University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1 111 Ljubljana, Slovenia

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ON THE CROSSING NUMBERS OF CARTESIAN PRODUCTS WITH TREES

Drago Bokal

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Drago Bokal†

Department of Mathematics
Institute of Mathematics, Physics and Mechanics
Jadranska 19, Ljubljana, Slovenia
drago.bokal@imfm.uni-lj.si

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Abstract

Zip product was recently used in a note establishing the crossing number of the Cartesian product $K_{1,n} \Box P_m$. In this paper, we further investigate the relations of this graph operation with the crossing numbers of graphs. First, we use a refining of the embedding method bound for crossing numbers to weaken the connectivity condition under which the crossing number is additive for the zip product. Next, we deduce a general theorem for bounding the crossing numbers of (capped) Cartesian products of graphs with trees, which yields exact results under certain symmetry conditions. We apply this theorem to obtain exact and approximate results on crossing numbers of various graphs with trees.

Keywords: crossing number, Cartesian product, tree, embedding method.

1 Introduction

The richness of repetitive patterns in Cartesian products of graphs reflects in their drawings and makes Cartesian products one of the few graph classes, for which exact crossing number results are known. Their crossing numbers have been studied since 1973, when Harary, Kainen, and Schwenk established the crossing number of $C_3 \Box C_3$ and conjectured the value for the crossing number of $C_m \Box C_n$ in [11]. In [26], Ringeisen and Beineke proved their conjecture for $m = 3$ and in [5] for $m = 4$. The topic was put aside for a decade and a half, until Klešč and, independently, Richter and Stobert established the conjecture for $m = 5$ in [19]; Richter and Salazar for $m = 6$ in [24]; and Adamsson and Richter for $m = 7$ in [1]. All of them relied on induction as the main tool, and in all of the cases $m = 3, \dots, 7$, the base case $n = m$ was published separately [11, 9, 25, 2, 3]. The recent state-of-the-art on the problem is the result of Glebsky and Salazar from [10]: $\mathrm{cr}(C_m \Box C_n) = (m-2)n$ for $m \geq 3$ and $n \geq \frac{1}{2}(m+1)(m+2)$.

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Besides the Cartesian products of two cycles, there are several other exact results. Beineke and Ringeisen determined the crossing number of $G \square C_n$ for any graph G of order four, except $S_3 = K_{1,3}$ in [5]. This gap has been bridged by Jendrol' and Ščerbová, who determined the crossing number of $S_3 \square C_n$, $S_3 \square P_m$, and $S_4 \square P_2$ in [12]. They conjectured that $\operatorname{cr}(S_n \square P_m) = (m-1)\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ for any star $S_n = K_{1,n}$, $n \geq 3$, and any path P_m of length $m \geq 1$. Klešč proved this conjecture for $n = 4$ in [14], where he also determined $\operatorname{cr}(S_4 \square C_m)$ for $m \geq 3$. For general n, the conjecture was recently confirmed by the author in [7]. In [18], Klešč determined the crossing number of $G \square P_m$ and $G \square S_n$ for any graph G of order four and, in [15], the crossing number of $G \square P_m$ for any graph G of order five. For several graphs of order five, the crossing numbers of their Cartesian product with C_n or S_n are also known, as well as some other Cartesian products, most of which are due to Klešč [14, 15, 16, 17, 19]. The methods rely on the high level of symmetry that these Cartesian products exhibit and mostly apply the following approach: if no copy of G has a sufficient number of crossings, then enough crossings can be found in the drawing, otherwise, the claim follows by induction. Also, a carefully chosen partition of edges of the Cartesian product is usually applied.

In [7], we suggested another approach: instead of taking the graph G as it is, one could iteratively construct G from smaller graphs and a drawing of G from drawings of smaller graphs. Then the crossing number of G can be established by studying how the crossing number of smaller graphs changes with the operation. Pinontoan, Richter, and the author used similar techniques in combination with restricting graph drawings to the disk to prove crossing-criticality of certain graph families in [6, 23]. In combination with restricting graph drawings to the anulus, Pelsmajer, Schaefer, and Štefankovič used such techniques to prove that the odd crossing number is different from the pair crossing number in [22].

The above exact results on the crossing number of $G \square H$ fit into three categories: (i) G and H are two small graphs, (ii) G is a small graph and H is a graph from some infinite family, or (iii) G belongs to one and H to a possibly different infinite family. The results of type (i) are mainly used as the induction basis for establishing results of type (ii), and the crossing numbers of $C_m \square C_n$ and $K_{1,n} \square P_m$ are the only known results of type (iii).

This contribution builds on the Zip product operation, introduced in [7]. Using a sharpening of the embedding method for proving lover bounds on the crossing numbers of general graphs, we relax the condition under which the crossing number is additive for the zip product. We apply the capped Cartesian product operation from [7] in combination with a newly introduced π-subdivision to establish several new exact results of type (ii) and (iii): we express the crossing number of the Cartesian product of any tree T and any star $K_{1,n}$ as a sum of crossing numbers of Cartesian products of two stars, and obtain closed formulas for $\operatorname{cr}(K_{1,n} \square T)$ for any $n \geq 1$ and any tree T with $\Delta(T) \leq 3$, $\operatorname{cr}(K_{1,3} \square T)$ for any tree, $\operatorname{cr}(W_n \square P_m)$ for any wheel W_n and any path, and $\operatorname{cr}(W_3 \square T)$ for any tree T with $\Delta(T) \leq 3$.

2 The zip product and the embedding method

Let G_i, $i = 1, 2$, be a graph whith a vertex $v_i \in V(G_i)$ whose neighborhood $N_i = N_{G_i}(v_i)$ has size d. A *zip function* of graphs G_1 and G_2 at vertices v_1 and v_2 is a bijection $\sigma : N_1 \to N_2$. The *zip product* $G_1 \odot_\sigma G_2$ of graphs G_1 and G_2 according to σ is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the edges $u\sigma(u)$, $u \in N_1$. Let $G_1 \, {}_{v_1}\!\odot_{v_2} G_2$ denote the set of all graphs obtained as a zip product $G_1 \odot_\sigma G_2$ for some zip function $\sigma : N_1 \to N_2$.

