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# FAULT-DIAMETER OF CARTESIAN PRODUCT OF GRAPHS AND CARTESIAN GRAPH BUNDLES

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# Fault-diameter of Cartesian product of graphs and Cartesian graph bundles

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**Abstract**

Cartesian graph bundles is a class of graphs that is a generalization of the Cartesian graph products. Let $G$ be a $k_G$-connected graph and $\mathcal{D}_c(G)$ denote the diameter of $G$ after deleting any of its $c < k_G$ vertices. We prove that if $G_1$, $G_2$, $\ldots$, $G_q$ are $k_1$-connected, $k_2$-connected,…, $k_q$-connected graphs and $0 \leq a_1 < k_1$, $0 \leq a_2 < k_2$,…, $0 \leq a_q < k_q$ and $a = a_1 + a_2 + \ldots + a_q + (q-1)$, then the fault diameter of $G$, a Cartesian product of $G_1$, $G_2$, $\ldots$, $G_q$, with $a$ faulty nodes is $\mathcal{D}_a(G) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1$. We also show that $\mathcal{D}_{a+b+1}(G) \leq \mathcal{D}_a(F) + \mathcal{D}_b(B) + 1$ if $G$ is a graph bundle with fibre $F$ over base $B$, $a \leq k_F$, and $b \leq k_B$. As an auxiliary result we prove that connectivity of graph bundle $G$ is at least $k_F + k_B$.

*Key words:* Cartesian graph bundles, Cartesian graph products, fault diameter, interconnection network.

## 1 Introduction

In the design of large interconnection networks several factors have to be taken into account. A usual constraint is that each processor can be connected to a limited number of other processors and the delays in communication must not be too long. Extensively studied network topologies in this context include graph products and bundles. For example the meshes, tori, hypercubes

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and some of their generalizations are Cartesian products. It is less known that some well-known topologies are Cartesian graph bundles, i.e. some twisted hypercubes [5,8] and multiplicative circulant graphs [16]. Other graph products, sometimes under different names, have been studied as interesting communication network topologies [16,13,4].

Furthermore, an interconnection network should be fault-tolerant. Since nodes of a network do not always work, if some nodes are faulty, some information may not be transmitted by some of these nodes. The fault diameter has been determined for many important networks recently [7,6,11,18]. The concept of fault diameter of Cartesian product graphs was first described in [10], but the upper bound was wrong, as shown by Xu, Xu and Hou who corrected the mistake [18]. An upper bound for the fault diameter of Cartesian graph bundles was given in [1].

In this paper we generalize the result of [18] to arbitrary number of factors. As a $k$-connected graph remains connected if up to $k-1$ vertices are missing, we study the diameter of a graph with any permitted number of vertices deleted.

First, we consider generalization of the upper bound for fault diameter to Cartesian graph bundles. Our result reads

**Theorem 1.1** *Let $F$ and $B$ be $k_F$-connected and $k_B$-connected graphs respectively, $0 \leq a < k_F$, $0 \leq b < k_B$, and $G$ a Cartesian bundle with fibre $F$ over the base graph $B$. Then*

$$\mathcal{D}_{a+b+1}(G) \leq \mathcal{D}_a(F) + \mathcal{D}_b(B) + 1.$$

We also prove the three theorems listed below.

**Theorem 1.2** *Let $G_1$, $G_2$, $\ldots$, $G_q$ be $k_1$-connected, $k_2$-connected,$\ldots$, $k_q$-connected graphs. Let $0 \leq a_1 < k_1$, $0 \leq a_2 < k_2, \ldots$, $0 \leq a_q < k_q$ and $a = a_1 + a_2 + \ldots + a_q + (q-1)$. Then the fault diameter of a a graph $G$, the Cartesian product of $G_1$, $G_2$, $\ldots$, $G_q$, with a faulty nodes is*

$$\mathcal{D}_a(G) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1.$$

On products with more than two factors, this result clearly improves previous bounds. Namely, if we applied the main theorem [18] on a Cartesian product $G$ of 3 factors $G_1, G_2, G_3$ then we would get

$$\mathcal{D}_{a+b+c+2}(G) \leq \mathcal{D}_{a+b+1}(G_1 \Box G_2) + \mathcal{D}_c(G_3) + 1 \leq \mathcal{D}_a(G_1) + \mathcal{D}_b(G_2) + \mathcal{D}_c(G_3) + 2,$$

which is of course more than $\mathcal{D}_a(G_1) + \mathcal{D}_b(G_2) + \mathcal{D}_c(G_3) + 1$.

In fact, Theorem 1.2 implies a more precise result for an upper bound of the fault diameter of a Cartesian product of graphs:

**Theorem 1.3** *Let $G_1$, $G_2$, ..., $G_q$ be $k_1$-connected, $k_2$-connected,..., $k_q$-connected graphs. Let $0 \leq a < k_1 + k_2 + \ldots + k_q$. Then*

$$\mathcal{D}_a(G) \leq \min\{\mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1 \mid$$

$$a_1 + a_2 + \ldots + a_q = a - (q-1), 0 \leq a_1 < k_1, 0 \leq a_2 < k_2, \ldots, 0 \leq a_q < k_q\}.$$

Furthermore, for cases with a small number of faulty vertices we prove the exact formula for computing the fault diameter:

**Theorem 1.4** *Let $G_1, G_2, \ldots, G_q$ be connected graphs, and $G = \Box_{i=1}^{q} G_i$. Then*

*(1)* $\mathcal{D}_a(G) = \sum_{i=1}^{q} \mathcal{D}_0(G_i) = \mathcal{D}_0(G)$ *for* $0 \leq a < q-1$;
*(2)* $\mathcal{D}_0(G) \leq \mathcal{D}_{q-1}(G) \leq \mathcal{D}_0(G) + 1$.

*where $\mathcal{D}_0(G)$ is the diameter of $G$.*

This text is a summary of our recent results on fault diameter. Theorem 1.1 also appears in [1], Theorems 1.2, 1.3 and 1.4 also appear in [3,2].

## 2 Preliminaries

Throughout the paper we will use the following definitions and notations.

**Definition 2.1** *A* simple graph $G = (V, E)$ *is determined by a* vertex set $V = V(G)$ *and a set $E = E(G)$ of (unordered) pairs of vertices, called the set of* edges.

As usual, we will use the short notation $uv$ for edge $\{u, v\}$. Two graphs are *isomorphic*, if there is a bijection between the vertex sets that preserves adjacency.

**Definition 2.2** *Let $G_1$ and $G_2$ be graphs. The* Cartesian product *of graphs $G_1$ and $G_2$, $G = G_1 \Box G_2$, is defined on the vertex set $V(G_1) \times V(G_2)$. Vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$.*

For $u \in V(G_1)$ define the *layers* $G_2(u) = \{(u, x) \mid x \in V(G_2)\}$ and for $v \in V(G_2)$ the *layers* $G_1(v) = \{(x, v) \mid x \in V(G_1)\}$. The layers are clearly isomorphic to factors, $G_1(u) \simeq G_1$ and $G_2(v) \simeq G_2$. For further reading on graph products we recommend [9].

