mathcal{A}^r(D,Y)$ which are $\mathcal{C}^0$-close to a given $f_0 \in \mathcal{A}^r(D,Y)$ is isomorphic to the set of all $\mathcal{C}^0$-small sections of the complex vector bundle $f_0^*(TY) \to \bar D$ which are holomorphic in D and of class $\mathcal{C}^r(\bar D)$. This gives a local Banach

Date: July 7, 2006.
2000 *Mathematics Subject Classification.* 32E10, 32E30, 32H02, 46G20, 58B12.
Key words and phrases. Stein manifolds, Banach manifolds, approximation.
Research supported by grants P1-0291 and J1-6173, Republic of Slovenia.

chart around $f_0 \in \mathcal{A}^r(D,Y)$ with values in the Banach space $\Gamma^r_{\mathcal{A}}(D, f_0^*(TY))$ of all sections $\bar{D} \to f_0^*(TY)$ of class $\mathcal{C}^r(\bar{D})$ which are holomorphic in D, and it is easily seen that these charts are holomorphically compatible. This Banach manifold structure is natural in the sense that the evaluation map $D \times \mathcal{A}^r(D,Y) \to Y$ is holomorphic.

For the disc $\triangle = \{z \in \mathbb{C}\colon |z| < 1\}$, the Banach manifold $\mathcal{A}(\triangle, Y)$ is locally modeled on the complex Banach space $\mathcal{A}(\triangle)^n$ with $n = \dim Y$. In view of the Oka-Grauert principle for vector bundles of class $\mathcal{A}(D)$, due to Leiterer [34], the same holds whenever every continuous complex vector bundle over $\bar{D}$ is topologically trivial.

Lempert [36] obtained analogous results for the space of all $\mathcal{C}^k$ maps from a compact $\mathcal{C}^k$ manifold V into a complex manifold M, showing that $\mathcal{C}^k(V,M)$ is a Banach manifold if $k \in \mathbb{Z}_+$, resp. a Fréchet manifold if $k = \infty$. He also obtained a complex manifold structure on the space of real analytic maps by considering Stein complexifications. Like in our case, maps $f\colon V \to M$ near a given map f_0 correspond to $\mathcal{C}^0$-small sections of the vector bundle f_0^*TM. (In this direction see also [27], [28], [30], [31].)

We now turn to holomorphic approximation theorems.

Theorem 1.2. *Assume that $D \Subset S$ is a strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell \geq 2$) in a Stein manifold S, Y is a complex manifold and $r \in \{0, 1, \ldots, \ell\}$. Any map in $\mathcal{A}^r(D,Y)$ can be approximated in the $\mathcal{C}^r(\bar{D}, Y)$ topology by maps holomorphic in open neighborhoods of $\bar{D}$.*

When $S = \mathbb{C}^n$, $Y = \mathbb{C}$ and $r = 0$, this classical result follows from the Henkin-Ramírez integral kernel representation of functions in $\mathcal{A}(D)$ (Henkin [20], Ramírez [42], Kerzman [29], [21, p. 87]). Another approach is to cover bD by finitely many open charts in which bD is strongly convex, approximate the given function in $\mathcal{A}(D)$ by a holomorphic function in each of the charts and patch these local approximations into a global one by solving a Cousin problem with bounds. This method also works for $r > 0$ by using a solution operator to the $\bar{\partial}$-equation for $(0,1)$-forms with $\mathcal{C}^r$ estimates (Range and Siu [43], Lieb and Range [38], Michel and Perotti [39]). For approximation on certain weakly pseudoconvex domains see Fornaess and Nagel [6].

The kernel approach essentially depends on the linear structure of the target manifold and does not seem amenable to generalizations. In this paper we adapt the second approach mentioned above to an arbitrary target manifold Y by using the method of sprays. A crucial ingredient is a result from [5] concerning gluing of pairs of sprays on special configurations of domains called *Cartan pairs*. Using this gluing technique we show that any $f_0 \in \mathcal{A}^r(D,Y)$ is the core map of a dominating spray (Corollary 4.2). This immediately implies Theorem 1.1, and Theorem 1.2 is then obtained by the bumping method alluded to above. We prove the analogous approximation result for sections of holomorphic submersions (Theorem 5.1). When $r \geq 2$, a different proof can obtained by using the fact that the graph $G_f =$

$\{(z,f(z))\colon z\in \bar D\}\subset S\times Y$ of any map $f\in \mathcal{A}^2(D,Y)$ admits a basis of open Stein neighborhoods in $S\times Y$ [5, Theorem 2.6].

For approximation of holomorphic maps from planar Jordan domains to (almost) complex manifolds see D. Chakrabarti [3, Theorem 1.1.4], [4].

For maps from domains in open Riemann surfaces we obtain the following stronger result which extends Theorem 5.1 in [5].

Theorem 1.3. *Let D be a relatively compact domain with $\mathcal{C}^2$ boundary in an open Riemann surface S and let Y be a (reduced, paracompact) complex space. Every continuous map $\bar D\to Y$ which is holomorphic in D can be approximated uniformly on $\bar D$ by maps which are holomorphic in open neighborhoods of $\bar D$ in S. The analogous result holds for maps of class $\mathcal{A}^r(D,Y)$.*

The domains of the approximating maps in Theorems 1.2 and 1.3 must in general shrink to $\bar D$. For target manifolds Y satisfying the following *Convex approximation property* we also obtain global approximation results.

Definition 1.4. ([12, Def. 1.1]) A complex manifold Y enjoys the *Convex Approximation Property* (CAP) if any holomorphic map from a neighborhood of a compact convex set $K\subset \mathbb{C}^n$ ($n\in\mathbb{N}$) to Y can be approximated uniformly on K by entire maps $\mathbb{C}^n\to Y$.

Recall that a compact set K in a complex manifold S is $\mathcal{O}(S)$-convex if for every point $p\in S\backslash K$ there is a holomorphic function $g\in\mathcal{O}(S)$ satisfying $|g(p)|>\sup_{z\in K}|g(z)|$. If $D\Subset S$ then a compact subset $K\subset\bar D$ is $\mathcal{A}(D)$-*convex* if for every $p\in\bar D\backslash K$ the above holds for some $g\in\mathcal{A}(D)$.

The following is a consequence of Theorem 1.2 and the main result of [12]; for a more general result see Corollary 5.3.

Corollary 1.5. *Assume that S is a Stein manifold and $D\Subset S$ is a strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell\ge 2$) whose closure $\bar D$ is $\mathcal{O}(S)$-convex. Let $r\in\{0,1,\ldots,\ell\}$ and let $f\colon S\to Y$ be a $\mathcal{C}^r$ map which is holomorphic in D. If Y enjoys* CAP *then f can be approximated in the $\mathcal{C}^r(\bar D,Y)$ topology by holomorphic maps $S\to Y$ which are homotopic to f.*

We shall extend Corollary 1.5 to sections of certain holomorphic fiber bundles over compact strongly pseudoconvex domains.

Definition 1.6. Assume that $D\Subset S$ is domain with $\mathcal{C}^\ell$ boundary ($\ell\ge 2$) in a Stein manifold S, Y is a complex manifold and $r\in\{0,1,\ldots,\ell\}$. An $\mathcal{A}^r_Y(D)$-*bundle* is a fiber bundle $h\colon X\to\bar D$ with fiber $h^{-1}(z)\simeq Y$ ($z\in\bar D$) which is smooth of class $\mathcal{C}^r(\bar D)$ and is holomorphic over D. More precisely, every point $z_0\in\bar D$ admits a relatively open neighborhood U in $\bar D$ and a $\mathcal{C}^r$ fiber bundle isomorphism $\Phi\colon X|_U=h^{-1}(U)\to U\times Y$ which is holomorphic over $U\cap D$.

We denote by $\Gamma^r(\bar D,X)$ the set of all sections $f\colon\bar D\to X$ of class $\mathcal{C}^r$, and by $\Gamma^r_{\mathcal{A}}(D,X)$ the set of all $f\in\Gamma^r(\bar D,X)$ which are holomorphic over D.

Theorem 1.7. (The homotopy principle for sections of $\mathcal{A}^r$-bundles) *Assume that $D_0 \subset D \Subset S$ are strongly pseudoconvex domains with $\mathcal{C}^\ell$ boundaries in a Stein manifold S ($\ell \geq 2$) and that $h\colon X \to \bar{D}$ is an $\mathcal{A}^r_Y(D)$-bundle for some $r \in \{0, 1, \ldots, \ell\}$. If $\bar{D}_0$ is $\mathcal{A}(D)$-convex and the fiber Y enjoys* CAP *then sections $\bar{D} \to X$ satisfy the following:*

- (i) *Every continuous section $f_0 \in \Gamma(\bar{D}, X) = \Gamma^0(\bar{D}, X)$ is homotopic to a section $f_1 \in \Gamma^r_{\mathcal{A}}(\bar{D}, X)$; if f_0 is holomorphic in D_0 then a homotopy from f_0 to f_1 can be chosen to consist of sections which are holomorphic in D_0 and uniformly close to f_0 on $\bar{D}_0$.*
- (ii) *Every homotopy $\{f_t\}_{t\in[0,1]} \in \Gamma(\bar{D}, X)$ with $f_0, f_1 \in \Gamma^r_{\mathcal{A}}(D, X)$ can be deformed with fixed ends to a homotopy in $\Gamma^r_{\mathcal{A}}(D, X)$.*

Theorem 1.7 also applies if the base is a compact complex manifold with Stein interior and smooth strongly pseudoconvex boundary since such manifold is diffeomorphic to a strongly pseudoconvex domain in a Stein manifold by a diffeomorphism which is holomorphic in the interior [26], [41] (some loss of smoothness may occur). In this connection we mention Catlin's boundary version [2] of the Newlander-Nirenberg integrability theorem [40].