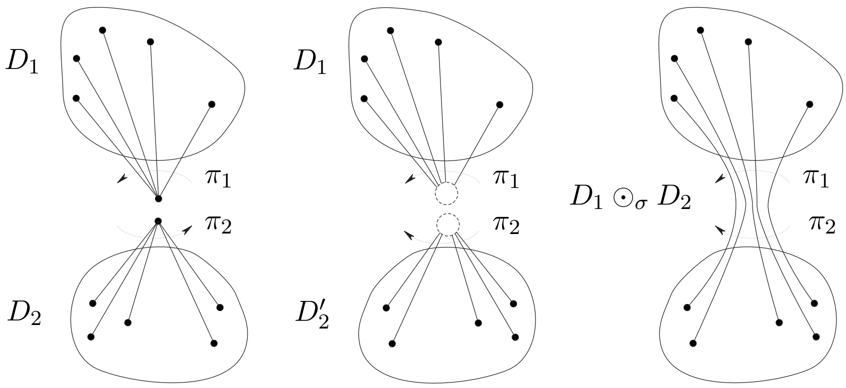

Figure 1: Zip product of drawings D_1 and D_2.

A drawing D_i of the graph G_i defines (up to a circular permutation) a bijection $\pi_i : N_i \to \{1, \ldots, d\}$, which respects the edge rotation around v_i in D_i. The *zip function* of drawings D_1 and D_2 at vertices v_1 and v_2 is $\sigma : N_1 \to N_2$, $\sigma = \pi_2^{-1}\pi_1$. The *zip product* $D_1 \odot_\sigma D_2$ of D_1 and D_2 according to σ is obtained from D_1 by adding a mirrored copy of D_2 that has v_2 incident with the infinite face disjointly into some face of D_1 incident with v_1, by removing vertices v_1 and v_2 together with small disks around them from the drawings, and then by joining the edges according to function σ, cf. Figure 1. For this construction, the following Lemmas hold:

Lemma 1 ([7]) *For $i = 1, 2$, let D_i be an optimal drawing of G_i, let $v_i \in V(G_i)$ be a vertex of degree d, and let σ be a zip function of D_1 and D_2 at v_1 and v_2. Then,* $\mathrm{cr}(G_1 \odot_\sigma G_2) \le \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$.

Lemma 2 *Let G_1 and G_2 be two graphs with vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ of degree d, and let $G \in G_1 \, {}_{v_1}\!\odot_{v_2} G_2$. Then,* $\mathrm{cr}(G) \le \mathrm{cr}(G_1) + \mathrm{cr}(G_2) + \frac{1}{2}\binom{d}{2}$.

Proof. Let $G = G_1 \odot_\sigma G_2$. For $i = 1, 2$, let D_i be an optimal drawing of G_i with v_i in the infinite face, and let $u_1, \ldots, u_d$ be the vertex rotation of the G_1-neighbors around v_1 in D_1. We draw D_1 in the $y \ge 1$ half-plane Π^+, remove the vertex v_1, and connect each vertex u_i, $i = 1, \ldots, d$, with the point $(i, 1)$. This can be done in Π^+ without introducing new crossings.

Similarly, for the G_2-neighbors $u'_1, \ldots, u'_d$ of v_2, we draw D_2 in the $y \le -1$ half-plane Π^-, remove the vertex v_2, and connect each vertex u'_i, $i = 1, \ldots, d$, with the point $(i, -1)$ in Π^- without introducing new crossings. For $i = 1, \ldots, d$ and $u'_j = \sigma(u_i)$, we connect $(i, 1)$ with $(j, -1)$ using a straight line. This introduces $c \le \binom{d}{2}$ new crossings.

If $c \le \frac{1}{2}\binom{d}{2}$, we are done. If $c > \frac{1}{2}\binom{d}{2}$, then the above construction with a mirrored copy of D_2 that has vertex rotation $u''_1 = u'_d, \ldots, u''_d = u'_1$ around v_2 requires $\binom{d}{2} - c < \frac{1}{2}\binom{d}{2}$ new crossings. □

In specific cases, the number of crossings can be further minimized by rearranging the vertices or using the symmetry of graphs. Pelsmajer, Schaefer, and Štefankovič proved that for any specific case, the smallest number of crossings needed to combine two drawings can be computed by solving a linear program [22].

Let H, G be two graphs. An *embedding* of H into G is a pair of injections $\omega = \langle \lambda, \Lambda \rangle$, $\lambda : V(H) \to V(G)$, $\Lambda : E(H) \to \{P \mid P \text{ is a path in } G\}$, such that $\Lambda(e)$ is a path in G from $\lambda(u)$ to $\lambda(v)$ for any edge $e = uv \in E(H)$. The paths $\{\Lambda(e) \mid e \in E(G)\}$ are called *ω-active paths*. The *edge congestion* μ_ω is the maximum number of active paths using some edge of G, and the *vertex congestion* m_ω is the maximum number of active paths using some vertex of G.