**Definition 2.3** *Let $B$ and $F$ be graphs. A graph $G$ is a* Cartesian graph bundle with fibre $F$ over the base graph $B$ *if there is a* graph map $p : G \to B$ *such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to $F$, and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ is isomorphic to $F \square K_2$.*

More precisely, the mapping $p : G \to B$ maps graph elements of $G$ to graph elements of $B$, i.e. $p : V(G) \cup E(G) \to V(B) \cup E(B)$. In particular, here we also assume that the vertices of $G$ are mapped to vertices of $B$ and the edges of $G$ are mapped either to vertices or to edges of $B$. The mapping $p$ will also be called the *projection* (of the bundle $G$ to its base $B$). Note that each edge $e = uv \in E(B)$ naturally induces an isomorphism $\varphi_e : p^{-1}(\{u\}) \to p^{-1}(\{v\})$ between two fibres. It may be interesting to note that while it is well-known that a graph can have only one representation as a product (up to isomorphism and up to the order of factors) [9], there may be many different graph bundle representations of the same graph [19]. Here we assume that the bundle representation is given. (Graph bundles were first studied in [14].)

**Definition 2.4** *Let $G$ be a Cartesian graph bundle with fibre $F$. The* fibre of vertex $x \in G$ *will be denoted by $F_x$. Formally, $F_x = p^{-1}(p(x))$.*

For a later reference note that a graph bundle over a tree is isomorphic to a product, i.e. we can assume that all isomorphisms $\varphi_e$ are identities.

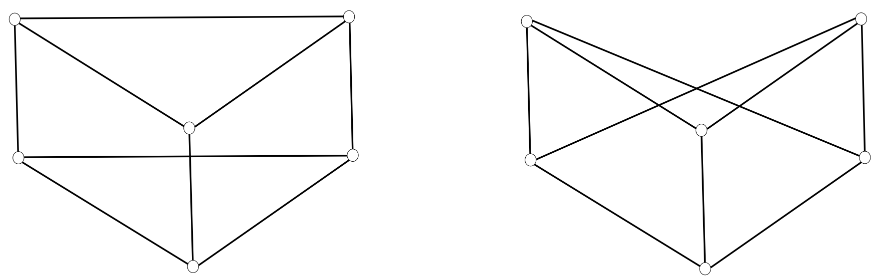

Fig. 1. Nonisomorphic bundles from Example 2.5

**Example 2.5** *Let $F = K_2$ and $B = C_3$. On Figure 1 we see two nonisomorphic bundles with fibre $F$ over the base graph $B$. Informally, one can say that bundles are "twisted products".*

**Example 2.6** *While it is well-known that the hypercube is a Cartesian product of edges, i.e. $Q_n = K_2 \square K_2 \square \ldots \square K_2$, it is less known that graph bundles also appear as computer topologies. A well known example is the twisted torus on Figures 2 and 3.*

**Definition 2.7** *A* walk *between $x$ and $y$ is a sequence of vertices $v_0$, $e_1$, $v_1$,*

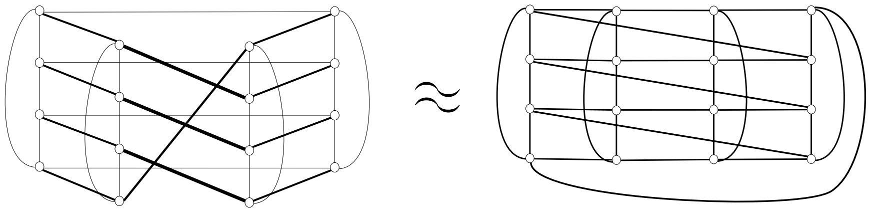

Fig. 2. Twisted torus: Cartesian graph bundle with fibre $C_4$ over base $C_4$.

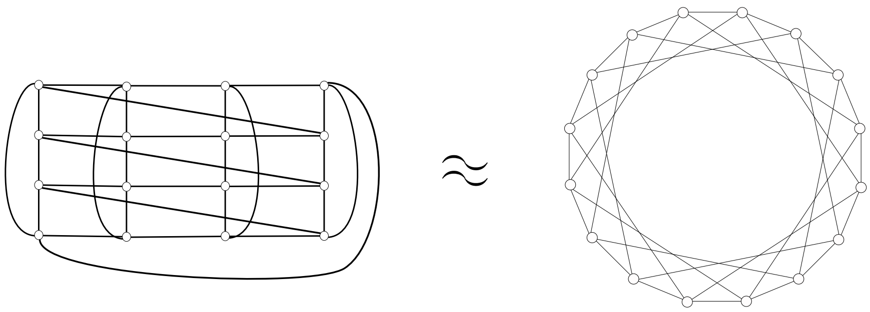

Fig. 3. Cartesian graph bundle with fibre $C_4$ over base $C_4$ is the ILIAC IV architecture.

$e_2$, $v_2$, $\ldots$, $v_{k-1}$, $e_k$, $v_k$ *where* $x = v_0$, $y = v_k$, *and* $e_i = v_{i-1}v_i$ *for each* $i$. *A walk with all vertices distinct is called a* path. *The* length *of a path* $P$, *denoted by* $\ell(P)$, *is the number of edges in* $P$. *The* distance *between vertices* $x$ *and* $y$, *denoted by* $\mathrm{d}_G(x, y)$ *is the length of a shortest path between* $x$ *and* $y$ *in* $G$. *The* diameter *of a graph* $G$, $\mathrm{d}(G)$, *is the maximum distance between any two vertices in* $G$.

A path $P$ in $G$, defined by a sequence $x = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = y$ can alternatively be seen as a subgraph of $G$ with $V(P) = \{v_1, v_2, \ldots, v_k\}$ and $E(P) = \{e_1, e_2, \ldots, e_k\}$.

**Definition 2.8** *The* connectivity *of a graph* $G$, $\kappa(G)$, *is the minimum cardinality over all vertex-separating sets in* $G$, *if* $G$ *is not a complete graph* $K_n$, *otherwise we define* $\kappa(K_n) = n - 1$. *A graph* $G$ *is said to be* $k$-connected, *if* $\kappa(G) \geq k$.

One of the Menger's theorems (see, for example, [17], page 167) reads:

**Theorem 2.9** (Menger) *If* $x$ *and* $y$ *are vertices of a graph* $G$ *and* $(x, y) \notin E(G)$, *then the minimum size of an* $x, y$*-cut equals the maximum number of pairwise internally disjoint* $x, y$*-paths.*

The following well-known corollary easily follows

**Corollary 2.10** *Let* $G$ *be a* $k$*-connected graph and* $\delta_G$ *be its minimum degree. Then* $\delta_G \geq k$.