Theorem 1.7 is proved in §6 (a more precise result is Theorem 6.1). For fiber bundles over *open* Stein manifolds (without boundary) the corresponding result was obtained in [12] (see also [19], [13], [32]). The classical case when $X \to S$ is a principal G-bundle, with fiber a complex Lie group G, is due to H. Grauert [17], [18]. His results imply that the holomorphic classification of principal G-bundles over Stein spaces agrees with the topological classification. (See also the exposition of H. Cartan [1] and the papers [7], [9], [23].) A. Sebbar [44] proved the analogous result for principal G-bundles which are smooth on the closure of a pseudoconvex domain $D \subset \mathbb{C}^n$ and holomorphic over D.

The CAP property of the fiber Y is both necessary and sufficient for validity of Theorem 1.7 (i). In the absence of topological obstructions, CAP of Y also implies extendibility of holomorphic maps from closed complex subvarieties in Stein manifolds to Y [10], [33]. Among the conditions implying CAP are (from strongest to weakest):

- (a) homogeneity under a complex Lie group (Grauert [17]),
- (b) the existence of a dominating spray on Y (Gromov [19]), and
- (c) the existence of a finite dominating family of sprays, [8].

Further examples of manifolds enjoying CAP can be found in [11] and [12].

The paper is organized as follows. In §2 we recall the notion of a (dominating) holomorphic spray and some of the related technical tools developed in [5]. A key ingredient is Proposition 2.4 concerning gluing of holomorphic sprays on Cartan pairs (Def. 2.3); it depends on a certain nonlinear version of Cartan's lemma with control up to the boundary. In sections 3 and 4 we obtain some technical results on the existence and approximation of sprays.

Theorem 1.1 is a consequence of Corollary 4.3 which provides linearization of a neighborhood of any section of class $\mathcal{A}^r$. In §5 we prove Theorems 1.2 and 1.3, and in §6 we prove Theorem 1.7.

2. Gluing sprays on Cartan pairs

In this section D is a relatively compact, strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell \geq 2$) in a Stein manifold S and $h\colon X \to \bar{D}$ is either an $\mathcal{A}^r_Y(D)$-bundle for some $r \in \{0, \ldots, \ell\}$, or the restriction to $\bar{D}$ of a holomorphic submersion $\widetilde{h}\colon \widetilde{X} \to S$. (Our results also apply if $h\colon X \to \bar{D}$ is a smooth submersion which is holomorphic over D.)

Let $VT_xX = \ker dh_x$ denote the *vertical tangent space* at a point $x \in X$.

Definition 2.1. Assume that P is an open set in $\mathbb{C}^N$ containing the origin. An *h-spray of class $\mathcal{A}^r(D)$ with the parameter set P* is a map $f\colon \bar{D} \times P \to X$ of class $\mathcal{C}^r$ which is holomorphic on $D \times P$ and satisfies $h(f(z,t)) = z$ for all $z \in \bar{D}$ and $t \in P$. We call $f_0 = f(\cdot\,,0)$ the *central* (or *core*) section of the spray. A spray f is *dominating* on a subset $K \subset \bar{D}$ if the partial differential

$$\partial_t|_{t=0} f(z,t)\colon T_0\mathbb{C}^N = \mathbb{C}^N \to VT_{f(z,0)}X$$

is surjective for all $z \in K$; if this holds with $K = \bar{D}$ then f is said to be dominating.

A (dominating) *1-parametric h-spray* of class $\mathcal{A}^r(D)$ and parameter set P is a $\mathcal{C}^r$ map $f\colon [0,1] \times \bar{D} \times P \to X$ such that $f^s = f(s,\cdot\,,\cdot)\colon \bar{D} \times P \to X$ is a (dominating) h-spray of class $\mathcal{A}^r(D)$ for each $s \in [0,1]$.

For a product fibration $h\colon X = \bar{D} \times Y \to \bar{D}$, $h(z,y) = z$, we can identify an h-spray $\bar{D} \times P \to X$ with a *spray of maps* $f\colon \bar{D} \times P \to Y$ by composing with the projection $\bar{D} \times Y \to Y$, $(z,y) \to y$. Such a spray is dominating on $K \subset \bar{D}$ if $\partial_t|_{t=0} f(z,t)\colon T_0\mathbb{C}^N \to T_{f(z,0)}Y$ is surjective for all $z \in K$.

In applications the parameter set P will be allowed to shrink around $0 \in \mathbb{C}^N$. If f is dominating on a compact subset K of $\bar{D}$ then by shrinking P we may assume that $\partial_t f(z,t)\colon T_t\mathbb{C}^N \to VT_{f(z,t)}X$ is surjective for all $(z,t) \in K \times P$.

Remark 2.2. Our definition of a spray is similar to Def. 4.1 in [5], but is adapted for use in this paper. Sprays were introduced to the Oka-Grauert theory by Gromov [19] to obtain approximation and gluing theorems for holomorphic sections. In Gromov's terminology (which was taken up in [13], [14], [15]), a dominating spray on a complex manifold Y is a holomorphic map $s\colon E \to Y$ from the total space of a holomorphic vector bundle $\pi\colon E \to Y$ such that $s(0_y) = y$ and $ds_{0_y}(E_y) = T_yY$ for every point $y \in Y$. When $E = Y \times \mathbb{C}^N$, we get for any holomorphic map $f_0\colon D \to Y$ a dominating spray $f\colon D \times \mathbb{C}^N \to Y$ (in the sense of Def. 2.1) by setting $f(z,t) = s(f_0(z),t)$. Global dominating sprays (with the parameter set $P = \mathbb{C}^N$) exist only rarely.

Sprays whose parameter set is a (small) open subset of $\mathbb{C}^N$ have been called *local sprays* in the literature on the Oka-Grauert problem.

Definition 2.3. A pair of open subsets $D_0, D_1 \Subset S$ in a Stein manifold S is said to be a *Cartan pair* of class $\mathcal{C}^\ell$ ($\ell \geq 2$) if

(i) D_0, D_1, $D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$ are strongly pseudoconvex with $\mathcal{C}^\ell$ boundaries, and

(ii) $\overline{D_0 \backslash D_1} \cap \overline{D_1 \backslash D_0} = \emptyset$ (the separation property).

We say that D_1 *is a convex bump on* D_0 if, in addition to the above, there is a biholomorphic map from an open neighborhood of $\bar{D}_1$ in S onto an open subset of $\mathbb{C}^n$ ($n = \dim S$) which maps D_1 and $D_{0,1}$ onto strongly convex domains in $\mathbb{C}^n$.

The following proposition and its proof are crucial for all that follows.

Proposition 2.4. (Gluing of sprays) *Let (D_0, D_1) be a Cartan pair of class $\mathcal{C}^\ell$ ($\ell \geq 2$) in a Stein manifold S and set $D = D_0 \cup D_1$, $D_{0,1} = D_0 \cap D_1$. Assume that $h\colon X \to \bar{D}$ is either an $\mathcal{A}^r_Y(D)$-bundle for some $r \in \{0, \ldots, \ell\}$, or the restriction to $\bar{D}$ of a holomorphic submersion $\widetilde{X} \to S$.*

Given an h-spray $f\colon \bar{D}_0 \times P_0 \to X$ ($P_0 \subset \mathbb{C}^N$) of class $\mathcal{A}^r(D_0)$ which is dominating on $\bar{D}_{0,1}$, there is an open set $0 \in P \Subset P_0$ satisfying the following. For every h-spray $f'\colon \bar{D}_1 \times P_0 \to X$ of class $\mathcal{A}^r(D_1)$ which is sufficiently $\mathcal{C}^r$ close to f on $\bar{D}_{0,1} \times P_0$ there exists an h-spray $g\colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$, close to f in the $\mathcal{C}^r$ topology on $\bar{D}_0 \times P$, such that $g_0 = g(\cdot\,, 0)$ is homotopic to f_0 on $\bar{D}_0$ and is homotopic to f'_0 on $\bar{D}_1$. If f and f' agree to order $m \in \mathbb{N}$ along $\bar{D}_{0,1} \times \{0\}$ then g can be chosen to agree to order m with f along $\bar{D}_0 \times \{0\}$ and with f' along $\bar{D}_1 \times \{0\}$.

If in addition $\sigma \subset \bar{D}_0$ is a common zero set of finitely many $\mathcal{A}^r(D_0)$ functions and $\sigma \cap \bar{D}_{0,1} = \emptyset$ then g can be chosen such that g_0 agrees with f_0 to a given finite order on σ.

The analogous result holds for 1-parametric sprays.

Proposition 2.4 essentially follows from Proposition 4.3 in [5]. For later reference we recall the main steps of the proof:

(i) Lemma 4.4 in [5] gives a domain $P_1 \Subset P_0$ containing the origin and a *transition map* $\gamma\colon \bar{D}_{0,1} \times P_1 \to \mathbb{C}^N$ of class $\mathcal{A}^r(D_{0,1} \times P_1)$, close to the map $(z,t) \to t$ in the $\mathcal{C}^r$ topology (the closeness depending on the $\mathcal{C}^r$ distance of f' to f on $\bar{D}_{0,1} \times P_0$) and satisfying
$$f(z,t) = f'(z, \gamma(z,t)), \quad z \in \bar{D}_{0,1},\ t \in P_1.$$

(ii) Let $P \Subset P_1$ be a domain containing the origin $0 \in \mathbb{C}^N$. If γ is sufficiently $\mathcal{C}^r$-close to $\gamma_0(z,t) = t$ on $\bar{D}_{0,1} \times P_1$ then Theorem 3.2 in [5] furnishes maps $\alpha\colon \bar{D}_0 \times P \to \mathbb{C}^N$ and $\beta\colon \bar{D}_1 \times P \to \mathbb{C}^N$, of class $\mathcal{A}^r$ on their respective domains, satisfying
$$\gamma(z, \alpha(z,t)) = \beta(z,t), \quad z \in \bar{D}_{0,1},\ t \in P.$$

(iii) It follows that the sprays $f(z,\alpha(z,t))$ and $f'(z,\beta(z,t))$, defined on $\bar D_0 \times P$ resp. on $\bar D_1 \times P$, amalgamate into a spray $g\colon \bar D \times P \to Z$ with the stated properties. (Compare with (4.4) in [5].)

