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MANIFOLDS OF HOLOMORPHIC MAPPINGS FROM STRONGLY PSEUDOCONVEX DOMAINS

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Abstract. Let D be a bounded strongly pseudoconvex domain in a Stein manifold S, and let Y be a complex manifold. We prove that the graph of any continuous map $\bar{D} \to Y$ which is holomorphic in D admits a basis of open Stein neighbourhoods in $S \times Y$, each of them isomorphic to an open set with convex fibers in the total space of a holomorphic vector bundle. Using this result we describe the (infinite dimensional) complex manifold structure on certain classical spaces of maps $\bar{D} \to Y$ which are holomorphic in D (Theorem 2.1).

1. Introduction

An important problem in complex analysis is to understand whether a given set in a complex manifold admits an open Stein neighbourhood, or perhaps even a fundamental basis of such neighbourhoods. Due to their rich function theoretic properties, Stein domains are enormously useful when solving complex analytic problems. In this paper we prove that any holomorphic graph with continuous boundary values over a strongly pseudoconvex domain enjoys this property. Our main result is the following.

Theorem 1.1. *Assume that $h\colon X \to S$ is a holomorphic submersion of a complex manifold X onto a Stein manifold S. Let $D \Subset S$ be a strongly pseudoconvex domain with $\mathcal{C}^2$ boundary in S, and let $f\colon \bar{D} \to X$ be a continuous section of h (i.e., $h \circ f = id_{\bar{D}}$) which is holomorphic in D. There exists a holomorphic vector bundle $\pi\colon E \to U$ over an open set $U \subset S$ containing $\bar{D}$, and for every open set $\Omega_0 \subset X$ containing $f(\bar{D})$ there exist a Stein open set $\Omega \subset X$ with $f(\bar{D}) \subset \Omega \subset \Omega_0$ and a fiber preserving biholomorphic map $\Theta\colon \Omega \to \widetilde{\Omega} \subset E$ onto an open subset with convex fibers in E.*

By 'fiber preserving' we mean that Θ maps the fiber $\Omega_z = h^{-1}(z) \cap \Omega$ over each point $z \in h(\Omega) \subset U$ biholomorphically onto an open convex set $\widetilde{\Omega}_z$ in the fiber $E_z = \pi^{-1}(z)$. The map $z \to \widetilde{f}(z) = \Theta(f(z)) \in E_z$ is a continuous section of the restricted bundle $E_{\bar{D}} = \pi^{-1}(\bar{D})$ which is holomorphic in D.

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Furthermore, Θ furnishes an isomorphism between the space of sections of $h^{-1}(\bar{D}) \to \bar{D}$ which are sufficiently uniformly close to f and the space of sections of $E_{\bar{D}}$ close to $\widetilde{f}$, a fact which will be exploited in §2 below.

If f extends holomorphically to a neighbourhood of $\bar{D}$ in S then Theorem 1.1 follows from Siu's theorem [30]. The main problem are of course holomorphic graphs with low boundary regularity. In §5 we show that the closure of the graph of a bounded holomorphic function need not admit a Stein neighborhood basis, so it is impossible to omit boundary continuity.

The following special case might be of particular interest; it suffices to apply Theorem 1.1 to the projection $h\colon X = S \times Y \to S$, $(z, y) \to z$.

Corollary 1.2. *If $D \Subset S$ is as in Theorem 1.1, Y is a complex manifold and $f\colon \bar{D} \to Y$ is a continuous map which is holomorphic in D then $G_f = \{(z, f(z))\colon z \in \bar{D}\}$ admits a basis of open Stein neighbourhoods in $S \times Y$.*

Corollary 1.2 is immediate for $Y = \mathbb{C}^n$ (consider a family of translates of an approximating holomorphic map over a small Stein open neighbourhood of $\bar{D}$ in S), and for a Stein manifold Y the result follows by embedding Y into a Euclidean space. In [9, Theorem 2.6] this was proved for holomorphic maps with $\mathcal{C}^2$ boundary values to an arbitrary complex manifold Y by adapting Demailly's proof [5] of Siu's theorem [30]; we do not know whether that method could be adapted to maps which are merely continuous at the boundary. Corollary 1.2 fails in general for *images* of holomorphic maps as opposed to their graphs; see the examples in §5.

In §2 we apply Theorem 1.1 to describe a natural complex manifold structure on certain classical spaces of holomorphic maps from strongly pseudoconvex domains to complex manifolds. Each of these manifolds of maps is infinite dimensional and is locally modeled on a locally convex topological vector space (Banach, Hilbert or Fréchet). Analogous results are known for spaces of smooth maps from a smooth compact manifold, possibly with boundary, to smooth or complex manifolds; see Lempert [27] and the references therein. It is expected that these *generalized loop spaces* could be useful in the study of geometric properties of the target manifold.

We prove Theorem 1.1 in §4 by the method of *holomorphic sprays* developed in [9] and [10], although the paper is essentially self-contained since we reprove the key technical ingredient, Lemma 3.2. In [10] the authors constructed dominating holomorphic sprays over $\bar{D}$ with a given central section by successively gluing together sprays defined over subsets of $\bar{D}$; using sprays they obtained a linearization of a relative neighbourhood of $f(\bar{D})$ in $h^{-1}(\bar{D}) \subset X$ [10, Corollary 4.3]. Here we improve the gluing lemma and thereby obtain sufficiently good control of the range of sprays in order to find open Stein neighbourhoods of a given section.