For Cartesian product graphs, there is a well-known bound for the connectivity of the product (see for example [7]).

**Theorem 2.11** *Let $G_1$ and $G_2$ be $k_1$ and $k_2$-connected graphs, respectively. Then $G_1 \Box G_2$ is at least $(k_1 + k_2)$-connected.*

As a curiosity let us remark that only recently, a complete formula for the connectivity of the Cartesian product has been proved [15]. Namely, the connectivity of the Cartesian product is

$$\kappa(G_1 \Box G_2) = \min\{\kappa(G_1)|G_2|, \kappa(G_2)|G_1|, \delta_{G_1} + \delta_{G_2}\}$$

as claimed already in [12].

**Definition 2.12** *Let $G$ be a graph and $x \in V(G)$ a vertex. The* neighborhood *of the vertex $x$ in the graph $G$, $N_G(x)$, is the set of all vertices in $G$ that are adjacent to $x$.*

**Definition 2.13** *Let $G$ be a graph, $x, y \in V(G)$ distinct vertices, $P$ a path from $x$ to $y$ in $G$, and $z \in V(P) \setminus \{x, y\}$. We will use $x \xrightarrow{P} z$ to denote the subpath $\tilde{P} \subseteq P$ from $x$ to $z$. If $z$ is adjacent to $x$ in $P$, we will simply use $x \to z$.*

**Definition 2.14** *Let $G$ be a graph and $X \subseteq V(G)$. A path $P$ from a vertex $x$ to a vertex $y$* avoids *$X$ in $G$, if $V(P) \cap X = \emptyset$, and it* internally avoids *$X$, if $(V(P) \setminus \{x, y\}) \cap X = \emptyset$.*

Let $G = G_1 \Box G_2$, $P$ a path in $G_2$, and $v$ a vertex of $G_1$. For simplicity of notation, we will also use $P$ to denote the path $\{v\} \Box P$ in the layer $G_2(v)$.

**Theorem 2.15** *Let $F$ and $B$ be $k_F$-connected and $k_B$-connected graphs respectively, and $G$ a Cartesian bundle with fibre $F$ over the base graph $B$. Let $\kappa(G)$ be the connectivity of $G$. Then $\kappa(G) \geq k_F + k_B$.*

**Proof.** Let $p : G \to B$ be a projection such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to $F$, and for each edge $e = (u, v) \in E(B)$, $p^{-1}(\{e\})$ induces an isomorphism $\varphi_e : F_u \to F_v$. Let $x$ and $y$ be two distinct vertices in $G$. By Theorem 2.9 it is enough to construct $k_F + k_B$ of pairwise internally disjoint paths from $x$ to $y$ in $G$.

Suppose $p(x) \neq p(y)$. As $B$ is $k_B$-connected, there are $k_B$ pairwise internally disjoint paths $Q'_1, Q'_2, \ldots, Q'_{k_B}$ from $p(x)$ to $p(y)$ in $B$. As $p^{-1}(Q'_1)$ is isomorphic to $Q'_1 \Box F$, hence by Theorem 2.11 and Theorem 2.9, there are $k_F + 1$ pairwise internally disjoint paths $P_1, P_2, \ldots, P_{k_F+1}$ from $x$ to $y$ in $p^{-1}(Q'_1) \subseteq G$.

Now let $Q_i$ be a path from $y$ to a point $z_i \in F_x$ in $G$ such that $p(Q_i) = Q'_i$, for each $i = 2, 3, \ldots, k_B$. As all the paths $Q'_i$ are pairwise internally disjoint, it follows that for each $i = 2, 3, \ldots, k_B$, $Q_i$ intersects $p^{-1}(Q'_1)$ only at the end vertices. Furthermore, by construction, $Q_2, Q_3, \ldots, Q_{k_B}$ are pairwise internally disjoint paths. Let $N = p^{-1}(N_B\ (p(x)))$. For each $i = 2, 3, \ldots, k_B$, let $x_i$ in $N \cap Q_i$. As the paths $Q'_2, Q'_3, \ldots, Q'_{k_B}$ are pairwise internally disjoint, and as $p(x_i) \in Q'_i \setminus \{p(x), p(y)\}$, therefore $F_{x_1}$, $F_{x_2}$, $\ldots$, $F_{x_{k_B}}$ are pairwise disjoint fibres in $G$, such that each of these fibres is disjoint with $p^{-1}(Q'_1)$. For each $i = 2, 3, \ldots, k_B$, let $w_i$ be the vertex in $F_{x_i}$, adjacent to $x$. As $F_{x_i}$ is connected for each $i = 2, 3, \ldots, k_B$, there is a path $R'_i : x_i \to w_i$ in $F_{x_i}$. Observe, that

$$R_2 : y \overset{Q_2}{\to} x_2 \overset{R'_2}{\to} w_2 \to x,$$

$$R_3 : y \overset{Q_3}{\to} x_3 \overset{R'_3}{\to} w_3 \to x,$$

$$\vdots$$

$$R_{k_B} : y \overset{Q_{k_B}}{\to} x_{k_B} \overset{R'_{k_B}}{\to} w_{k_B} \to x$$

are pairwise internally disjoint paths from $y$ to $x$ in $G$. This way we constructed $k_F + k_B$ pairwise internally disjoint paths $P_1, P_2, \ldots, P_{k_F+1}$, $R_2, R_3, \ldots, R_{k_B}$ from $x$ to $y$ in $G$.

If $p(x) = p(y)$ then recall that there are $k_F$ internally disjoint paths from $x$ to $y$ within the fibre $F_x$. Additional $k_B$ paths can be constructed by taking $k_B \le \delta_B$ neighbors of $p(x)$ in $B$, say $z_1, z_2, \ldots z_{k_B}$ and observing that the paths

$$x \to \varphi_{xz_i}(x) \overset{P}{\to} \varphi_{xz_i}(y) \to y,$$

where $P$ is any path from $\varphi_{xz_i}(x)$ to $\varphi_{xz_i}(y)$ in $F_{z_i}$, are pairwise disjoint and are disjoint with all paths in $F_x$. $\square$

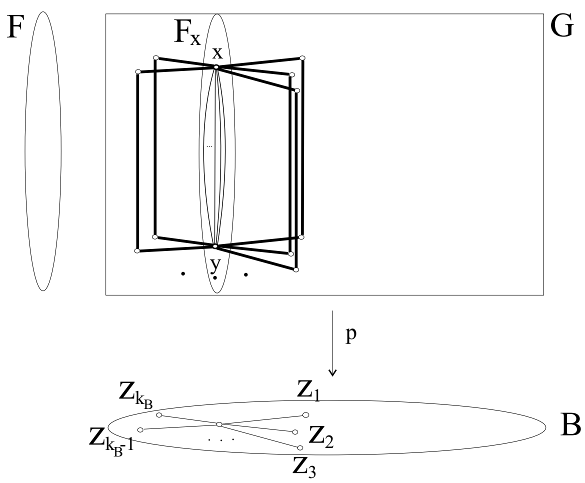


Fig. 4. Construction of the $k_B + k_F$ paths in the case, where $p(x) = p(y)$.