The additional properties of g follow from the construction in [5].

3. Approximation of sprays on convex domains

The following approximation result will be used in §4.

Lemma 3.1. *Let $C \subset B \Subset \mathbb{C}^n$ be a pair of strongly convex domains with $\mathcal{C}^2$ boundaries in $\mathbb{C}^n$, let $P \subset \mathbb{C}^N$ be an open set containing the origin, let $r \ge 2$ be an integer, and let Y be a complex manifold. Given a spray of maps $f\colon \bar C \times P \to Y$ of class $\mathcal{A}^r(C,Y)$ whose central map $f_0 = f(\cdot\,,0)$ extends to a map $\bar B \to Y$ of class $\mathcal{A}^r(B,Y)$, there exist*

(i) *a sequence of open sets $P_0 \supset P_1 \supset P_2 \supset \cdots$ in $\mathbb{C}^N$, with $P_0 \subset P$ and $0 \in P_j$ for all $j \in \mathbb{Z}_+$,*

(ii) *a sequence of relatively open domains $\Omega_j \subset \bar B \times \mathbb{C}^N$ satisfying*

$$(\bar C \times \bar P_0) \cup (\bar B \times \bar P_j) \subset \Omega_j, \quad j = 0,1,2,\ldots,$$

(iii) *a sequence of maps $F_j\colon \Omega_j \to Y$ of class $\mathcal{A}^r(\Omega_j, Y)$ $(j \in \mathbb{Z}_+)$*

such that $F_j(\cdot\,,0) = f_0$ on $\bar B$ for all $j \in \mathbb{Z}_+$ and the sequence F_j converges to f in the $\mathcal{C}^r$ topology on $\bar C \times \bar P_0$ as $j \to \infty$.

If in addition we are given a vector bundle map $L\colon \bar B \times \mathbb{C}^N \to TY$ of class $\mathcal{A}^r(B)$ covering f_0 (i.e., such that the linear map $L_z\colon \{z\} \times \mathbb{C}^N \to T_{f(z,0)}Y$ is of class $\mathcal{C}^r$ in $z \in \bar B$ and is holomorphic in B) and satisfying $L_z = \partial_t f(z,t)|_{t=0}$ for $z \in \bar C$ then the sequence F_j can be chosen such that, in addition to the above, $\partial_t F_j(z,t)|_{t=0} = L_z$ for every $z \in \bar B$ and $j \in \mathbb{Z}_+$.

If $Y = \mathbb{C}^M$ then all of the above hold for all $r \ge 0$.

Proof. Consider first the case $Y = \mathbb{C}$. We may assume that C contains the origin $0 \in \mathbb{C}^n$ in its interior. Taylor expansion in the t variable gives

$$f(z,t) = f_0(z) + \sum_{j=1}^{N} g_j(z,t)\, t_j, \quad (z,t) \in \bar C \times P,$$

for some $g_j \in \mathcal{A}^r(C \times P)$. For each $s < 1$ the function $g_j^s(z,t) = g_j(sz,t)$ is holomorphic in $\frac{1}{s}C \times P \supset \bar C \times P$; by choosing s close to one we insure that the approximation is as close as desired in $\mathcal{C}^r(\bar C \times P)$. Fix an s, choose a polydisc $P_0 \Subset P$ containing $0 \in \mathbb{C}^N$ and apply Runge's theorem to approximate g_j^s in $\mathcal{C}^r(\bar C \times \bar P_0)$ by an entire function $\widetilde g_j$. The function

$$F(z,t) = f_0(z) + \sum_{j=1}^{N} \widetilde g_j(z,t)\, t_j, \quad (z,t) \in \bar B \times \mathbb{C}^N,$$

then approximates f on $\bar{C} \times \bar{P}_0$, and it agrees with f_0 on $\bar{B} \times \{0\}$. This gives a desired sequence F_j on the fixed domain $\Omega = \bar{B} \times P$.

Assume in addition that $L\colon B \times \mathbb{C}^N \to \mathbb{C}$ is a map of class $\mathcal{A}^r(B \times \mathbb{C}^N)$ which is linear in the second variable and satisfies $L_z = \frac{\partial}{\partial t}|_{t=0} f(z,t)\colon \mathbb{C}^N \to \mathbb{C}$ for $z \in \bar{C}$. By Taylor's formula we have

$$f(z,t) = f_0(z) + L_z(t) + \sum_{j,k=1}^{N} g_{j,k}(z,t) t_j t_k, \quad (z,t) \in \bar{C} \times P,$$

for some $g_{j,k} \in \mathcal{A}^r(C \times P)$. Approximating each $g_{j,k}$ in $\mathcal{C}^r(\bar{C} \times \bar{P}_0)$ by an entire function $\widetilde{g}_{j,k}\colon \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{C}$ and setting

$$F(z,t) = f_0(z) + L_z(t) + \sum_{j,k=1}^{N} \widetilde{g}_{j,k}(z,t) t_j t_k, \quad (z,t) \in \bar{B} \times \mathbb{C}^N,$$

gives the desired approximation.

The case $Y = \mathbb{C}^M$ follows by applying the above result to each component.

Assume now that Y is an arbitrary complex manifold and that $r \geq 2$. By [5, Theorem 1.6] the graph $\{(z, f_0(z))\colon z \in \bar{B}\}$ admits an open Stein neighborhood $U \subset \mathbb{C}^n \times Y$. Choose a properly holomorphic embedding $\Phi\colon U \to \mathbb{C}^M$, an open neighborhood $V \subset \mathbb{C}^M$ of $\Phi(U)$ and a holomorphic retraction $\iota\colon V \to \Phi(U)$ onto $\Phi(U)$. We apply the already proved approximation result to the map $(z,t) \to \widetilde{f}(z,t) := \Phi(z, f(z,t)) \in \mathbb{C}^M$ to get a sequence $\widetilde{F}_j$ satisfying the conclusion of Lemma 3.1 with respect to $\widetilde{f}$. Let $pr_Y\colon \mathbb{C}^n \times Y \to Y$ denote the projection onto the second factor. Assuming that $\widetilde{F}_j$ approximates $\widetilde{f}$ sufficiently closely on $(\bar{C} \times \bar{P}_0) \cup (\bar{B} \times \{0\})$, the latter set has an open neighborhood Ω_j on which the map $F_j = pr_Y \circ \Phi^{-1} \circ \iota \circ \widetilde{F}_j$ is defined, and the resulting sequence F_j satisfies the conclusion. □

4. Linearization around a section

We shall prove that each holomorphic section over a strongly pseudoconvex domain which is continuous up to the boundary can be embedded into a dominating spray of sections (Corollary 4.2). This is the key to all main results of the paper. Applying Cartan's Theorem B for holomorphic vector bundles which are continuous (or smooth) up to the boundary we then obtain an up to the boundary version of Grauert's tubular neighborhood theorem (Corollary 4.3), and Theorem 1.1 easily follows.

The main step is provided by the following result obtained by combining Propositions 2.4 and 3.1. (Compare with [5, Lemma 4.2].)

Proposition 4.1. *Assume that D is a relatively compact, strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell \geq 2$) in a Stein manifold S, $r \in \{0, 1, \ldots, \ell\}$, and $h\colon X \to \bar{D}$ is either an $\mathcal{A}^r_Y(D)$-bundle (Def. 1.6) or the restriction to $\bar{D}$ of a holomorphic submersion $\widetilde{h}\colon \widetilde{X} \to S$. Given a section*

$f_0 \in \Gamma^r_{\mathcal{A}}(D, X)$ and a surjective complex vector bundle map $L\colon \bar{D} \times \mathbb{C}^N \to VT(X)|_{f_0(\bar{D})}$ of class $\mathcal{C}^r$ which covers f_0 and is holomorphic over D, there exist a domain $P \subset \mathbb{C}^N$ containing the origin and a (dominating) h-spray $f\colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$ satisfying

$$f(z, 0) = f_0(z), \quad \partial_t|_{t=0} f(z, t) = L_z, \quad z \in \bar{D}. \tag{4.1}$$

Furthermore, given a $\mathcal{C}^r$ homotopy of sections $f_0^s \in \Gamma^r_{\mathcal{A}}(D, X)$ ($s \in [0, 1]$), covered by a $\mathcal{C}^r$ homotopy of surjective complex vector bundle maps $L^s\colon \bar{D} \times \mathbb{C}^N \to VT(X)|_{f_0^s(\bar{D})}$ which are holomorphic over D such that for $s = 0, 1$ the map f_0^s is the central map of a spray f^s over $\bar{D} \times P$ and $\partial_t|_{t=0} f^s(z, t) = L_z^s$ ($z \in \bar{D}$), there exist a domain $P_1 \subset P$ containing the origin and a 1-parametric spray $F\colon [0, 1] \times \bar{D} \times P_1 \to X$ of class $\mathcal{A}^r(D)$ such that F^s agrees with f^s on $\bar{D} \times P_1$ for $s = 0, 1$ and

$$F^s(z, 0) = f_0^s(z), \quad \partial_t|_{t=0} F^s(z, t) = L_z^s, \quad z \in \bar{D}, \ s \in [0, 1]. \tag{4.2}$$

Proof. The conditions on $h\colon X \to \bar{D}$ imply that each point $x_0 \in X$ admits an open neighborhood $W \subset X$ isomorphic to a product $U \times V$, where U is a (relatively) open subset of $\bar{D}$ and V is an open subset of a Euclidean space $\mathbb{C}^l$, such that in the coordinates $x = (z, w)$ ($z \in U, \ w \in \mathbb{C}^l$), h is the projection $(x, w) \to x$. Such coordinate neighborhoods in X will be called *special* (with respect to h).