Leighton used the embedding method to estimate the crossing numbers of shuffle-exchange graphs and meshes of trees [21]. Shahrokhi, Sýkora, Székely, and Vrťo generalized his approach to the following result:

Theorem 3 ([27]) *Let G be a graph of order n, ω an embedding of a graph H into G with edge-congestion μ_ω and vertex congestion m_ω, and Σ any surface. Then,*

$$\mathrm{cr}_\Sigma(G) \geq \frac{\mathrm{cr}_\Sigma(H)}{\mu_\omega^2} - \frac{n}{2}\left(\frac{m_\omega}{\mu_\omega}\right)^2 .$$

For our purposes, we refine the above statement. Let ω be an embedding of a graph H into a graph G and let Ω_v be the set of Λ-images of edges of H that contain the vertex $v \in V(G)$. In the proof of Theorem 3, it is estimated that the paths of Ω_v contribute at most $\binom{|\Omega_v|}{2}$ crossings to the drawing of H at vertex v. This bound can be tightened, since adjacent edges and edges embedded along paths of $\bigcup_{v \in V(G)} \Omega_v$ sharing more than one vertex can be uncrossed. Let o_ω be the number of pairs $\{e, f\} \subseteq E(H)$, such that e and f are not adjacent, and $V(\Lambda(e)) \cap V(\Lambda(f))$ is nonempty.

Theorem 4 *Let G be a graph of order n and $\omega = (\lambda, \Lambda)$ an embedding of a graph H into G with edge-congestion μ_ω. Then there are at least $\frac{1}{\mu_\omega^2}(\mathrm{cr}(H) - o_\omega)$ crossings on the edges of $\Lambda(E(H))$ in every drawing D of G. If $e, f \in E(H)$ are adjacent, none of these crossings appears between the edges of $\Lambda(e)$ and $\Lambda(f)$.*

A proof of Theorem 4 can be found in [8], where we study extensions of the embedding method and other general crossing number lower bounds to the minor crossing number.

Let $v \in V(G)$ be a vertex of degree d in G. A *bundle* of v is a set B of d edge disjoint paths from v to some vertex $u \in V(G)$, $u \neq v$. Vertex v is the *source* of the bundle and u is its *sink*. Other vertices on the paths of B are *internal vertices* of the bundle. Let $\breve{E}(B) = E(B) \cap E(G - v)$ denote the set of edges of B that are not incident with v. They are called *distant edges* of B. Two bundles B_1 and B_2 of v are *coherent* if their sets of distant edges are disjoint.

Lemma 5 *For $i = 1, 2$, let G_i be a graph, $v_i \in V(G_i)$, $\deg(v_i) = d$, $N_i = N_{G_i}(v_i)$. Assume that v_2 has a bundle B in G_2. For every bijection $\sigma : N_1 \to N_2$ and every drawing D of $G = G_1 \odot_\sigma G_2$, there are at least $\mathrm{cr}(G_1)$ crossings in D of an edge from $E(G_1 - v_1)$ with an edge from $E(G_1 - v_1) \cup \breve{E}(B) \cup \{u\sigma(u) \mid u \in N_1\}$.*

Proof. Let ω be the embedding of G_1 into G, constructed as follows: $G - v_1$ is embedded into itself using the identity, v_1 is embedded into the sink v of B, and the edges $v_1 u$, $u \in N_1$, are embedded into distinct edge disjoint paths $v\sigma(u)u$ of B. For this embedding, $\mu_\omega = 1$ and

$o_\omega = 0$. By Theorem 4, the claim follows, as the edges of $\breve{E}(B) \cup \{u\sigma(u) \mid u \in N_1\}$ lie on the active paths all emanating from v. □

This lemma suffices to establish a lower bound on the crossing number of the zip product of a graph and a planar graph whose vertex has a bundle. In the general case, we require two coherent bundles at both vertices v_i.

Lemma 6 *For $i = 1, 2$, let G_i be a graph, $v_i \in V(G_i)$ its vertex of degree d, and $N_i = N_{G_i}(v_i)$. Assume that G_1 is planar and that v_1 has a bundle B_1 in G_1. Then, $\mathrm{cr}(G_1 \odot_\sigma G_2) \geq \mathrm{cr}(G_2)$ for any bijection $\sigma : N_1 \to N_2$. Equality holds if, for $i = 1, 2$, σ respects the edge rotation around v_i in some optimal drawing D_i of G_i.*

Lemma 7 *For $i = 1, 2$, let G_i be a graph, $v_i \in V(G_i)$ its vertex of degree d, and $N_i = N_{G_i}(v_i)$. Also assume that v_i has two coherent bundles $B_{i,1}$ and $B_{i,2}$ in G_i. Then, $\mathrm{cr}(G_1 \odot_\sigma G_2) \geq \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$ for any bijection $\sigma : N_1 \to N_2$.*

Proof. Let $G = G_1 \odot_\sigma G_2$ and $F = \{v\sigma(v) \mid v \in N_1\} \subseteq E(G)$. For $i, j = 1, 2$, let $E_i = E(G_i - v_i) \subseteq E(G)$ and let G_{ij} be the subgraph of G spanned by the edges of E_i, $F_{ij} = \breve{E}(B_{ij})$, and F.

For an optimal drawing D of G, let $\mathrm{cr}_D(A, B)$ denote the number of crossings of an edge from $A \subseteq E(G)$ with an edge from $B \subseteq E(G)$. Lemma 5 implies $\mathrm{cr}(G_1) \leq \mathrm{cr}_D(E_1, E_1) + \mathrm{cr}_D(E_1, F) + \mathrm{cr}_D(E_1, F_{2j})$, thus also $\mathrm{cr}(G_1) \leq \mathrm{cr}_D(E_1, E_1) + \mathrm{cr}_D(E_1, F) + \frac{1}{2}\,\mathrm{cr}_D(E_1, F_{21} \cup F_{22})$.

A similar inequality holds for $\mathrm{cr}(G_2)$, and, as in [7], the sum of the two inequalities counts each crossing of D at most once. □

Although Lemma 7 does generalize the lower bound from [7], it seems that requiring two bundles is still too strong condition. We propose the following question:

Question 8 *Does one bundle vertex $v_i \in V(G_i)$, $i = 1, 2$, imply the claim of Lemma 7?*

Besides the trivial cases $d = 0, 1$ and the case when G_1 is planar, also a recent positive answer for $d = 2$ of Leaños and Salazar [20] suggests that the question could probably be answered affirmatively. Such a positive answer would allow us to efficiently eliminate caps in the results on the crossing numbers of Cartesian products.