**Definition 2.16** *Let $G$ be a $k$-connected graph and $0 \leq a < k$. Then we define the $a$-*fault diameter *of $G$ as*

$$\mathcal{D}_a(G) = \max\{\mathrm{d}(G \setminus X) \mid X \subseteq V(G), |X| = a\}.$$

Note that $\mathcal{D}_a(G)$ is the largest diameter among subgraphs of $G$ with $a$ vertices deleted, hence $\mathcal{D}_0(G)$ is just the diameter of $G$, $\mathrm{d}(G)$. For $a \geq k$, the fault diameter of $k$-connected graph does not exist. In other words, $\mathcal{D}_a(G) = \infty$ as some of the graphs are not connected. This definition slightly differs from the one of [18], see also the conclusions section.

## 3 Fault diameter of Cartesian graph bundles

One of our main results is an upper bound for the fault diameter of Cartesian graph bundles in terms of fault diameters of the fibre and the base graph.

We will use the following technical lemma in the proof of Theorem 1.1. The lemma follows from the upper bound for the Cartesian product [18], but as the proof is short we decided to write it for completeness.

**Lemma 3.1** *Let $G = Q \Box F$ be a Cartesian product of a path $Q$ with vertices $V(Q) = \{v_0, v_1, \ldots, v_k\}$ and a graph $F$ with $\mathcal{D}_a(F) < \infty$. Let $s$ and $t$ be vertices of $G$ with coordinates $s = (s_1 = v_0, s_2)$ and $t = (t_1 = v_k, t_2)$ and let $X \subseteq V(G) \setminus \{s, t\}$ be a set of vertices with $|X| \leq a + 1$. Then $\mathrm{d}_{G \setminus X}(s, t) \leq \mathcal{D}_a(F) + \ell(Q) + 1$.*

**Proof.** Let $X \subseteq V(G) \setminus \{s, t\}$ be a set of vertices with $|X| \leq a + 1$. For $u \in V(F)$ define the *layers* $Q(u) = \{(x, u) \mid x \in Q\}$ and the weights of layers $w(Q(u)) = w(u) = |Q(u) \cap X|$. For each $v \in V(B)$, let $F(v) = p^{-1}\{v\}$

We distinguish two cases. First, if $|F(v_k) \cap X| > a$ then $|F(v_0) \cap X| = 0$ and $w(Q(t_2)) = w(t_2) = |Q(t_2) \cap X| = 0$. Therefore there is a path from $s = (s_1, s_2)$ to $(s_1, t_2)$ of length $\mathcal{D}_0(F) \leq \mathcal{D}_a(F)$ and a path from from $(s_1, t_2)$ to $t = (t_1, t_2)$ of length $\ell(Q)$.

Second, if $|F(v_k) \cap X| \leq a$. As $F$ is at least $a+1$ connected, there are at least $a + 1$ neighbors of $s_2$ in $V(F)$. Denote the neighbors by $u_i$, $i = 1, 2 \ldots, a + 1$. Among the $a+2$ layers $Q(s_2), Q(u_1), \ldots Q(u_{a+1})$, at least one does not intersect $X$. If $w(Q(s_2)) = 0$ then the path $Q(s_2)$ from $s$ to $(t_1, s_2)$ has the length $\ell(Q(s_2)) = \ell(Q)$, and because of $|F(v_k) \cap X| \leq a$, there is a path $R$ from $(t_1, s_2)$ to $t = (t_1, t_2)$ of length $\ell(R) \leq \mathcal{D}_a(F)$. If $w(Q_{u_i}) = 0$ for one of the neighbors then we may construct the path $P$ as

$$P : s = (s_1, s_2) \to (s_1, u_i) \stackrel{Q(u_i)}{\to} (t_1, u_i) \stackrel{R}{\to} t = (t_1, t_2)$$

and $\ell(P) \leq 1 + \ell(Q(u_i)) + \mathcal{D}_a(F) = 1 + \ell(Q) + \mathcal{D}_a(F)$. $\square$

**Theorem 1.1** *Let $F$ and $B$ be $k_F$-connected and $k_B$-connected graphs respectively, $0 \leq a < k_F$, $0 \leq b < k_B$, and $G$ a Cartesian bundle with fibre $F$ over the base graph $B$. Then*

$$\mathcal{D}_{a+b+1}(G) \leq \mathcal{D}_a(F) + \mathcal{D}_b(B) + 1.$$

**Proof.** Let $k = a + b + 2$. By Theorem 2.15, $\kappa(G) \geq k$, hence $\mathcal{D}_{k-1}(G)$ is well-defined. Let $\delta_F$ be the minimum degree of $F$ and $\delta_B$ be the minimum degree of $B$. Recall that $\delta_F \geq \kappa(F) > a$ and $\delta_B \geq \kappa(B) > b$. Finally, let $X \subseteq V(G)$ such that $|X| = k - 1$ and $x, y \in V(G \setminus X)$ be two distinct vertices. We shall construct a path $P$ from $x$ to $y$ in $G \setminus X$ such that the length $\ell(P) \leq \mathcal{D}_a(F) + \mathcal{D}_b(B) + 1$.

We first assume that $x$ and $y$ are in distinct fibres, i.e. $p(x) \neq p(y)$.

As before, let $p : G \to B$ be a projection from $G$ to its base $B$. For each $v \in V(B)$, let $w(v) = |p^{-1}(\{v\}) \cap X|$.

Let $\mathcal{V} = \{v \mid v \in V(B) \setminus \{p(x), p(y)\}, w(v) > 0\}$, and $b_0 = |\mathcal{V}|$. As $B$ is $k_B$-connected, and $0 \leq b < k_B$, there are at least $b + 1$ internally disjoint paths $Q_0, Q_1, Q_2, \ldots, Q_b$ between $p(x)$ and $p(y)$ in $B$.

We now distinguish two cases.

CASE 1. $b_0 < b$. If $p(x)$ and $p(y)$ are not adjacent in $B$, then there is a path $Q$ from $p(x)$ to $p(y)$ in $B$ that internally avoids $p(X)$, $|X \cap p^{-1}(u)| = 0$ for each $u \in V(Q) \setminus \{p(x), p(y)\}$, with length $\ell(Q) \leq \mathcal{D}_b(B)$. Let $\tilde{Q}$ be a path from $x$ to a vertex $\tilde{y} \in F_y$, such that $p(\tilde{Q}) = Q$, and $\ell(\tilde{Q}) = \ell(Q)$. Let $v_1 \in V(\tilde{Q})$ be a vertex, adjacent to $\tilde{y}$, and $v_2 \in F_{v_1}$ a vertex, adjacent to $y$. As $|X \cap F_{v_1}| = 0$, there is a path $R$ in $F_{v_1}$ from $v_1$ to $v_2$, such that $\ell(R) \leq \mathcal{D}_0(F) \leq \mathcal{D}_a(F)$. We may construct the path $P$ as

$$P : x \stackrel{\tilde{Q}}{\to} v_1 \stackrel{R}{\to} v_2 \to y$$

and $\ell(P) \leq \mathcal{D}_b(B) - 1 + \mathcal{D}_0(F) + 1 \leq \mathcal{D}_a(F) + \mathcal{D}_b(B)$.