By Lemma 12.3 in [22] there exist strongly pseudoconvex domains $D_0 \subset D_1 \subset \cdots \subset D_m = D$ with $\mathcal{C}^\ell$ boundaries such that $\bar{D}_0 \subset D$, and for every $j = 0, 1, \ldots, m-1$ we have $D_{j+1} = D_j \cup B_j$ where B_j is a convex bump on D_j (Def. 2.3). Each of the sets B_j may be chosen so small that $f_0(\bar{B}_j)$ is contained in a special coordinate neighborhood of X. The essential ingredient in the proof is Narasimhan's lemma on local convexification.

By Lemma 5.3 in [13] there exists a dominating h-spray f with core f_0 over a neighborhood of $\bar{D}_0$. We recall the main idea of the proof and show that f can be chosen to satisfy $\partial_t|_{t=0} f(z, t) = L_z$ for $z \in \bar{D}_0$. Let $\{e_j\}_{j=1}^N$ be the standard basis of $\mathbb{C}^N$. Set

$$L_j(f_0(z)) := L_z e_j \in VT_{f_0(z)} X, \quad j = 1, \ldots, N.$$

Note that $f_0(D)$ is a closed complex submanifold of $X|_D = h^{-1}(D)$ and hence admits an open Stein neighborhood $\Omega \subset X|_D$ [45]. Each L_j is a holomorphic section of the vertical tangent bundle $VT(X)$ on the submanifold $f_0(D)$ of Ω, and by Cartan's Theorem B it extends to a holomorphic vertical vector field on Ω. Denote by θ_t^j its flow. The map

$$f(z, t_1, \ldots, t_N) = \theta_{t_N}^N \circ \cdots \circ \theta_{t_2}^2 \circ \theta_{t_1}^1 (f_0(z)),$$

which is well defined and holomorphic for z in a neighborhood of $\bar{D}_0$ and for $t = (t_1, \ldots, t_N)$ in an open set $P \subset \mathbb{C}^N$ containing the origin, is then a dominating spray satisfying $f(\cdot\,, 0) = f_0$ and $\partial_t|_{t=0} f(z, t) = L_z$.

To find a desired spray on $\bar{D}$ we perform a stepwise extension of f over the convex bumps $B_0, \ldots, B_{m-1}$. At the j-th step we assume that we have a spray $\bar{D}_j \times P_j \to X$ with the required properties, and we shall approximate it by a spray $\bar{D}_{j+1} \times P_{j+1} \to X$ with a possibly smaller parameter set $0 \in P_{j+1} \subset P_j$. Since all steps are of the same kind, it suffices to explain the details for the first step $j = 0$.

Applying Lemma 3.1 with the sets $C = D_0 \cap B_0$, $B = B_0$ (which are strictly convex in local holomorphic coordinates) and $Y = \mathbb{C}^M$ (since $f_0(\bar{B}_0)$ is contained in a special coordinate chart of X), we find an open set $0 \in P' \subset P_0$, a relatively open set $\Omega \subset \bar{B} \times \mathbb{C}^N$ containing $(\bar{C} \times \bar{P}_0) \cup (\bar{B} \times \bar{P}')$, and a map $f' \colon \Omega \to \mathbb{C}^M$ of class $\mathcal{A}^r(\Omega)$ which approximates f in the $\mathcal{C}^r$ topology on $\bar{C} \times P_0$, such that $f(z,0) = f'(z,0)$ and $\partial_t|_{t=0} f(z,t) = \partial_t|_{t=0} f'(z,t)$ for $z \in \bar{C}$. We now proceed as in the proof of Proposition 2.4. Assuming as we may that the approximation of f by f' is sufficiently close in the $\mathcal{C}^r$ topology on $\bar{C} \times P_0$, Lemma 4.4 in [5] furnishes a transition map $\gamma(z,t) = z + c(z,t)$ of class $\mathcal{A}^r$ between f and f', defined for $z \in \bar{C}$ and t in a smaller parameter set $0 \in P' \Subset P_0 \subset \mathbb{C}^N$, such that γ is $\mathcal{C}^r$ close to $\gamma_0(z,t) = t$ (depending on the closeness of f' to f), γ agrees with γ_0 to order one at $t = 0$, and $f(z,t) = f'(z, \gamma(z,t))$ for $(z,t) \in \bar{C} \times P'$.

Applying [5, Theorem 3.2] to γ on the Cartan pair (D_0, B_0) we obtain a smaller parameter set $P_1 \subset P'$ and maps $\alpha(z,t) = t + a(z,t)$ on $\bar{D}_0 \times P_1$, $\beta(z,t) = t + b(z,t)$ on $\bar{B}_0 \times P_1$, of class $\mathcal{A}^r$ and close to $\gamma_0(z,t) = t$ on their respective domains, agreeing with γ_0 to order 1 at $t = 0$, and such that $\gamma(z, \alpha(z,t)) = \beta(z,t)$ for $z \in \bar{C}_0$ and $t \in P_1$. (See Step (ii) in the proof of Proposition 2.4 above.) Now $(z,t) \to f(z, \alpha(z,t))$ is a spray of class $\mathcal{A}^r$ on $\bar{D}_0 \times P_1$ which is $\mathcal{C}^r$-close to f and agrees with f to order one at $t = 0$, $f'(z, \beta(z,t))$ is a spray with the analogous properties on $\bar{B}_0 \times P_1$, and by construction the two sprays agree on $\bar{C}_0 \times P_1$; hence they define a spray satisfying (4.1) on the set $\bar{D}_1 = \bar{D}_0 \cup \bar{B}_0$. After m steps of this kind we obtain the first part of the Proposition.

We continue with the parametric case (this will only be used in the proof of Theorem 1.7). Fix an $s \in [0,1]$. By the first part of the Proposition there exists an h-spray $f^s \colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$ satisfying

$$f^s(z,0) = f^s_0(z), \quad \partial_t|_{t=0} f^s(z,t) = L^s_z, \quad z \in \bar{D}.$$

For $s = 0,1$ we use the already given sprays. We wish to choose these sprays to depend smoothly on the parameter s. To do this, we shall first use a fixed spray f^s to find a solution in an open interval $I_s \subset \mathbb{R}$ around s, and finally we shall patch these solutions together.

Fix a number $u \in [0,1]$. Since L^u is surjective, there is a direct sum splitting $\bar{D} \times \mathbb{C}^N = E \oplus G$, where E and G are vector bundles of class $\mathcal{A}^r(D)$ and $E_z = \ker L^u_z$ for each $z \in \bar{D}$. (We use Theorem B for $\mathcal{A}(D)$-bundles, due to Leiterer [35], and Heunemann's approximation theorem [24]; compare with the proof of Lemma 4.4 in [5].) We split the fiber variable on $\{z\} \times \mathbb{C}^N$

accordingly as $t = t'_z \oplus t''_z \in E_z \oplus G_z$. Note that $L^u\colon G \to VT(X)|_{f^u(\bar D)}$ is a complex vector bundle isomorphism of class $\mathcal{A}^r(D)$. By the inverse function theorem there is an open interval $I_u = (u-\delta, u+\delta) \subset \mathbb{R}$ such that for each $s \in I_u \cap [0,1]$ there exists a unique section $g_s\colon \bar D \to G$ of class $\mathcal{A}^r(D)$ satisfying

$$f^u(z, 0'_z \oplus g_s(z)) = f^s_0(z), \quad z \in \bar D.$$

It follows that the map

$$H^s(z,t) \mapsto f^u\big(z, t'_z \oplus (t''_z + g_s(z))\big)$$

is a dominating 1-parametric spray with the core f^s_0 for $s \in I_u \cap [0,1]$. Note that for $s = u$ we have $g_u = 0$ and $H^u = f^u$.

It remains to adjust the t-differential at $t = 0$. Elementary linear algebra shows that, after shrinking the interval I_u if necessary, there exist for every $s \in I_u$ a unique complex vector bundle automorphism $A^s\colon G \to G$ and a unique complex vector bundle map $B^s\colon E \to G$, both of class $\mathcal{A}^r(D)$, such that the map

$$F^s(z,t) = H^s\big(z, t'_z \oplus (B^s_z t'_z + A^s_z t''_z)\big), \quad s \in I_u \cap [0,1],$$

is a 1-parametric spray satisfying $\partial_t|_{t=0} F^s(z,t) = L^s_z(t)$ for all $z \in \bar D$ and $s \in I_u \cap [0,1]$.

The above argument gives a finite covering of $[0,1]$ by intervals $I_j = [a_j, b_j]$ ($j = 0,1,\ldots,m$), where $a_0 = 0 < a_1 < b_0 < a_2 < b_1 < \cdots < b_m = 1$, and for each j a 1-parametric spray $F_j\colon I_j \times \bar D \times P \to X$ satisfying the conclusion of the Proposition on I_j. It remains to patch the sprays $\{F_j\}_{j=0}^m$ into a 1-parametric spray F on $[0,1]$.