We use the following observation in iterative applications of the zip product.

Lemma 9 *Let G_1 and G_2 be disjoint graphs, $v_i \in V(G_i)$, $\deg_{G_i}(v_i) = d$, and $G \in G_1\, {}_{v_1}\!\odot_{v_2} G_2$. If v_2 has a bundle in G_2 and $v \in V(G_1)$, $v \neq v_1$, has k pairwise coherent bundles in G_1, then v has k pairwise coherent bundles in G.*

Proof. Assume $G = G_1 \odot_\sigma G_2$ and let $B_1, \ldots, B_k$ be the bundles of v in G_1 and B the bundle of v_2 in G_2. For an edge $e = uv_2$, let P_e be the path of B containing e. A path $P \in \bigcup_{i=1}^k B_i$ can have at most two edges that are in P incident with v_1. If there are none, define $P' = P$. If there is only one such edge wv_1, define $P' = Pw\sigma(w)P_e$, $e = \sigma(w)v_2$. For two such edges $wv_1 \neq zv_1$ on P, define $P' = Pw\sigma(w)P_eP_f\sigma(z)zP$, $e = \sigma(w)v_2$, $f = \sigma(z)v_2$. As the paths of B are pairwise edge disjoint and each of them is used at most once in the construction of some P', the paths P'_1 and P'_2 are edge disjoint for edge disjoint paths $P_1, P_2 \in \bigcup_{i=1}^k B_i$. If P_1, P_2 in G_1 share only the edge incident with v, so do P'_1 and P'_2 in G. If $u \neq v_1$ is the endvertex of the path $P \in \bigcup_{i=1}^{k} B_i$, then u is the endvertex of P'. If v_1 is the endvertex of the path $P \in \bigcup_{i=1}^{k} B_i$, then the endvertex of P' is the sink of B. These three statements imply that the sets $B_i' = \{P' \mid P \in B_i\}$, $i = 1, \ldots, k$, are pairwise coherent bundles of v in G. $\square$

3 General graphs

Let $G^{(i)}$ be the *suspension* of order i of a graph G, i.e. the complete join of G and an empty graph on i vertices $\{v_1, \ldots, v_i\}$, called the *apices* of $G^{(i)}$. For a multiset $L \subseteq V(G_2)$, we denote with $G_1 \square_L G_2$ the *capped Cartesian product* of graphs G_1 and G_2, i.e. the graph obtained by adding a distinct vertex v' to $G_1 \square G_2$ for each copy of a vertex $v \in L$ and joining v' to all the vertices of $G_1 \square \{v\}$. We call each v' a *cap* of v. When L contains precisely all vertices of degree one in G_2, we use $G_1 \widehat{\square} G_2$ in place of $G_1 \square_L G_2$. For $v \in V(H)$, let $\chi_L(v)$ denote the multiplicity of v in L and let $\ell(v) := \deg_{G_2}(v) + \chi_L(v)$. An edge $uv \in E(G_2)$ is *unbalanced* if $\ell(u) \neq \ell(v)$. Let $\beta(G_2)$ be the number of unbalanced edges of G_2.

A drawing D of $G^{(i)}$ is *apex-homogeneous* if there exists a permutation ρ of the vertices of G such that the vertex rotation around every apex in D is ρ or ρ^{-1}. Two drawings $D^{(i)}$ of $G^{(i)}$ and $D^{(j)}$ of $G^{(j)}$ are *pairwise apex-homogeneous*, if they are apex-homogeneous with respect to the same permutation ρ. A graph G has *all apex-homogeneous drawings* if there exist drawings $D^{(i)}$ of $G^{(i)}$, $i \geq 1$, such that every two of them are pairwise apex-homogeneous.

Theorem 10 *Let G be a graph of order n, let T be a tree, and let $L \subseteq V(T)$ be a multiset with either $\ell(v) \geq 3$ or, if G has a dominating vertex, $\ell(v) \geq 2$ for every $v \in V(T)$. Define*

$$B = \sum_{v \in V(T)} \operatorname{cr}(G^{(\ell(v))}).$$

Then, $B \leq \operatorname{cr}(G \square_L T) \leq B + \frac{\beta(T)}{2}\binom{n}{2}$. Also, $\operatorname{cr}(G \square_L T) = B$ whenever G has all apex-homogeneous drawings.

Proof. For each value l of $\ell(v)$, $v \in V(T)$, let $D^{(l)}$ be a fixed optimal drawing of $G^{(l)}$. If possible, let these drawings be pairwise apex-homogeneous. Let $v_1, \ldots, v_m$ be some depth-first search ordering of vertices of T, set $l_i = \ell(v_i)$, and let $e_i = v_i u_i$ be the edge connecting v_i with $T[\{v_1, \ldots, v_{i-1}\}]$.

Using this setup we construct $G \square_L T$ as a sequence of zip products of suspensions of G. Let $G_1 = G^{(l_1)}$ and define $G_i = G^{(l_i)} \odot_{\iota_i} G_{i-1}$ for $i = 2, \ldots, m$, where ι_i maps a vertex of the G subgraph of $G^{(l_i)}$ to its counterpart in the neighborhood of some cap u' of u_i, $u' \in V(G_{i-1})$. The graph G_m is isomorphic to $G \square_L T$. Since G has a dominating vertex and $\ell(v) \geq 2$, or since $\ell(v) \geq 3$, the apices in the suspensions have two coherent bundles. Lemmas 7 and 9 imply $\operatorname{cr}(G \square_L T) \geq B$.