If $p(x)$ and $p(y)$ are adjacent in $B$, there is a neighbor $u$ of $p(x)$ in $B$, such that $w(u) = 0$ (note that $p(x)$ has at least $b+1$ neighbors). If $u$ is adjacent also to $p(y)$, then we may proceed as before, using for $Q$ the path $Q : x \to u \to y$. If $u$ is not adjacent to $p(y)$, then there is a path $Q$ from $u$ to $p(y)$ in $B$ that internally avoids $p(X)$, $|X \cap p^{-1}(u)| = 0$ for each $u \in V(Q) \setminus \{p(x), p(y)\}$, with length $\ell(Q) \leq \mathcal{D}_b(B)$. Now let $u'$ be the vertex in $p^{-1}(u)$, adjacent to $x$ in $G$. Let $\tilde{Q}$ be a path from $u'$ to a vertex $\tilde{y} \in F_y$, such that $p(\tilde{Q}) = Q$, and $\ell(\tilde{Q}) = \ell(Q)$. Let $v_1 \in V(\tilde{Q})$ be a vertex, adjacent to $\tilde{y}$, and $v_2 \in F_{v_1}$ a vertex, adjacent to $y$. As $|X \cap F_{v_1}| = 0$, there is a path $R$ in $F_{v_1}$ from $v_1$ to $v_2$, such that $\ell(R) \leq \mathcal{D}_0(F) \leq \mathcal{D}_a(F)$. We may construct the path $P$ as

$$P : x \to u' \xrightarrow{\tilde{Q}} v_1 \xrightarrow{R} v_2 \to y$$

and $\ell(P) \leq 1 + \mathcal{D}_b(B) - 1 + \mathcal{D}_0(F) + 1 \leq \mathcal{D}_a(F) + \mathcal{D}_b(B) + 1$.

CASE 2. If $b_0 \geq b$, then there is a path in $B$ which does not intersect $p(X)$ in too many vertices. To see this, delete $b$ vertices of $B \setminus \{p(x), p(y)\}$ with positive $w$. As $B$ is $k_B$-connected and $k_B > b$, there is a path $Q$ between $p(x)$ and $p(y)$ in $B$ of length $\ell(Q) \leq \mathcal{D}_b(B)$ that avoids deleted vertices. Formally, $\sum_{v_i \in V(Q)} w(v_i) \leq a + 1$. The subgraph $p^{-1}(Q)$ intersects $X$ in at most $a + 1$ vertices and is isomorphic to a Cartesian product, $p^{-1}(Q) \simeq Q \square F$. By Lemma 3.1, a path of length $\leq \mathcal{D}_b(B) + \mathcal{D}_a(F) + 1$ from $x$ to $y$ can be constructed, concluding the argument for subcase 2.

To complete the proof, we have to consider the case where $x$ and $y$ are in the same fibre, i.e. $p(x) = p(y)$, and $F_x = F_y$. The idea, without details, is as follows. If $|X \cap F_x| \leq a$ then there is a path of length at most $\mathcal{D}_a(F)$ within the fibre. If $|X \cap F_x| > a$, then there are at least $b+1$ neighbors of $x$ in $B$ and hence there are at least $b + 1$ fibres that can be used to construct internally disjoint paths from $x$ to $y$. At least one of these paths does not intersect $X$, and we have a path of distance at most $1 + \mathcal{D}_0(F) + 1 \leq \mathcal{D}_b(B) + \mathcal{D}_a(F) + 1$. This completes the proof. $\square$

## 4 Product of $q$ factors

Before proving Theorem 1.2, let us prove Lemma 4.1, which will be useful in the proof of Theorem 1.2.

**Lemma 4.1** *Let $G_1, G_2, \ldots, G_q$ be 1-connected graphs, and $G = \square_{i=1}^q G_i$. Then*

*(1) $\mathcal{D}_a(G) = \sum_{i=1}^q \mathcal{D}_0(G_i) = \mathcal{D}_0(G)$ for $0 \leq a < q - 1$;*
*(2) if there is a factor $G_i$ such that $\mathrm{d}(G_i) < 2$, then $\mathcal{D}_{q-1}(G) = \mathcal{D}_0(G) + 1$;*

*(3) if* $\mathrm{d}(G_i) \geq 2$ *for all factors* $G_i$*, then* $\mathcal{D}_{q-1}(G) = \mathcal{D}_0(G)$*.*

**Proof.** For each $i = 1, 2, 3, \ldots, q$, let $p_i : G \to G_i$ be the projection on $G_i$.

CASE 1. $p_i(x) \neq p_i(y)$ for all $i$. Then there are at least $q$ internally disjoint paths $P$ of length $\ell(P) \leq \sum_{i=1}^{q} \mathcal{D}_0(G_i) = \mathcal{D}_0(G)$. As $a < q$, at least one of them avoids faulty vertices. Therefore there is a path $P$ in $G$ such that $P$ avoids faulty vertices, and $\ell(P) \leq \mathcal{D}_0(G)$. Therefore $\mathcal{D}_a(G) \leq \mathcal{D}_0(G)$.

CASE 2. $p_i(x) = p_i(y)$ for at least two indexes $i$. Without loss of generality, assume that $p_i(x) \neq p_i(y)$ for $i$=1,2,…,$k$ and $p_i(x) = p_i(y)$ for $i$=$k$+1,…, $q$. There are at least $k$ vertex disjoint shortest paths between $x$ and $y$ within the first $k$ factors. The length of these paths is at most $\ell = \sum_{i=1}^{k} \mathcal{D}_0(G_i)$. We can construct additional $q - k$ vertex disjoint paths from $x$ to $y$ of length $\ell + 2$ as follows. Take any of the shortest paths $P$, choose a neighbor in the $i$-th factor (for $i = q, q-1, \ldots, k+1$) and construct a new path.

$$x \to u \xrightarrow{P} v \to y$$

Precisely, for $i = q$, take a neighbor of $x$, $u = (x_1, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, u_q)$. Then $v = (y_1, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, u_q)$ is a neighbor of $y$ and there is a path of length $1+\ell+1$ from $x$ to $y$. Clearly $\ell+2 \leq \sum_{i=1}^{k} \mathcal{D}_0(G_i) + q - k \leq \sum_{i=1}^{q} \mathcal{D}_0(G_i)$.