Note that each pair of adjacent intervals I_j, I_{j+1} intersect in the segment $[a_{j+1}, b_j]$, while each three intervals are disjoint. Hence it suffices to explain how to patch F_j and F_{j+1} to a spray over $I_j \cup I_{j+1}$. Choose a point $u \in (a_{j+1}, b_j)$. Using a decomposition $\bar D \times \mathbb{C}^N = E \oplus G$ as above, with $E = \ker \partial_t|_{t=0} F_j(u, \cdot\,, t)$, the implicit function theorem gives a segment $J_j = [\alpha_j, \beta_j] \subset (a_{j+1}, b_j)$ with $\alpha_j < u < \beta_j$, a polydisc $P_0 \subset P$ containing $0 \in \mathbb{C}^N$, and a unique map $\gamma(s,z,t) = t'_z \oplus (t''_z + c(s,z,t))$ of class $\mathcal{C}^r(J_j \times \bar D \times P_0)$, holomorphic with respect to $(z,t) \in D \times P_0$, such that

$$F_j(s,z,\gamma(s,z,t)) = F_{j+1}(s,z,t), \quad s \in I_j,\ z \in \bar D,\ t \in P_0$$

and $c(s,z,t)$ vanishes to second order at $t = 0$. (The special form of the transition map γ is insured by the fact that the first order jets of the sprays F_j and F_{j+1} with respect to t agree at $t = 0$ for every $s \in I_j$. For more details see Lemma 4.4 in [5].) Choose a smooth function $\chi\colon \mathbb{R} \to [0,1]$ such that $\chi(s) = 0$ for $s \le \alpha_j$ and $\chi(s) = 1$ for $s \ge \beta_j$. The map

$$(s,z,t) \mapsto F_j\big(s, z, t'_z \oplus (t''_z + \chi(s) c''(s,z,t))\big)$$

is then a 1-parametric spray of class $\mathcal{A}^r(D)$ satisfying the Proposition on $I_j \cup I_{j+1}$. After m steps we obtain a solution on $[0,1]$. □

A map L as in Proposition 4.1 always exists for a sufficiently large N as follows from Cartan's Theorem A for coherent sheaves of $\mathcal{A}^r$ modules on strongly pseudoconvex domains ([25, Theorem 6], [38]). Hence we get

Corollary 4.2. (Existence of dominating sprays.) *Given $h\colon X \to \bar{D}$ and a section $f_0 \in \Gamma^r_{\mathcal{A}}(D,X)$ as in Proposition 4.1, there exist a domain $0 \in P \subset \mathbb{C}^N$ for some $N \in \mathbb{N}$ and a dominating h-spray $f\colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$ with the core section f_0. Furthermore, for any homotopy of sections $f_0^s \in \Gamma^r_{\mathcal{A}}(D,X)$ $(s \in [0,1])$ there exists a homotopy of dominating h-sprays $f^s\colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$ such that the core of f^s equals f_0^s for every $s \in [0,1]$.*

Corollary 4.2 implies the following 'up to the boundary' version of Grauert's tubular neighborhood theorem which allows linearization of analytic problems near a given section.

Corollary 4.3. (Linearizing a neighborhood.) *Let $h\colon X \to \bar{D}$ be as in Proposition 4.1. Given a section $f \in \Gamma^r_{\mathcal{A}}(D,X)$ there exist a holomorphic vector bundle $\pi\colon E \to \widetilde{D}$ over an open neighborhood $\widetilde{D} \subset S$ of $\bar{D}$, a relatively open neighborhood Ω of the zero section of the restricted bundle $E|_{\bar{D}} := \pi^{-1}(\bar{D})$ and a fiber preserving $\mathcal{C}^r$ diffeomorphism $\Phi\colon \Omega \to \Phi(\Omega) \subset X$ which is biholomorphic on $\Omega\cap\pi^{-1}(D)$ and maps the zero section of $E|_{\bar{D}}$ onto $f(\bar{D})$.*

Proof. Let $f\colon \bar{D} \times P \to X$ be a dominating spray furnished by Corollary 4.2. There is a splitting $\bar{D} \times \mathbb{C}^N = E \oplus E'$ of $\mathcal{A}^r$ vector bundles over $\bar{D}$, where $E'_z = \ker \partial_t|_{t=0} f(z,t)$ and E is a complementary bundle (Theorem B for $\mathcal{A}^r$ bundles; see [25]). The restriction of f to $\Omega = E \cap P$ satisfies the conclusion of Corollary 4.3. ☐

Note that the bundle E in Corollary 4.3 is just the normal bundle of the given section $f\colon \bar{D} \to X$, and it can be identified with the restriction of the vertical tangent bundle $VT(X) = \ker dh$ to the image $f(\bar{D})$. In the special case when $r \geq 2$ and h extends to a holomorphic submersion $\widetilde{X} \to \widetilde{D}$ over an open neighborhood $\widetilde{D}$ of $\bar{D}$ in S, $f(\bar{D})$ admits a basis of open Stein neighborhoods in $\widetilde{X}$ [5, Theorem 2.6] and the conclusion of Corollary 4.3 easily follows from standard methods.

Corollary 4.3 implies the following result concerning deformations of a given map in $\mathcal{A}^r(D,Y)$. For related results concerning maps in L^2-Sobolev classes from certain bordered Riemann surfaces to (almost) complex manifolds see Ivashkovich and Shevchishin [27], [28].

Corollary 4.4. (The deformation space of an $\mathcal{A}^r$ map.) *Assume that D is a relatively compact domain with strongly pseudoconvex boundary of class $\mathcal{C}^\ell$ $(\ell \geq 2)$ in a Stein manifold. If $r \in \{0,1,\ldots,\ell\}$ and Y is an arbitrary complex manifold then for any $f_0 \in \mathcal{A}^r(D,Y)$ the space of all $f \in \mathcal{A}^r(D,Y)$ which are sufficiently $\mathcal{C}^0$-close to f_0 is isomorphic to the space of $\mathcal{C}^0$-small sections in $\Gamma^r_{\mathcal{A}}(D,E)$, where $E = f_0^*(TY)$.*

Proof. We may consider maps $f\colon \bar{D} \to Y$ as sections of the product fibration $h\colon X = \bar{D} \times Y \to \bar{D}$. Fix $f_0 \in \Gamma^r_{\mathcal{A}}(D, X)$. Let $\Phi\colon \Omega \to \Phi(\Omega) \subset X$ be as in Corollary 4.3, where Ω is an open neighborhood of the zero section in the complex vector bundle $E = f_0^*(TY)$ and Φ maps the zero section of E onto $f_0(\bar{D})$. If $f\colon \bar{D} \to X$ is a section in $\Gamma^r_{\mathcal{A}}(D, X)$ which is sufficiently uniformly close to f_0 then $f(\bar{D}) \subset \Phi(\Omega)$, and hence $f = \Phi \circ f'$ for a unique $f' \in \Gamma^r_{\mathcal{A}}(D, E)$ with $f'(\bar{D}) \subset \Omega$. □

Proof of Theorem 1.1. The proof is similar to Lempert's construction in [36, §2], and our Corollary 4.4 plays a similar role as Lemma 2.1 in that paper. A main difference is that the sprays in our paper must be holomorphic also in the base variable $z \in D$, and their construction is a main technical difficulty in our proof.

For each $f_0 \in \mathcal{A}^r(D, Y)$ Corollary 4.4 furnishes a contractible local chart $\mathcal{U} \subset \mathcal{A}^r(D, Y)$, consisting of all maps $z \to \Phi(z, \xi(z))$ $(z \in \bar{D})$, where $\xi \in \Gamma^r_{\mathcal{A}}(D, f_0^*(TY))$ is a section with range in an open, fiberwise contractible set $\Omega_0 \subset E_0 = f_0^*(TY)$ containing the zero section. The transition map between any pair of such charts is of the form

$$\Gamma^r_{\mathcal{A}}(D, E_0) \ni \xi \to \widetilde{\xi} \in \Gamma^r_{\mathcal{A}}(D, E_1),$$

where $\widetilde{\xi}(z) = \Psi(z, \xi(z))$ $(z \in \bar{D})$ for a fiber preserving diffeomorphism Ψ of class $\mathcal{A}^r$ from an open subset of $\Omega_0 \subset E_0$ onto an open subset of $E_1 = f_1^*(TY)$. (Actually E_0 and E_1 are isomorphic since f_1 is isotopic to f_0 due to fiber contractibility of the set Ω_0.) In local coordinates (z, w) on E_0 resp. E_1, Ψ is of the form $\Psi(z, w) = (z, \psi(z, w))$. The differential of the transition map $\xi \to \widetilde{\xi}$ at ξ_0, applied to a tangent vector $\eta \in \Gamma^r_{\mathcal{A}}(D, E_0)$, equals

$$z \to \partial_2\Psi(z, \xi(z))\, \eta(z), \quad z \in \bar{D}.$$

Since the partial differential $\partial_2\Psi$ is nondegenerate and the $\mathcal{C}^r$ regularity up to the boundary is preserved when differentiating on the fiber variable, we see that the differential is a complex Banach space isomorphism effected by $\partial_2\Psi(\cdot\,, \xi(\cdot))$, and hence the transition map is biholomorphic. □

5. Uniform approximation of holomorphic sections

In this section we prove the following approximation theorem which includes Theorem 1.2 as a special case.

Theorem 5.1. *Assume that $D \Subset S$ is a strongly pseudoconvex domain of class $\mathcal{C}^\ell$ $(\ell \ge 2)$ in a Stein manifold S. Let $h\colon X \to S$ be a holomorphic submersion of a complex manifold X onto S and let $r \in \{0, 1, \ldots, \ell\}$. Every section $f \in \Gamma^r_{\mathcal{A}}(D, X)$ can be approximated in the $\mathcal{C}^r$ topology on $\bar{D}$ by sections which are holomorphic in open neighborhoods of $\bar{D}$ in S.*

Proof. Let $n = \dim S$ and $n + m = \dim X$. Denote by pr_1 resp. pr_2 the coordinate projections of $\mathbb{C}^n \times \mathbb{C}^m$ onto the respective factors $\mathbb{C}^n$, $\mathbb{C}^m$, and let $B \subset \mathbb{C}^n$, $B' \subset \mathbb{C}^m$ denote the unit balls.