We construct a drawing of $G \square_L T$ that establishes the upper bound using the drawings $D^{(l)}$. We define $D_0 = D^{(l_0)}$ and, for $i = 2, \ldots, m$, let $D_i = D^{(l_i)} \odot_{\iota_i} D_{i-1}$. If G has all apex-homogeneous drawings or if the edge $v_i u_i$ is balanced, then there is an apex in $D^{(l_i)}$ that has the same (or inverse) vertex rotation as some cap of u_i in D_{i-1}. We perform the zip product using this apex (and possibly a mirrored drawing) and we introduce no new crossings by Lemma 1. If neither of the conditions is satisfied, then Lemma 2 asserts that at most $\frac{1}{2}\binom{n}{2}$ new crossings need to be introduced. The claim follows. $\square$

In the rest of the paper, we list several special cases of the above theorem. The following two corollaries imply equality of $\mathrm{cr}(G \,\square\, P_m)$ and $\mathrm{cr}(G \,\widehat{\square}\, P_m)$ up to an additive constant, whenever G has a dominating vertex.

Corollary 11 ([7]) $\mathrm{cr}(G \,\widehat{\square}\, P_m) = (m+1)\,\mathrm{cr}(G^{(2)})$ *for a graph G with a dominating vertex and $m \geq 0$.*

Corollary 12 *The following inequality holds for a graph G of order n with a dominating vertex and $m \geq 2$:*

$$(m-1)\,\mathrm{cr}(G^{(2)}) \leq \mathrm{cr}(G \,\square\, P_m) \leq (m-1)\,\mathrm{cr}(G^{(2)}) + 2\,\mathrm{cr}(G^{(1)}) + \binom{n}{2}.$$

Proof. $G \,\square\, P_m$ contains $G \,\widehat{\square}\, P_{m-2}$ as a subdivision, thus $\mathrm{cr}(G \,\square\, P_m) \geq (m-1)\,\mathrm{cr}(G^{(2)})$ by Corollary 11. Let u, v be the caps of $\bar{G} = G \,\widehat{\square}\, P_{m-2}$ and v', v'' the apices of two disjoint copies G', G'' of $G^{(1)}$. The graph $G \,\square\, P_m$ is isomorphic to some graph in $(\bar{G}\ {}_u\!\odot_{v'} G')\ {}_v\!\odot_{v''} G''$ and the upper bound follows by Lemma 2. $\square$

4 Cycles and paths

Lemma 13 *Whenever (i) $3 \leq n$ and $1 \leq d \leq 6$, (ii) $3 \leq n \leq 6$ and $1 \leq d$, (iii) $3 \leq n \leq 8$ and $1 \leq d \leq 10$, or (iv) $3 \leq n \leq 10$, $1 \leq d \leq 8$, then $\mathrm{cr}(C_n^{(d)}) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{d}{2} \rfloor \lfloor \frac{d-1}{2} \rfloor$. Moreover, under the above conditions, there exist pairwise apex-homogeneous drawings of $C_n^{(d)}$.*

Proof. The graph $C_n^{(d)}$ has $K_{n,d}$ as a subgraph, thus $\mathrm{cr}(C_n^{(d)}) \geq \mathrm{cr}(K_{n,d})$. In [13], Kleitman established the crossing number of the latter when (i) or (ii) apply, and, in [29], Woodall established it under conditions (iii) or (iv). In each of the cases, we can add the edges of the cycle into an optimal drawing of $K_{n,d}$ without introducing new crossings, cf. Figure 2 for an example with $n = 7$, $d = 3$. The vertex rotations around the apices in these drawings respect the ordering imposed by C_n, thus they are pairwise apex-homogeneous. $\square$

Corollary 14 *Let d be the maximum degree in a tree T and n an integer. If one of the conditions (i)–(iv) applies to n, d, then, for $d_v = \deg_T(v)$, $v \in V(T)$,*

$$\mathrm{cr}(C_n \,\widehat{\square}\, T) = \mathrm{cr}(P_{n-1} \,\widehat{\square}\, T) = \mathrm{cr}(\bar{K}_n \,\widehat{\square}\, T) = \sum_{v \in V(T)} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{d_v}{2} \right\rfloor \left\lfloor \frac{d_v - 1}{2} \right\rfloor.$$

Proof. The lower bound follows by Theorem 10 and Lemma 13. Using consistency of vertex rotations around apices in optimal drawings of $C_n^{(d)}$ implied by Lemma 13, we can maintain the tightness of the lower bound in the construction of an optimal drawing. For vertices of

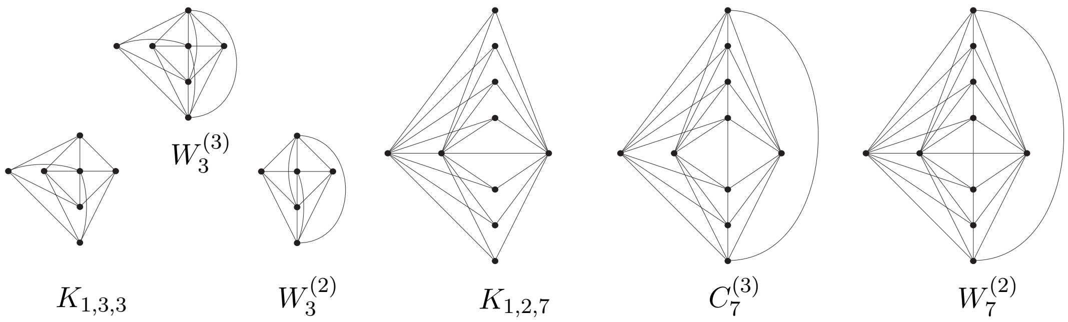

Figure 2: Several optimal drawings.

degree one and two in T, the apices of the graph $C_n^{(2)}$ have only one coherent bundle, and since $C_n^{(2)}$ is planar, Lemma 6 applies.