CASE 3. $p_i(x) = p_i(y)$ for exactly one $i$. Say $p_q(x) = p_q(y)$. Then there are at least $q-1$ paths $P$ from $x$ to $y$ in the layer $L(x) = p_q^{-1}(p_q(x))$ with length $\ell(P) \leq \sum_{i=1}^{q-1} \mathcal{D}_0(G_i) < \sum_{i=1}^{q} \mathcal{D}_0(G_i)$. If $a < q-1$, then there is a path $P$ in $G$ that avoids faulty vertices and $\ell(P) \leq \sum_{i=1}^{q} \mathcal{D}_0(G_i) = \mathcal{D}_0(G)$. Therefore $\mathcal{D}_a(G) \leq \mathcal{D}_0(G)$. If $a = q-1$, then either one of the paths has no faulty vertices or all the faulty vertices appear in $L(x)$. In the worst case (if all the faulty vertices appear in $L(x)$) there is a path $P$ from $x$ to $y$ with length $\ell(P) \leq 1 + \sum_{i=1}^{q-1} \mathcal{D}_0(G_i) + 1 \leq \mathcal{D}_0(G) + 1$. Therefore $\mathcal{D}_a(G) \leq \mathcal{D}_0(G) + 1$.

Summarizing, we have $\mathcal{D}_a(G) \leq \mathcal{D}_0(G)$ for $a < q-1$. As $\mathcal{D}_a(G) \geq \mathcal{D}_0(G)$ for each $a$, hence $\mathcal{D}_a(G) = \mathcal{D}_0(G) = \sum_{i=1}^{q} \mathcal{D}_0(G_i)$ for all $a$, $0 < a < q-1$. If $a = q-1$, then we have $\mathcal{D}_a(G) \leq \mathcal{D}_0(G) + 1$ and hence $\mathcal{D}_0(G) \leq \mathcal{D}_a(G) \leq \mathcal{D}_0(G) + 1 = \sum_{i=1}^{q} \mathcal{D}_0(G_i) + 1$. Furthermore, if there is an integer $i$ such that $\mathrm{d}(G_i) < 2$, then $\mathcal{D}_{q-1}(G) = \mathcal{D}_0(G) + 1$. If $\mathrm{d}(G_i) \geq 2$ for all integers $i$, then $\mathcal{D}_{q-1}(G) = \mathcal{D}_0(G)$. □

**Example 4.2** *Let* $G = Q_q = \square_{n=1}^{q} K_2$*. Then* $G$ *is* $q$*-connected. It follows from Theorem 4.1 that* $\mathcal{D}_{q-1}(G) = q+1$*, and* $\mathcal{D}_a(G) = q$ *for* $0 \leq a < q-1$*.*

In the proof of Lemma 4.1 we only used the assumption that the factors are connected, therefore essentially the same proof gives

**Theorem 1.4** *Let $G_1, G_2, \ldots, G_q$ be connected graphs, and $G = \square_{i=1}^{q} G_i$. Then*

*(1) $\mathcal{D}_a(G) = \sum_{i=1}^{q} \mathcal{D}_0(G_i) = \mathcal{D}_0(G)$ for $0 \leq a < q-1$;*
*(2) $\mathcal{D}_0(G) \leq \mathcal{D}_{q-1}(G) \leq \mathcal{D}_0(G) + 1$.*

Now we recall and prove Theorem 1.2.

**Theorem 1.2** *Let $G_1$, $G_2$, $\ldots$, $G_q$ be $k_1$-connected, $k_2$-connected,$\ldots$, $k_q$-connected graphs. Let $0 \leq a_1 < k_1$, $0 \leq a_2 < k_2, \ldots$, $0 \leq a_q < k_q$ and $a = a_1 + a_2 + \ldots + a_q + (q-1)$. Then the fault diameter of $G$, a Cartesian product of $G_1$, $G_2$, $\ldots$, $G_q$, with $a$ faulty nodes is*

$$\mathcal{D}_a(G) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_3) + 1.$$

**Proof.** We prove the result by induction on the number of factors. The assertion holds for $q = 1$, trivially, and for $q = 2$ it is a special case of Theorem 1.1. A sketch of proof for the case $q = 3$ appears in [3].

Let $G = \square_{i=1}^{q} G_i$, $X \subseteq V(G)$, such that $|X| = a$, and let $x = (x_1, x_2, x_3, \ldots, x_q)$ and $y = (y_1, y_2, y_3, \ldots, y_q) \in V(G \setminus X)$. We shall construct a path $P$ from $x$ to $y$ in $G \setminus X$ such that the length $\ell(P) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1$. We assume that the theorem holds for all $n = 1, 2, \ldots, q-1$, and show that it holds also for $n = q > 2$.

In the rest of the proof below we also assume that $q > 2$ and there is at least one factor which is more than 1-connected. (Recall that we proved the claim of Theorem 1.2 for a product of 1-connected graphs by proving Lemma 4.1.)

Let for each $i = 1, 2, 3, \ldots, q$, $p_i : G \to G_i$ be the $i$-th projection on $G_i$.

CASE 1. $p_i(x) = p_i(y)$ for some $i$. Without loss of generality, we can say $i = q$. For each $u \in G_q$ let $w(u) = |X \cap p_q^{-1}(u)|$. If $w(p_q(x)) < a - a_q$ then there is a path $P$ in $p_q^{-1}(p_q(x)) \subseteq G$ from $x$ to $y$ with required length, by induction. Assume $w(p_q(x)) \geq a - a_q$. Vertex $p_q(x)$ has at least $a_q + 1$ neighbors $z_1, z_2, z_3, \ldots, z_{a_q+1}$ in $G_q$. As $w(p_q^{-1}(p_q(x))) \geq a - a_q$, there is an index $i \in \{1, 2, 3, \ldots, a_q + 1\}$ such that $w(z_i) = 0$. Therefore there is a path $\tilde{P}$ in $p_q^{-1}(z_i)$ from $(x_1, x_2, \ldots, x_{q-1}, z_i)$ to $(y_1, y_2, \ldots, y_{q-1}, z_i)$ with length $\ell(\tilde{P}) \leq \mathcal{D}_0(G_1) + \mathcal{D}_0(G_2) + \ldots + \mathcal{D}_0(G_{q-1})$ and

$$P : x \to (x_1, x_2, \ldots, x_{q-1}, z_i) \xrightarrow{\tilde{P}} (y_1, y_2, \ldots, y_{q-1}, z_i) \to y$$

is a path from $x$ to $y$ in $G$ with length $\ell(P) \leq \mathcal{D}_0(G_1) + \mathcal{D}_0(G_2) + \ldots + \mathcal{D}_0(G_{q-1}) + 2 \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1$.

CASE 2. $p_i(x) \neq p_i(y)$ for all $i$. Let $G_q$ be one of the factors which is more than 1-connected. As before, let $w(u) = |X \cap p_q^{-1}(u)|$ for each $u \in G_q$. We distinguish two subcases.