Since $h\colon X \to S$ is a holomorphic submersion, there exist for each point $x_0 \in X$ open neighborhoods $x_0 \in W \subset X$, $h(x_0) \in V \subset S$, and biholomorphic maps $\phi\colon V \to B \subset \mathbb{C}^n$, $\Phi\colon W \to B \times B' \subset \mathbb{C}^n \times \mathbb{C}^m$, such that $\phi(h(x)) = pr_1(\Phi(x))$ for every $x \in W$. Thus Φ is of the form

$$\Phi(x) = (\phi(h(x)), \phi'(x)) \in B \times B', \quad x \in W,$$

where $\phi' = pr_2 \circ \Phi$. Let us call such (W, V, Φ) a *special coordinate chart* on X (with respect to h). Note that $h(W) = V$.

Fix a section $f\colon \bar{D} \to X$ in $\Gamma^r_{\mathcal{A}}(D, X)$. Using Narasimhan's lemma on local convexification of a strongly pseudoconvex hypersurface we find finitely many special coordinate charts (W_j, V_j, Φ_j) on X, with $\Phi_j = (\phi_j \circ h, \phi'_j)$, such that $bD \subset \cup_{j=1}^{j_0} V_j$ and the following hold for $j = 1, \ldots, j_0$:

(i) $\phi_j(bD \cap V_j)$ is a strongly convex hypersurface in the ball $B \subset \mathbb{C}^n$,
(ii) $f(\bar{D} \cap V_j) \subset W_j$, and
(iii) $\overline{\phi'_j(f(D \cap V_j))} \subset B'$.

Choose a number $c < 1$ sufficiently close to 1 such that the sets $U_j = \phi_j^{-1}(cB) \Subset V_j$ $(j = 1, \ldots, j_0)$ still cover bD.

By a finite induction we shall construct an increasing sequence of strongly pseudoconvex domains with $\mathcal{C}^\ell$ boundaries $D = D_0 \subset D_1 \subset \cdots \subset D_{j_0} \Subset S$ and sections $f_j \in \Gamma^r_{\mathcal{A}}(D_j, X)$ $(j = 0, 1, \ldots, j_0)$, with $f_0 = f$, such that for every $j = 1, \ldots, j_0$ the restriction of f_j to $\bar{D}_{j-1}$ will be close to the previous section f_{j-1} in $\Gamma^r_{\mathcal{A}}(\bar{D}_{j-1}, X)$. The domain D_j of f_j will in general depend on the rate of approximation on D_{j-1} and will be chosen such that

$$D_{j-1} \subset D_j \subset D_{j-1} \cup V_j, \quad bD_{j-1} \cap U_j \subset D_j$$

for $j = 1, \ldots, j_0$; that is, we enlarge D_{j-1} inside the j-th coordinate neighborhood V_j so that the part of bD_{j-1} which lies in the smaller set U_j is contained in the next domain D_j. The final domain D_{j_0} will contain $\bar{D}$ in its interior, and the section $f_{j_0} \in \Gamma^r_{\mathcal{A}}(D_{j_0}, X)$ will approximate f as close as desired in $\mathcal{C}^r(\bar{D}, X)$. To keep the induction going we will also insure at every step that the properties (ii) and (iii) above remain valid with (D, f) replaced by (D_j, f_j) for all $j = 1, \ldots, j_0$.

Since all steps will be of exactly the same kind, we shall explain how to get (D_1, f_1) from $(D, f) = (D_0, f_0)$. We begin by finding a domain D'_1 with $\mathcal{C}^\ell$ boundary in S which is a convex bump on $D = D_0$ (Def. 2.3) such that $\overline{U}_1 \cap \bar{D}_0 \subset \bar{D}'_1 \subset V_1$. We shall first find a set $\widetilde{D}'_1 \Subset B$ with desired properties and then take $D'_1 = \phi_1^{-1}(\widetilde{D}'_1)$.

Choose a smooth function $\chi \geq 0$ with compact support contained in $B \subset \mathbb{C}^n$ such that $\chi = 1$ on cB. Let $\tau\colon B \to \mathbb{R}$ be a strongly convex defining function for the domain $\phi_1(D \cap V_1) \subset B$. Choose $c' \in (c, 1)$ close to 1 such that the hypersurface $\phi_1(bD \cap V_1) = \{\tau = 0\}$ intersects the sphere $\{|z| = c'\}$ transversely. If $\delta > 0$ is chosen sufficiently small then the set

$$\{z \in \mathbb{C}^n \colon |z| < c',\ \tau(z) + \delta\chi(z) < 0\}$$

satisfies the required properties, except that it is not smooth along the intersection of the hypersurfaces $\{|z| = c'\}$ and $\{\tau + \delta\chi = 0\}$. By rounding off the corners of the intersection we get a strongly convex set $\widetilde{D}'_1$ in B such that $D'_1 = \phi_1^{-1}(\widetilde{D}'_1) \subset V_1$ satisfies the desired properties (fig. 1).

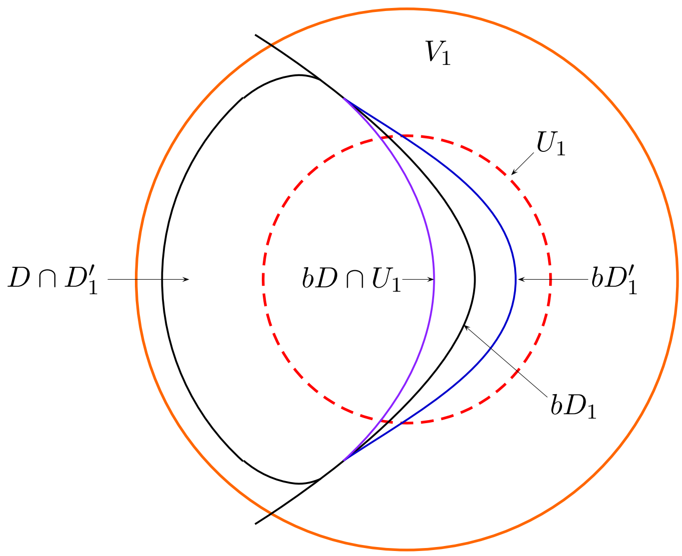

FIGURE 1. The domains D'_1 and D_1

Proposition 4.1 furnishes a dominating h-spray $F\colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$ with the core section $F_0 = f$. By shrinking its parameter set $P \subset \mathbb{C}^N$ if necessary we can insure that properties (ii) and (iii) above are satisfied if we replace $f = F_0$ by any section $F_t = F(\cdot\,, t)$, $t \in P$.

By using the special coordinate chart (W_1, V_1, Φ_1) we find an open set $\Omega \subset V_1$ containing $\bar{D} \cap V_1$ and a holomorphic h-spray $G\colon \Omega \times P \to X|_\Omega = h^{-1}(\Omega)$, with range contained in W_1, such that the restriction of G to $(\bar{D} \cap V_1) \times P$ approximates F as close as desired in the $\mathcal{C}^r$ topology. (The size of Ω will depend on G.)

We wish to use Proposition 2.4 to glue F and G into a single h-spray. We cannot do this directly since their domains do not form a Cartan pair. (The domain of G need not contain all of the set $\bar{D}'_1$ which forms a Cartan pair with D.) We proceed as follows. If G is sufficiently close to F in the $\mathcal{C}^r$ topology on the intersection of their domains, Lemma 4.4 in [5] furnishes a transition map $\gamma(z,t) = z + c(z,t)$ of class $\mathcal{A}^r$ between F and G, defined for $z \in \bar{D} \cap \bar{D}'_1$ and t in a smaller parameter set $P' \Subset P \subset \mathbb{C}^N$, such that γ is $\mathcal{C}^r$ close to $\gamma_0(z,t) = t$ (depending on the closeness of G to F), and $F(z,t) = G(z, \gamma(z,t))$ for $z \in \bar{D} \cap \bar{D}'_1$ and $t \in P'$.

Theorem 3.2 in [5], applied to γ on the Cartan pair (D, D'_1), furnishes a smaller parameter set $P_1 \subset P'$ and maps $\alpha(z,t) = t + a(z,t)$ on $\bar{D} \times P_1$,

$\beta(z,t)=t+b(z,t)$ on $\bar{D}_1'\times P_1$, of class $\mathcal{A}^r$ and close to γ_0 on their respective domains, such that $\gamma(z,\alpha(z,t))=\beta(z,t)$ for $z\in \bar{D}\cap\bar{D}_1'$ and t close to 0. The spray $F(z,\alpha(z,t))$ is then defined on $\bar{D}\times P_1$ and is close to F, the spray $G(z,\beta(z,t))$ is defined on $(\bar{D}_1'\cap\Omega)\cap P_1$, and by the construction these sprays agree for $z\in\bar{D}\cap\bar{D}_1'\subset\Omega$. The central section f_1 of the new amalgamated spray, obtained by setting $t=0$, is then of class $\mathcal{A}^r$ on $\bar{D}\cup(\bar{D}_1'\cap\Omega)$.

It remains to restrict f_1 to a suitably chosen strongly pseudoconvex domain $D_1\Subset S$ contained in $D\cup(D_1'\cap\Omega)$ and satisfying the other required properties. We choose D_1 such that it agrees with D outside of V_1, while

$$D\cap V_1=\phi_1^{-1}(\{z\in B\colon \tau(z)+\epsilon\chi(z)<0\})$$

for a small $\epsilon>0$ (fig. 1). By choosing $\epsilon>0$ sufficiently small (depending on f_1) we insure that the hypotheses (i)–(iii) are satisfied by the pair (D_1,f_1). This completes the first step.