The above remains true for any spanning subgraph of the cycle C_n, in particular for the path P_{n-1} and for the empty graph $\bar{K}_n$. $\square$

5 Stars

The graph $S_n^{(d)}$ is isomorphic to the complete tripartite graph $K_{1,d,n}$, which can be obtained by contracting an edge of $K_{d+1,n+1}$. Also, the graph $S_n \,\square\, S_d$ is a subdivision of $S_n^{(d)}$. These observations enable us to express the crossing numbers of Cartesian products of stars with trees as a sum of the crossing numbers of Cartesian products of two stars:

Corollary 15 *Let T be a tree and $n \geq 1$. Then, for $d_v = \deg_T(v)$,*

$$\operatorname{cr}(S_n \,\square\, T) = \sum_{v \in V(T),\, d_v \geq 2} \operatorname{cr}(K_{1,d_v,n}).$$

Proof. Let T' be the tree obtained by deleting all the leaves of T and let L be the set of leaves of T', each leaf v with multiplicity equal to $\deg_T(v) - 1$. The graph $S_n \,\square\, T$ is a subdivision of $S_n \,\square_L\, T'$ and the lower bound follows by Theorem 10 since $S_n^{(d)}$ is isomorphic to $K_{1,d,n}$.

For the upper bound, we observe that the automorphism group of $S_n^{(i)}$ acts as a full symmetric group on the leaves of S_n. Thus, we can propagate the labelling of the leaves in S_n whenever applying the zip product to two drawings in the construction of Theorem 10. After subdividing the edges connecting the caps with the leaves of stars in $S_n \,\square_L\, T'$, the resulting graph will be isomorphic to $S_n \,\square\, T$. $\square$

Corollary 16 *Let T be a tree. Then,*

$$\operatorname{cr}(S_3 \,\square\, T) = \sum_{v \in V(T),\, d_v \geq 2} \left\lfloor \frac{d_v}{2} \right\rfloor \left(2 \left\lfloor \frac{d_v - 1}{2} \right\rfloor + 1 \right).$$

Proof. Asano proved $\mathrm{cr}(K_{1,3,n}) = \lfloor \frac{n}{2} \rfloor \left(2 \lfloor \frac{n-1}{2} \rfloor + 1\right)$ in [4]. The equality follows by Corollary 15. □

Corollary 17 *Let $n \geq 1$ be any integer and T a subcubic tree with n_2 vertices of degree two and n_3 vertices of degree three. Then,*

$$\mathrm{cr}(S_n \,\square\, T) = \left\lfloor \frac{n}{2} \right\rfloor \left((n_2 + 2n_3) \left\lfloor \frac{n-1}{2} \right\rfloor + n_3 \right).$$

Proof. The graph $K_{1,2,n}$ has $K_{3,n}$ as a subgraph. Kleitman [13] proved that $\mathrm{cr}(K_{3,n}) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. Figure 2 presents a drawing of $K_{1,2,n}$ with this many crossings ($n = 7$). Corollary 15 and the result of Asano imply the claim. □

6 Wheels

Let a vertex v have a bundle B with the sink u in a graph G. If there exists a cycle C in G, such that C intersects each path of B at most in one internal vertex of B, then the triple (v, B, C) forms a *wheel* in G. Cycle C is the *rim* of the wheel, an *inner spoke* is a subpath from v to C of some path in B, an *outer spoke* is a subpath from u to C of some path in B, and an *axis* is a path in B that has no vertex in common with C. A *wheel gadget* in G is a wheel in G that has at least one axis and at least three inner spokes that meet C in distinct vertices.

Lemma 18 *Let G be a simple graph, $W = (v, B, C)$ a wheel gadget in G, and D a drawing of G. Then, D contains (i) a crossing of some axis of W with the rim, (ii) a crossing of some spoke of W with the rim, or (iii) a crossing of an inner spoke with an outer spoke of W.*

Proof. Let $P_1, P_2, P_3 \in B$ be three paths containing three inner spokes of W and $Q \in B$ an axis of W. Let $H = K_5$ have vertices $v_1, \ldots, v_5$, and let ω be the following embedding of H into G: v_i, $i = 1, 2, 3$, is embedded into the vertex in $P_i \cap C$, v_4 is embedded into v, and v_5 is embedded into the sink of B. The edges of H are embedded into edge-disjoint paths, containing the edges of $C \cup Q \cup \bigcup_{i=1}^{3} P_i$ in such way that $\mu_\omega = 1$ and $o_\omega = 0$. Then, by Theorem 4, D contains at least $\mathrm{cr}(K_5) = 1$ crossing on the edges of two active paths of ω that do not share an endvertex. This implies one of (i), (ii), or (iii). □

We introduce another operation on graphs that allows us to study the crossing numbers of Cartesian products of wheels. Let $F \subseteq E(G)$ be a subset of edges of G and π a permutation of F. A *π-subdivision* G^π of G is the graph, obtained from G by subdividing every edge $e \in F$ with the vertex v_e and adding the edges $\{v_e v_{\pi(e)} \mid e \in F\}$.

Theorem 19 *Let v be a vertex that has a bundle B_v in a graph G and let π be a cyclic permutation of a subset F of all but one of the edges incident with v, $|F| \geq 3$. Then*

$$\mathrm{cr}(G^\pi) \geq \mathrm{cr}(G) + 1, \tag{1}$$

with equality if π respects the edge rotation around v in some optimal drawing of G.

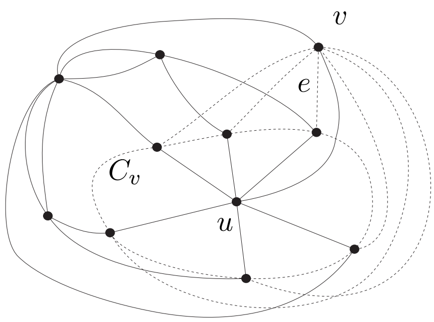

Figure 3: Drawings $\bar{D}^\pi$ and $\bar{D}$.