SUBCASE 2.1. $w(p_q(x)) \geq a - a_q$ or $w(p_q(y)) \geq a - a_q$. We may assume $w(p_q(x)) \geq a - a_q$. Then there is a neighbor $u$ of $p_q(y)$ in $G_q$ such that $w(u) = 0$ (recall that $p(y)$ has at least $a_q + 1$ neighbors in $G_q$). Therefore there is a path $\tilde{P}$ in $p_q^{-1}(u)$ from $(y_1, y_2, \ldots, y_{q-1}, u)$ to $(x_1, x_2, \ldots, x_{q-1}, u)$ with length $\ell(\tilde{P}) \leq \mathcal{D}_0(G_1) + \mathcal{D}_0(G_2) + \mathcal{D}_0(G_{q-1})$. As $x \notin X$, there are at most $a_q$ faulty vertices in the layer of $x$, $|G_q(x) \cap X| \leq a_q < a_q + 1$, and hence there is a path $Q$ from $(x_1, x_2, \ldots, x_{q-1}, u)$ to $x$ in $G_q(x)$ with length $\ell(Q) \leq \mathcal{D}_{a_q}(G_q)$. Therefore the path (see Fig. 5)

$$P : y \to (y_1, y_2, \ldots, y_{q-1}, u) \xrightarrow{\tilde{P}} (x_1, x_2, \ldots, x_{q-1}, u) \xrightarrow{Q} x$$

is a path from $y$ to $x$ in $G$ with length $\ell(P) \leq 1 + \mathcal{D}_0(G_1) + \mathcal{D}_0(G_2) + \ldots + \mathcal{D}_0(G_{q-1}) + \mathcal{D}_{a_q}(G_q) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \mathcal{D}_{a_3}(G_3) + \ldots + \mathcal{D}_{a_q}(G_q) + 1$.

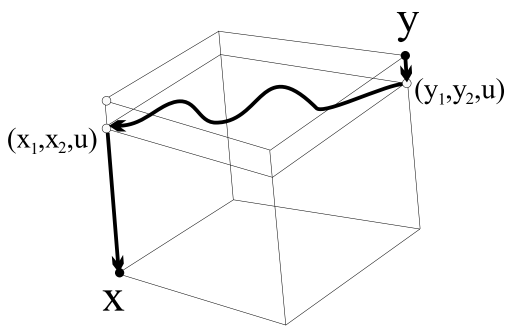


Fig. 5. The construction of a path in subcase 1 for $q = 3$.

SUBCASE 2.2. $w(p_q(x)) < a - a_q$ and $w(p_q(y)) < a - a_q$.

Assume first $w(p_q(x)) + w(p_q(y)) > a - a_q$. In $V(G_q) \setminus \{p_q(x), p_q(y)\}$ there are at most $a_q$ vertices $u$, such that $w(u) > 0$. After deleting them, we may construct a path $\tilde{P}$ in $G_q$ from $p_q(x)$ to $p_q(y)$ with length $\ell(\tilde{P}) \leq \mathcal{D}_{a_q}(G_q)$, such that for each $u \in V(\tilde{P} \setminus \{p_q(x), p_q(y)\})$, $w(u) = 0$. The existence of such $u$ follows from the assumption that $G_q$ is more than 1-connected,and hence there is at least one internal vertex along the path $\tilde{P}$. Let $v$ be the vertex in $\tilde{P}$, adjacent to $p_q(x)$. As $w(v) = 0$, there is a path $Q$ from $(x_1, x_2, \ldots, x_{q-1}, v)$ to $(y_1, y_2, \ldots, y_{q-1}, v)$ in $p_q^{-1}(v)$ with length $\ell(Q) \leq \mathcal{D}_0(G_1) + \mathcal{D}_0(G_2) + \ldots + \mathcal{D}_0(G_{q-1})$. Therefore the path (see Figure 6)

$$P : x \to (x_1, x_2, \ldots, x_{q-1}, v) \xrightarrow{Q} (y_1, y_2, \ldots, y_{q-1}, v) \xrightarrow{\tilde{P}} y$$

is the path from $x$ to $y$ in $G$ with length $\ell(P) \leq 1 + \mathcal{D}_0(G_1) + \mathcal{D}_0(G_2) + \ldots + \mathcal{D}_0(G_{q-1}) + \mathcal{D}_{a_q}(G_q) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1$.

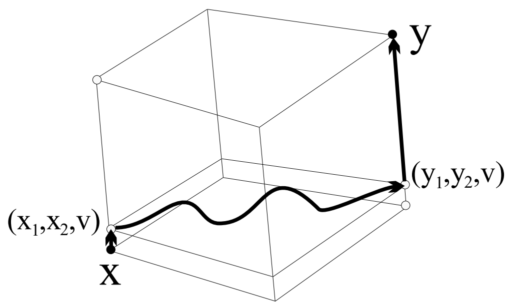


Fig. 6. The construction of the path $P$ for $q = 3$.

Now assume $w(p_q(x)) + w(p_q(y)) \leq a - a_q$. We claim that there is a path $\tilde{P}$ from $p_q(x)$ to $p_q(y)$ in $G_q$ such that $w(\tilde{P}) \leq a - a_q$. This is easily seen as follows: In the subgraph $G_q \setminus \{p(x), p(y)\}$, choose a set $Y$ of $a_q$ vertices with maximal $w$. Then $w(G_q \setminus \{p(x), p(y)\}) \leq a - a_q$ and hence there is a path $\tilde{P}$ of length at most $\ell(\tilde{P}) \leq \mathcal{D}_{a_q}(G_q)$ and with $w(\tilde{P}) \leq a - a_q$, which proves the claim. Let $p : G_1 \Box G_2 \Box \ldots \Box G_{q-1} \Box \tilde{P} \to G_1 \Box G_2 \Box \ldots \Box G_{q-1}$ be the projection, defined with $p(x_1, x_2, \ldots, x_{q-1}, x_q) = (x_1, x_2, \ldots, x_{q-1})$. For each $u \in G_1 \Box G_2 \Box \ldots \Box G_{q-1}$ let $W(u) = |p^{-1}(u) \cap X|$. We consider next two possibilities.

- $W(p(x)) = 0$ or $W(p(y)) = 0$. Say $W(p(x)) = 0$. Therefore there is a path $Q$ from $x$ to $(x_1, x_2, \ldots, x_{q-1}, y_q)$ in $p^{-1}(x)$, such that $\ell(Q) \leq \mathcal{D}_0(G_q)$. As $w(p_q(y)) < a - a_q$, there is a path $\tilde{Q}$ in $p_q^{-1}(p(y))$ from $(x_1, x_2, \ldots, x_{q-1}, y_3)$ to $y$ with length $\ell(\tilde{Q}) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_{q-1}}(G_{q-1}) + 1$. Hence

$$P : x \xrightarrow{Q} (x_1, x_2, \ldots, x_{q-1}, y_q) \xrightarrow{\tilde{Q}} y$$

  is a path from $x$ to $y$ in $G$, such that $\ell(P) \leq \mathcal{D}_0(G_q) + \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_{q-1}}(G_{q-1}) + 1 \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1$ (see Figure 7).