Applying the same procedure to (D_1,f_1) and the second chart (W_2,V_2,Φ_2) we get the next pair (D_2,f_2). After j_0 steps we find a section f_{j_0} over a neighborhood D_{j_0} of $\bar{D}$ in S which approximates f as close as desired in the $\mathcal{C}^r$ topology on $\bar{D}$. By construction all steps are made by homotopies, and hence the restriction of f_{j_0} to $\bar{D}$ is homotopic to $f=f_0$ in $\Gamma^r_{\mathcal{A}}(D,X)$. □

Proof of Theorem 1.3. Assume that D is a relatively compact connected domain with $\mathcal{C}^2$ boundary in an open Riemann surface S and $f_0\colon\bar{D}\to Y$ is a map of class $\mathcal{A}(D,Y)$ to an n-dimensional complex space Y. We wish to prove that f_0 can be uniformly approximated by maps which are holomorphic in open neighborhoods of $\bar{D}$ in S. (The analogous result for maps of class $\mathcal{A}^r(D,Y)$ can be proved in a similar way.)

We proceed by induction on the dimension of Y. Assume that the result already holds for complex spaces of dimension $<n=\dim Y$ (this is trivially satisfied when $n=1$). If $f_0(\bar{D})\subset Y_{sing}$, the inductive hypothesis gives approximation of f_0 by holomorphic maps to Y_{sing}. (It may happen that the image of any nearby holomorphic map is contained in Y_{sing}; see the example in [16].)

Suppose now that $f_0(\bar{D})\not\subset Y_{sing}$. The set $\sigma=\{z\in\bar{D}\colon f_0(z)\in Y_{sing}\}$ is locally the common zero set of finitely many functions of class $\mathcal{A}(D)$; hence σ is a closed subset of $\bar{D}$ such that $\sigma\cap D$ is discrete in D and $\sigma\cap bD$ has linear measure zero in bD.

We shall need the following result analogous to Corollary 4.2.

Lemma 5.2. *(Notation as above.) There exists a spray of maps $f\colon\bar{D}\times P\to Y$ of class $\mathcal{A}(D)$ which is dominating on $\bar{D}\backslash\sigma$ and whose core map is f_0.*

Proof. As in the proof of Proposition 4.1 we choose a finite sequence of domains $D_0\subset D_1\subset\cdots\subset D_m=D$ with $\mathcal{C}^2$ boundaries such that $\bar{D}_0\subset D$, and for every $j=0,1,\ldots,m-1$ we have $D_{j+1}=D_j\cup B_j$ where B_j is a convex bump on D_j (Def. 2.3). In addition, the properties of the set

$\sigma = f_0^{-1}(Y_{sing})$ imply that the sets B_j and D_j can be chosen such that the following hold for each $j = 0, \ldots, m-1$:

(i) $\sigma \cap bD_j \cap D = \emptyset$,
(ii) $\sigma \cap bD_j \cap bD$ is contained in the relative interior of $bD_j \cap bD$ (in bD),
(iii) $\sigma \cap \bar{D}_j \cap \bar{B}_j = \emptyset$.

Lemma 4.2 in [5] furnishes sprays $f\colon \bar{D}_0 \times P \to Y$ and $f'\colon \bar{B}_0 \times P \to Y$, satisfying the required properties over $\bar{D}_0$ resp. $\bar{B}_0$. In particular, the core of each of these two sprays is f_0, restricted to the respective domain, and the exceptional set (the set where the spray fails to dominate) is σ. Note that both sprays are dominating over the convex set $\bar{C}_0 = \bar{D}_0 \cap \bar{B}_0$ by (iii), and they agree for $t = 0$.

The proof of Lemma 4.4 in [5] gives a map $\gamma(z,t)$ of class $\mathcal{A}(C_0 \times P)$ (after shrinking P around 0), satisfying $f(z,t) = f'(z, \gamma(z,t))$ for $(z,t) \in \bar{C}_0 \times P$ and $\gamma(z,0) = 0$ for $z \in \bar{C}_0$. (In that proof we have assumed that the two sprays are close to each other, but in the present situation this is not necessary since $f(z,0) = f'(z,0)$ for $z \in \bar{C}_0$. It suffices to find a direct summand of class $\mathcal{A}(C_0)$ in $\bar{D}_0 \times \mathbb{C}^N$ to the kernel of the t-derivative of each of the two sprays and apply the implicit function theorem as in [5].)

As in the proof of Lemma 3.1 we approximate γ uniformly on $\bar{C}_0 \times P$ by an entire map $\gamma'(z,t)$ satisfying $\gamma'(z,0) = 0$ for $z \in \mathbb{C}^n$. The spray $(z,t) \to f'(z, \gamma'(z,t))$ is then defined on $(\bar{C}_0 \times P) \cup (\bar{B}_0 \times P_0)$ for some polydisc $P_0 \subset \mathbb{C}^n$ around 0 (which may depend on γ'), and it agrees with f for $t = 0$. If the two sprays are sufficiently uniformly close to each other on $\bar{C}_0 \times P$ (as we may assume to be the case), we can glue them into a new spray $\widetilde{f}$ with the required properties over $\bar{D}_1 = \bar{D}_0 \cup \bar{B}_0$. (See of Proposition 4.1 above, or the proof of Proposition 4.3 in [5]. The important point is that our sprays agree along $t = 0$, so we get a nonempty parameter set for the new spray.) Note also that the gluing process does not increase the exceptional set of the spray. After finitely many steps of this kind we get a desired spray f over $\bar{D}$. □

To complete the proof of Theorem 1.3 we proceed exactly as in the proof of Theorem 5.1 in [5]. The main idea is to attach to $\bar{D}$ a convex bump B such that $\sigma \cap \bar{D} \cap \bar{B} = \emptyset$ (this is possible since $\sigma \cap bD$ has empty interior in bD) and then approximate f by another spray over $K = \bar{D} \cup \bar{B}$. Let $g\colon K \to Y$ denote the core of the new spray. There exists a holomorphic vector field ξ in a neighborhood of K in S which points into D at every point of $bD \backslash B$ (we put no condition on ξ at the points of $bD \cap \bar{B}$; see [5]). Denoting its flow by ϕ_t, the map $g \circ \phi_t$ is defined and holomorphic in an open neighborhood of $\bar{D}$ for every sufficiently small $t > 0$, and it approximates g (and hence f_0) uniformly on $\bar{D}$. □

Combining Theorem 5.1 with the main result of [12] gives the following version of the Oka principle. Note that Corollary 1.5 is just Corollary 5.3 applied to the product submersion $X = S \times Y \to S$.

Corollary 5.3. *Assume that $D \Subset S$ is a strongly pseudoconvex domain with $\mathcal{O}(S)$-convex closure in a Stein manifold S and $h\colon X \to S$ is a holomorphic fiber bundle whose fiber Y enjoys* CAP *(Def. 1.4). For any continuous section $f_0\colon S \to X$ which is holomorphic on D there exists a homotopy $f_t\colon S \to X$ of continuous sections such that $f_t|_D$ is holomorphic and uniformly close to $f_0|_D$ for each $t \in [0,1]$, and f_1 is holomorphic on S.*

Furthermore, every homotopy of continuous sections $f_t\colon S \to X$, with $f_t|_D$ holomorphic for each $t \in [0,1]$ and f_0, f_1 holomorphic on S, can be deformed with fixed ends to a homotopy $\widetilde{f}_t\colon S \to X$ consisting of sections which are holomorphic on S such that the entire homotopy remains holomorphic on D.

The analogous results hold for sections of class $\mathcal{C}^r$ provided that $bD \in \mathcal{C}^\ell$ ($\ell \geq 2$) and $r \in \{0,1,\ldots,\ell\}$.

Proof. By Theorem 5.1 we can approximate f as close as desired in the $\mathcal{C}^r(\bar D, X)$ topology by a $\mathcal{C}^r$ section $g\colon U \to X$ which is holomorphic in an open neighborhood U of $\bar D$ in S. If the approximation of f by g is sufficiently close and U is chosen sufficiently small then g is homotopic to $f|_U$ by a homotopy of $\mathcal{C}^r$ sections $g_t\colon U \to X$ ($t \in [0,1]$, $g_0 = f$, $g_1 = g$) which are holomorphic in D (Corollary 4.3). Choose a smooth function $\chi\colon S \to [0,1]$ with compact support contained in U such that $\chi = 1$ in a smaller open neighborhood of $\bar D$. Set $\tilde g_t(z) = g_{\chi(z)t}(z)$; this is a homotopy of $\mathcal{C}^r$ sections which are holomorphic on D, they extend to all of S and equal f on $S \backslash U$.

Assuming that the fiber Y of $h\colon X \to S$ enjoys CAP, Theorem 1.2 in [12] shows that $\tilde g_1$ is homotopic to a holomorphic section $f_1\colon S \to X$ by a homotopy of sections $h_t\colon S \to X$ ($t \in [0,1]$, $h_0 = \tilde g_1$, $h_1 = f_1$) which are holomorphic and uniformly close to $\tilde g_1$ in an open neighborhood of $\bar D$. By combining the homotopies $\tilde g_t$ and h_t we obtain a homotopy from $f = f_0$ to f_1 satisfying the stated properties.

Similarly, a homotopy $\{f_t\}_{t\in[0,1]}$ in the second statement can be deformed (with fixed ends at $t = 0, 1$) to a homotopy $\{f'_t\}_{t\in[0,1]}$ consisting of sections which are holomorphic in an open neighborhood of $\bar D$ in S; it remains to apply the one-parametric Oka principle (Theorem 5.1 in [12]) to $\{f'_t\}$. □

6. Global approximation of sections of $\mathcal{A}^r$ bundles with flexible fibers

Let D be a relatively compact, strongly pseudoconvex domain in a Stein manifold S. Recall that $\mathcal{A}(D)$ denotes the algebra of all continuous functions on $\bar D$ which are holomorphic on D; we shall use the analogous notation for (open or closed) domains in $\bar D$.