Proof. Let C_v be the cycle on the edges $\{v_e v_{\pi(e)} \mid e \in F\}$ in G^π. Let D^π be an optimal drawing of G^π in the plane Π and D the induced subdrawing of G. The triple (v, B_v, C_v) is a wheel gadget in G^π. Assume it has no crossing on C_v in D^π. Then C_v is a simple closed curve in D^π and the whole drawing D^π lies in the same component of $\Pi - C_v$. Without loss of generality, D^π lies in the exterior of the disk bounded by C_v and an inner spoke must cross an outer spoke by Lemma 18. Let c be the number of crossings on the inner spokes.

Claim 1: *Under the above assumptions,* $\mathrm{cr}(D) \geq \mathrm{cr}(G) + c - \left\lfloor \frac{c}{|F|} \right\rfloor$. Let e be some inner spoke with the smallest number of crossings. We draw a new vertex u in the interior of C_v. It is possible to connect u with all the vertices of C_v without introducing new crossings and also to detach e from its endvertex on C_v and connect it with u (crossing C_v). Let $\bar{D}^\pi$ be the modified drawing, presented in Figure 3, and let $\bar{D}$ be its subdrawing obtained by removing the rim and all inner spokes but e from $\bar{D}^\pi$. In Figure 3, we indicate $\bar{D}$ with the solid edges. Since $\bar{D}$ is a drawing of a subdivision of G and has at least $c - \left\lfloor \frac{c}{|F|} \right\rfloor$ crossings less than D, the claim follows.

Either there is a crossing of D^π on C_v, which is not present in D, or $c - \left\lfloor \frac{c}{|F|} \right\rfloor \geq 1$. Inequality (1) follows.

If π respects the edge rotation around v in an optimal drawing D of G, we can draw C_v in D with at most one new crossing. □

Lemma 20 $\mathrm{cr}(W_n^{(2)}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1$ *for* $n \geq 3$.

Proof. The drawing of $G = W_n^{(2)}$ with $k = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1$ crossings presented in Figure 2 (with $n = 7$) establishes the upper bound.

Let D be an optimal drawing of G for some $n \geq 3$. Partition the edges of G into the edges E_1 of the $K_{3,n}$ subgraph of G, the edges E_2 of the path between the apices u, v of G containing the center w of the wheel, and the edges E_3 of the rim. There are at least $k - 1$ crossings between the edges of E_1. If there is a crossing involving an edge of E_3, the claim follows.

Otherwise, the rim is drawn as a simple closed curve γ and we may assume that G is drawn in the disk Δ bounded by γ. The edges emanating from each of the vertices u, v, and w do not cross and separate Δ into n disks. For distinct $x, y \in \{u, v, w\}$, there are at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings between edges incident with x and y, implying a contradiction $\mathrm{cr}(D) \geq 3(k-1)$. □

Corollary 21 $\mathrm{cr}(W_n \,\square\, P_m) = (m-1)\left(\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 1\right) + 2$ *for* $m \geq 1$ *and* $n \geq 3$.

Proof. Two applications of Theorem 19 to the graph with two vertices and $n+1$ parallel edges among them prove the claim in case $m = 1$.

Let u, v be the caps of $G = W_n \,\widehat{\square}\, P_{m-2}$ and let F_u (respectively, F_v) contain the edges incident with u (v) and a vertex on the rim of the corresponding wheel in G. Corollary 11 and Lemma 20 assert $\mathrm{cr}(G) = (m-1)\left(\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 1\right)$. The graph $W_n \,\square\, P_m$ is isomorphic to $G' = (G^{\pi})^{\pi'}$ for properly chosen permutations π of F_u and π' of F_v. Theorem 19 implies $\mathrm{cr}(G') = \mathrm{cr}(G) + 2$. □

Lemma 22 $\mathrm{cr}(W_3^{(3)}) = 5$.

Proof.

Let $G = W_3^{(3)}$, let D be an optimal drawing of G, and let C be the set of edges of the rim of W_3. In D, there is no crossing between two edges of C. These therefore bound a disk Δ in D. Then $G - C = S_3^{(3)}$; let F be the set of edges of S_3 in $G - C$. The result of Asano [4] implies at least three crossings on the edges of $G - C$. If there are two crossings on C, the claim follows.

Assume there is only one crossing on C. If it is a crossing with an edge of F, then the induced drawing of $G - F$ has no crossing on C and we may assume it lies in the interior of Δ. The graph $G - F - C$ is isomorphic to the graph $K_{3,4}$, of which three vertices are incident with C, and four are not, say $v_1, \ldots, v_4$. The edges emanating from v_i and v_j must cross for distinct $i, j \in \{1, 2, 3, 4\}$. This implies at least six crossings in D.

Assume the only crossing on C is not a crossing with an edge of F, but with an edge e emanating from one of v_i, $i \in \{1, 2, 3, 4\}$. The induced drawing of $G - e$ has no crossing on C, thus we may assume it is drawn in the interior of Δ. By the argument of the previous paragraph, there are at least three crossings between the edges of $K_{3,3}$ subgraph of $G - v_i - C$. If there is an additional crossing on the edges incident with v_i distinct from e, the claim follows. Assume there is none. Then there exists a simple closed curve γ that starts at the center w of the wheel, follows the edge wv_i, continues along an edge connecting v_i with the rim C, along the edges of C, along the other edge connecting v_i and C, and finally closes along $v_i w$, such that $G - v_i - C$ is drawn inside the disk Δ' bounded by γ. The vertex w is incident with other three vertices on the boundary of Δ' and we may assume two of the edges lie on this boundary. Let f be the third of these edges, which separates the other two neighbors of w in Δ'. There is a drawing of $K_{2,4}$ in Δ' with four vertices on the boundary of Δ', which has at least two crossings on its edges and at least two crossings with f.

If there are no crossings on C, then we may assume the whole drawing, and in particular the subdrawing of $K_{4,3}$, is in Δ and has at least six crossings. The claim about $\mathrm{cr}(W_3^{(3)})$ follows, since Figure 2 presents a drawing of $W_3^{(3)}$ with five crossings. □

Corollary 23 *Let T be a subcubic tree with n_i vertices of degree i, $i = 1, 2, 3$. Then,* $\mathrm{cr}(W_3 \,\Box\, T) = n_1 + 2n_2 + 5n_3$.