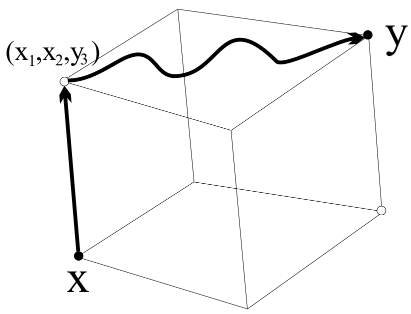


Fig. 7. The construction of the path $P$ for $q = 3$.

- $W(p(x)) > 0$ and $W(p(y)) > 0$. As $w(\tilde{P}) \leq a - a_q$ there is a path $Q$ from $p(x)$ to $p(y)$ in $G_1 \Box G_2 \Box \ldots \Box G_{q-1}$ with length $\ell(Q) \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_{q-1}}(G_{q-1}) + 1$ and for each $u \in V(Q) \setminus \{p(x), p(y)\}$, $W(u) = 0$. (Here we need the fact that $G_1 \Box G_2 \Box \ldots \Box G_{q-1}$ is not isomorphic to $K_2$, which is trivially true if $q > 2$. ) Let $u$ be the vertex, adjacent to $p(x)$ in $Q$. As

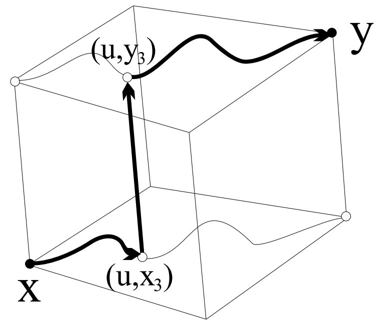


Fig. 8. The construction of the path $P$ for $q = 3$.

$W(u) = 0$, there is a path $\tilde{Q}$ from $(u, x_q)$ to $(u, y_q)$ in $p^{-1}(u)$ with length $\ell(\tilde{Q}) \leq \mathcal{D}_0(G_q)$. Finally we may construct the required path

$$P : x \rightarrow (u, x_q) \xrightarrow{\tilde{Q}} (u, y_q) \xrightarrow{Q} y$$

from $x$ to $y$, such that $\ell(P) \leq 1 + \mathcal{D}_0(G_q) + \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_{q-1}}(G_{q-1}) + 1 - 1 \leq \mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1$ (see Figure 8). □

In the proof of Theorem 1.2 we have assumed that each $G_i$ is at least $a_i + 1$ connected, and we only needed that $a_i + 1 \leq k_i$. Given $G$ and $X$ we may read the proof with arbitrary choice of $a_i$ which satisfies the conditions $a_1 + a_2 + \ldots + a_q = a - (q-1), 0 \leq a_1 < k_1, 0 \leq a_2 < k_2, \ldots, 0 \leq a_q < k_q\}$. This argument proves

**Theorem 1.3** *Let $G_1$, $G_2$, ..., $G_q$ be $k_1$-connected, $k_2$-connected,..., $k_q$-connected graphs. Let $0 \leq a < k_1 + k_2 + \ldots + k_q$. Then*

$$\mathcal{D}_a(G) \leq \min\{\mathcal{D}_{a_1}(G_1) + \mathcal{D}_{a_2}(G_2) + \ldots + \mathcal{D}_{a_q}(G_q) + 1 \mid$$

$$a_1 + a_2 + \ldots + a_q = a - (q-1), 0 \leq a_1 < k_1, 0 \leq a_2 < k_2, \ldots, 0 \leq a_q < k_q\}.$$

**Example 4.3** *Let $G = C_6 \Box C_{100}$. From Theorem 1.3,*

$$\mathcal{D}_2(G) \leq \min\{\mathcal{D}_1(C_6) + \mathcal{D}_0(C_{100}), \mathcal{D}_0(C_6) + \mathcal{D}_1(C_{100})\} + 1 =$$

$$= \min\{54, 101\} + 1 = 55.$$

*Clearly, the bound from Theorem 1.3 improves the upper bound from Theorem 1.2. For example, for $a = 0$ and $b = 1$, Theorem 1.2 gives $\mathcal{D}_{a+b+1}(G) \leq \mathcal{D}_a(C_6) + \mathcal{D}_b(C_{100}) + 1 = 102$. On the other hand, one can easily check that $\mathcal{D}_2(G) = 53$, therefore Corollary 1.3 does not give the exact formula for computing the fault diameter. In fact, the bound can be far from the exact* *value. For example, by Theorem 1.3,* $\mathcal{D}_3(G) \leq 103$*, while the exact value is* $\mathcal{D}_3(G) = 53$.

**Example 4.4** *We have computed the fault diameters of the hypercube using Theorem 1.4 in Example 4.2. Hypercube* $Q_q$ *can be represented also as* $(\square_{i=1}^{r} C_4)\square K_2$ *if* $q = 2r+1$ *or* $\square_{i=1}^{r} C_4$ *if* $q = 2r$*. Let us apply the Theorem 1.3.*

*First, let* $a = q-1$*. The only 'legal partitions' of* $a$ *are* $a = \underbrace{1+1+\ldots+1}_{r}+0+r$*, if* $q = 2r+1$*, and* $a = \underbrace{1+1+\ldots+1}_{r}+r-1$*, if* $q = 2r$*. By Theorem 1.3,* $\mathcal{D}_a(Q_q) \leq \underbrace{2+2+\ldots+2}_{r}+1+1 = 2r+2 = q+1$ *for* $q = 2r+1$*. For* $q = 2r$*,* $\mathcal{D}_a(Q_q) \leq \underbrace{2+2+\ldots+2}_{r}+1 = 2r+1 = q+1$.

*Second, let* $a < q-1$*. The same way as in the first case we get* $\mathcal{D}_a(Q_q) \leq q+1$*. For example, if* $a = q-2$*, we have* $a = \underbrace{1+1+\ldots+1}_{r-1}+0+0+r = 2r-1$*, if* $q = 2r+1$*, and* $a = \underbrace{1+1+\ldots+1}_{r-1}+0+r-1 = 2r-2$*, if* $q = 2r$*. In the first case we have* $\mathcal{D}_a(Q_q) \leq \underbrace{2+2+\ldots+2}_{r-1}+2+1+1 = 2r+2 = q+1$*. In the second case we have* $\mathcal{D}_a(Q_q) \leq \underbrace{2+2+\ldots+2}_{r-1}+2+1 = 2r+1 = q+1$.

*Hence Theorem 1.3 gives good upper bound for* $\mathcal{D}_a(Q_q)$ *while Theorem 1.4 gives the exact result.*

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