The following result, which includes a precise version of Theorem 1.7, is the *1-parametric Oka principle for sections of $\mathcal{A}^r(D)$-bundles.*

Theorem 6.1. *Assume that S is a Stein manifold, $D \Subset S$ is a strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary ($\ell \geq 2$), $r \in \{0, 1, \ldots, \ell\}$, and $h\colon X \to \bar{D}$ is an $\mathcal{A}^r_Y(D)$-bundle (Def. 1.6). Choose a distance function d on the manifold $J^r(\bar{D}, X)$ of all r-jets of sections $\bar{D} \to X$ of h. Let K be a compact $\mathcal{A}(D)$-convex subset of $\bar{D}$ and let $U \subset S$ be an open set containing K. If the fiber Y enjoys CAP (Def. 1.4) then sections $\bar{D} \to X$ satisfy the following:*

(i) *Given a continuous section $f_0\colon \bar{D} \to X$ which is of class $\mathcal{C}^r$ on $U \cap \bar{D}$ and holomorphic in $U \cap D$, there exist for every $\epsilon > 0$ a smaller open neighborhood $V \subset U$ of K and a homotopy of sections $f_t\colon \bar{D} \to X$ ($t \in [0, 1]$) which are of class $\mathcal{C}^r$ on $V \cap \bar{D}$ and holomorphic on $V \cap D$ such that $f_1 \in \Gamma^r_{\mathcal{A}}(\bar{D}, X)$ and*
$$\sup_{x \in K} d\big(j^r_x f_t, j^r_x f_0\big) < \epsilon, \quad t \in [0, 1].$$

(ii) *Given a homotopy of continuous sections $f_t\colon \bar{D} \to X$ such that f_t is holomorphic in $U \cap D$ and of class $\mathcal{C}^r$ on $U \cap \bar{D}$ for each $t \in [0, 1]$ (with continuous dependence of $j^r f_t$ on t on $U \cap \bar{D}$), and such that $f_0, f_1 \in \Gamma^r_{\mathcal{A}}(\bar{D}, X)$, there are a smaller open neighborhood $V \subset U$ of K and a homotopy of sections $g_{t,s}\colon \bar{D} \to X$ ($t, s \in [0, 1]$) satisfying*
   (1) *$g_{t,0} = f_t$, $g_{0,s} = f_0$, $g_{1,s} = f_1$ for all $t, s \in [0, 1]$,*
   (2) *$g_{t,1} \in \Gamma^r_{\mathcal{A}}(\bar{D}, X)$ for all $t \in [0, 1]$,*
   (3) *$g_{t,s}$ is of class $\mathcal{C}^r$ on $V \cap \bar{D}$, holomorphic on $V \cap D$, and*
$$\sup_{x \in K} d\big(j^r_x g_{t,s}, j^r_x f_t\big) < \epsilon, \quad s, t \in [0, 1].$$

Proof. Since $\bar{D}$ is a compact strongly pseudoconvex domain in a Stein manifold S, there exist a smoothly bounded, strongly pseudoconvex domain $D_0 \Subset D$ very close to D and a strongly pseudoconvex domain $B_0 \subset D$ such that (D_0, B_0) is a Cartan pair of class $\mathcal{C}^\ell$ and $K \subset \bar{B}_0 \subset U$. (See fig. 2. Note that K may intersect bD.)

Let $D_1 = D_0 \cup B_0$. A suitable choice of D_0 and B_0 insures that $\bar{D}$ is obtained from $\bar{D}_1$ by attaching finitely many convex bumps (Lemma 12.3 in [22]). More precisely, we get finitely many strongly pseudoconvex domains $D_1 \subset D_2 \subset \cdots \subset D_m = D$ with $\mathcal{C}^\ell$ boundaries such that for every $j = 1, \ldots, m-1$ we have $D_{j+1} = D_j \cup B_j$, where B_j is a convex bump on D_j (Def. 2.3). Each of the sets B_j ($j = 1, \ldots, m-1$) may be chosen so small that the fiber bundle is trivial over $\bar{B}_j$.

Choose a strongly pseudoconvex domain $D' \Subset U$ such that $\bar{B}_0 \subset D'$. By Corollary 4.2 there exist a domain $P_0 \subset \mathbb{C}^N$ containing the origin and a dominating h-spray $f\colon \overline{D' \cap D} \times P_0 \to X$ of class $\mathcal{A}^r(D' \cap D)$ with the central map f_0. Since f_0 is defined on $\bar{D}$, we may assume that f extends continuously to $\bar{D} \times P_0$ and f_t is a continuous section of X for each $t \in P_0$.

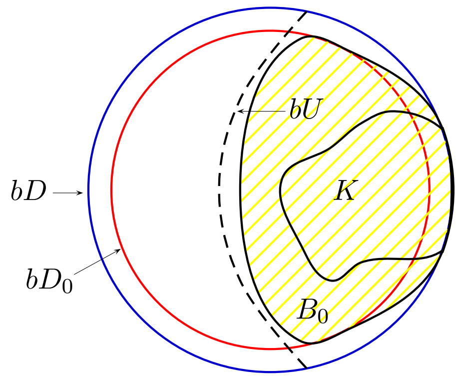

FIGURE 2. The sets K, D_0, B_0 and U

Let $\tilde f(z,t) = (f(z,t),t)$ for $(z,t) \in D \times P_0$. Define the fiber bundle map $\tilde h \colon X \times P_0 \to \bar D \times P_0$ by $\tilde h(x,t) = (h(x),t)$ and note that $\tilde f$ is a continuous section of $\tilde h$ which is holomorphic on $(D' \cap D) \times P_0$. Let $P \Subset P_0$ be a small ball centered at the origin. The set $\overline{D_0 \cap B_0} \times \overline{P}$ is a compact holomorphically convex subset of $D \times P_0$. By Oka principle with approximation on compact holomorphically convex sets [12, Theorem 1.2] there exists a homotopy of continuous sections $\tilde f^s \colon \bar D \times P_0 \to X \times P_0$ ($s \in [0,1]$) which are holomorphic and uniformly close to $\tilde f$ on $\overline{D_0 \cap B_0} \times \overline{P}$ and such that $\tilde f^1$ is holomorphic.

Denote by π the projection $D \times P_0 \to D$ onto the first factor. The map $\pi \circ \tilde f^1$ is a h-spray which approximates the spray f on $\overline{D_0 \cap B_0} \times P$. If the approximation is good enough, we can glue the sprays f and $\pi \circ \tilde f^1$ using Proposition 2.4 and thus obtain a smaller parameter set $P' \subset P$ and a h-spray f' of class $\mathcal{A}^r(D_1)$. Its central map g_1 belongs to $\Gamma^r_{\mathcal{A}}(D_1, X)$, it is close to f_0, and is homotopic to f_0 on $\bar D_1$.

In the sequel we perform finitely many steps of the same kind. At each step we find a section $g_j \in \Gamma^r_{\mathcal{A}}(D_j, X)$ ($j = 2,3,\ldots,m$) whose restriction to $\bar D_{j-1}$ is homotopic to, and close to, the section g_{j-1} in $\Gamma^r_{\mathcal{A}}(\bar D_{j-1}, X)$. Since all steps are of the same kind, it suffices to explain how to construct g_2.

Corollary 4.2 furnishes a parameter set P_1 and a dominating h-spray G_1 of class $\mathcal{A}^r(D_1)$ with the central section g_1. We may assume that P_1 is convex. As above $D_2 = D_1 \cup B_1$, where B_1 is a convex bump on D_1 (Def. 2.3) and the fiber bundle $X \to \bar D$ is trivial over $\bar B_1$. More precisely, there is a relatively open neighborhood W of $\bar B_1$ in $\bar D$ and a $\mathcal{C}^r$ fiber bundle isomorphism $\Phi \colon X|_W = h^{-1}(W) \to W \times Y$ which is holomorphic over $W \cap D$. Since G_1 is a h-spray on $\bar D_1 \times P_1$, we have

$$\Phi \circ G_1 = (G^1_D, G^1_Y) \colon \overline{D_1 \cap B_1} \times P_1 \to W \times Y$$

where $G^1_D(z,t) = z$ for $(z,t) \in \overline{D_1 \cap B_1} \times P_1$, and hence it extends to $\bar{B}_1 \times P_1$. Since B_1 is a convex bump on D_1, there is a biholomorphic map from an open neighborhood of $\bar{B}_1$ in S onto an open subset of $\mathbb{C}^n$ ($n = \dim S$) which maps B_1 and $D_1 \cap B_1$ onto strongly convex domains in $\mathbb{C}^n$. Since Y enjoys CAP, G^1_Y can be approximated in these coordinates arbitrarily well by entire maps. Hence G^1_Y can be approximated on $\overline{D_1 \cap B_1} \times P_1$ by a $\mathcal{C}^r$ map G' defined on $\bar{B}_1 \times P_1$ and holomorphic on $B_1 \times P_1$. The map $\Phi^{-1} \circ (G^1_D, G')$ is an h-spray of class $\mathcal{A}^r(B_1)$ which approximates the spray G_1 on $\overline{D_1 \cap B_1} \times P_1$. If the approximation is good enough, we can glue the sprays G_1 and $\Phi^{-1} \circ (G^1_D, G')$ using Proposition 2.4 and thus obtain a h-spray $\widetilde{G}_2$ of class $\mathcal{A}^r(D_2)$. Denote its central section by g_2. By construction g_2 approximates, and is homotopic to, the section g_1 on $\bar{D}_1$.

After m steps we reach a section $g_m \in \Gamma^r_{\mathcal{A}}(D, X)$ (which we now call f_1) satisfying the conclusion of Theorem 6.1 (i). A homotopy from f_0 to f_1 with the stated properties is obtained by combining the homotopies obtained in the individual steps of the proof.

The second part of the proof is obtained in essentially the same way by using the equivalence between CAP and the one-parametric Oka principle (Section 5 in [12]). A one-parametric spray with the given center is furnished by the parametric part of Corollary 4.2, and we glue sprays by the parametric version of Proposition 2.4. □

Acknowledgements. We wish to thank J. Leiterer for pointing out the works of Heunemann and Sebbar, and L. Lempert for helpful remarks concerning the theory of complex Banach manifolds.

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