Proof. We remove all the leaves from T and obtain a tree T'. Let L be the multiset of leaves of T', each $v \in L$ with multiplicity equal to $\deg_T(v) - 1$. Thus, $\ell(v)$ with respect to L and T' equals $\deg_T(v)$, which is at least two for every $v \in V(T')$. Since Lemmas 20 and 22 establish $\mathrm{cr}(W_3^{(2)}) = 2$ and $\mathrm{cr}(W_3^{(3)}) = 5$, Theorem 10 implies $\mathrm{cr}(W_3 \,\Box_L\, T') \geq 2n_2 + 5n_3$. Equality follows as the vertex rotations are consistent in the optimal drawings of $W_3^{(2)}$ and $W_3^{(3)}$ in Figure 2. This consistency in combination with Theorem 19 also implies that a properly chosen π-subdivision of edges connecting a cap of $W_3 \,\Box_L\, T'$ with the corresponding rim increases the crossing number by precisely one. To obtain $W_3 \,\Box\, T$ from $W_3 \,\Box_L\, T'$, we need one such subdivision for each leaf of T, and the claim follows. □

References

[1] J. Adamsson, R.B. Richter, Arrangements, circular arrangements, and the crossing number of $C_7 \,\Box\, C_n$, J. Combin. Theory Ser. B 90 (2004), 21–39.

[2] M. Anderson, R.B. Richter, P. Rodney, Crossing number of $C_6 \,\Box\, C_6$, Congr. Numer. 118 (1996), 97–107.

[3] M. Anderson, R.B. Richter, P. Rodney, Crossing number of $C_7 \,\Box\, C_7$, Congr. Numer. 119 (1997), 97–117.

[4] K. Asano, The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$, J. Graph Theory 10 (1986), 1–8.

[5] L.W. Beineke, R.D. Ringeisen, On the crossing number of product of cycles and graphs of order four, J. Graph Theory 4 (1980), 145–155.

[6] D. Bokal, Infinite families of crossing-critical graphs with prescribed average degree and crossing number, submitted.

[7] D. Bokal, On the crossing numbers of Cartesian products with paths, to appear in J. Combin. Theor. B. `http://lp.fmf.uni-lj.si/~drago/pathProducts.pdf`

[8] D. Bokal, É. Czabaraka, L.A. Székely, I. Vrťo, On lower bounds for the minor crossing number, in preparation. `http://lp.fmf.uni-lj.si/~drago/mcrBounds.pdf`

[9] A.M. Dean, R.B. Richter, The crossing number of $C_4 \,\Box\, C_4$, J. Graph Theory 19 (1995), 125–129.

[10] L.Y. Glebsky, G. Salazar, The crossing number of $C_m \Box C_n$ is as conjectured for $n \geq m(m+1)$, J. Graph Theory 47 (2004), 53–72.

[11] F. Harary, P.C. Kainen, A.J. Schwenk, Toroidal graphs with arbitrarily high crossing numbers, Nanta Math. 6 (1973), 58–67.

[12] S. Jendrol', M. Ščerbová, On the crossing numbers of $S_m \square P_n$ and $S_m \square C_n$; Čas. Pest. Mat. 107 (1982), 225–230.

[13] D.J. Kleitman, The crossing number of $K_{5,n}$, J. Combin. Theory Ser. B 9 (1971), 315–323.

[14] M. Klešč, On the crossing number of the Cartesian product of stars and paths or cycles, Math. Slovaca 41 (1991), 113–120.

[15] M. Klešč, The crossing numbers of Cartesian products of paths with 5-vertex graphs, Discrete Math. 233 (2001), 353–359.

[16] M. Klešč, The crossing number of $K_{2,3} \square C_3$, Discrete Math. 251 (2002), 109–117.

[17] M. Klešč, The crossing number of $K_{2,3} \square P_n$ and $K_{2,3} \square S_n$, Tatra Mt. Math. Publ. 9 (1996), 51–56.

[18] M. Klešč, The crossing numbers of products of paths and stars with 4-vertex graphs, J. Graph Theory 18 (1994), 605–614.

[19] M. Klešč, R.B. Richter, I. Stobert, The crossing number of $C_5 \square C_n$, J. Graph Theory 22 (1996), 239–243.

[20] J. Leaños, G. Salazar, On the additivity of crossing numbers of graphs, preprint.

[21] F.T. Leighton, Complexity Issues in VLSI, MIT Press, Cambridge, 1983.

[22] M.J. Pelsmajer, M. Schaefer, D. Štefanković, Odd crossing number is not crossing number, preprint.

[23] B. Pinontoan, R.B. Richter, Crossing numbers of sequences of graphs II: planar tiles, J. Graph Theory 42 (2003), 332–342.

[24] R.B. Richter, G. Salazar, The crossing number of $C_6 \square C_n$, Australas. J. Combin. 23 (2001), 135–143.

[25] R.B. Richter, C. Thomassen, Intersections of curve systems and the crossing number of $C_5 \square C_5$, Discrete Comput. Geom. 13 (1995), 149–159.

[26] R.D. Ringeisen, L.W. Beineke, The crossing number of $C_3 \square C_n$, J. Combin. Theory Ser. B 24 (1978), 134–136.

[27] F. Sharokhi, O. Sýkora, L.A. Székely, I. Vrťo, The crossing number of a graph on a compact 2-manifold, Adv. Math. 123 (1996), 105–119.

[28] A.T. White, L.W. Beineke, Topological Graph Theory, in: L.W. Beineke, R.J. Wilson, eds., Selected Topics in Graph Theory, Academic Press, New York, 1978, 15–50.

[29] D.R. Woodall, Cyclic-order graphs and Zarankiewicz's crossing-number conjecture, J. Graph Theory 17 (1993), 657–671.