For general theory of Stein manifolds we refer to [15] and [21]; for complex analysis in infinite dimensions see [7], [16], [26].

2. Complex manifolds of holomorphic maps

Let D be a relatively compact domain with Lipschitz boundary in $\mathbb{C}^n$. We consider the following function spaces on D:

(i) For $k \in \mathbb{Z}_+$ and $0 \le \alpha < 1$, $\mathcal{A}^{k,\alpha}(D)$ is the Banach space of all functions $\bar{D} \to \mathbb{C}$ in the Hölder class $\mathcal{C}^{k,\alpha}(\bar{D})$ which are holomorphic in D. When $\alpha = 0$ we shall write $\mathcal{A}^{k,0} = \mathcal{A}^k$ and $\mathcal{A}^0 = \mathcal{A}$.

(ii) $\mathcal{A}^\infty(D) = \cap_{k=0}^\infty \mathcal{A}^k(D)$ is the Fréchet space consisting of all $\mathcal{C}^\infty$ function $\bar{D} \to \mathbb{C}$ which are holomorphic in D.

(iii) For $k \in \mathbb{Z}_+$ and $p \ge 1$, $L^{k,p}_{\mathcal{O}}(D)$ is the Banach space (Hilbert if $p = 2$) consisting of all holomorphic functions $D \to \mathbb{C}$ whose partial derivatives of order $\le k$ belong to $L^p(D)$. These are Sobolev spaces of holomorphic functions on D.

If $L(D)$ is any of the above function spaces, we denote by $L(D, \mathbb{C}^m)$ the locally convex topological vector space consisting of maps whose components belong to $L(D)$. In the case (iii) we shall assume $kp > 2n$, so the Sobolev embedding theorem [28] provides a continuous inclusion $L^{k,p}_{\mathcal{O}}(D) \hookrightarrow \mathcal{A}(D)$.

Given a complex manifold Y of dimension m (without boundary), one defines the associated mapping space $L(D, Y)$ as follows. (See Lempert [27, §2] for the case when $\bar{D}$ is a compact smooth manifold and we are considering the space $\mathcal{C}^k(\bar{D}, Y)$, $k \in \mathbb{Z}_+ \cup \{\infty\}$.) Fix a continuous map $f\colon \bar{D} \to Y$. Choose finitely many holomorphic coordinates systems $\phi_j\colon U_j \to \widetilde{U}_j \subset \mathbb{C}^n$ on S, and $\psi_j\colon W_j \to \widetilde{W}_j \subset \mathbb{C}^m$ on Y, such that $f(\bar{D} \cap U_j) \subset W_j$ for all j. Also choose open subsets $V_j \Subset U_j$ such that $\bar{D} \subset \cup_j V_j$ and $V_j \cap D$ has Lipschitz boundary for each j. Then $f \in L(D, Y)$ precisely when for each j the restriction f_j of the map $\psi_j \circ f \circ \phi_j^{-1}$ to the set $\phi_j(\overline{D \cap V_j}) \Subset \widetilde{U}_j$ belongs to $L(\phi_j(D \cap V_j), \mathbb{C}^m)$; the definition is independent of the choices of charts. Further, given an open neighbourhood $\mathcal{U}_j \subset L(\phi_j(D \cap V_j), \mathbb{C}^m)$ of f_j for every j, the corresponding neighbourhood of f in $L(D, Y)$ consists of all maps $g\colon \bar{D} \to Y$ such that $g(\overline{D \cap V_j}) \subset W_j$ and the restriction g_j of $\psi_j \circ g \circ \phi_j^{-1}$ to $\phi_j(\overline{D \cap V_j})$ belongs to $\mathcal{U}_j$ for all j.

After this introduction we can state the following result.

Theorem 2.1. *Let D be a relatively compact strongly pseudoconvex domain in a Stein manifold S, and let Y be a complex manifold.*

(i) *For every $k \in \mathbb{Z}_+$ and $0 \le \alpha < 1$ the space $\mathcal{A}^{k,\alpha}(D, Y)$ is a complex Banach manifold.*

(ii) *The space $\mathcal{A}^\infty(D, Y)$ is a complex Fréchet manifold.*

(iii) *If $k \in \mathbb{N}$, $p \ge 1$ and $kp > 2 \dim S$ then $L^{k,p}_{\mathcal{O}}(D, Y)$ is a complex Banach manifold (a Hilbert manifold if $p = 2$).*

The analogous conclusions hold if $\bar{D}$ is a compact complex manifold with Stein interior D and smooth strongly pseudoconvex boundary bD; according to Heunemann [20] and Ohsawa [29] (see also Catlin [2]) such $\bar D$ embeds as a smoothly bounded strongly pseudoconvex domain in a Stein manifold S.

The special case of Theorem 2.1 (i) with $\alpha = 0$ was proved in [10]. Ivashkovich and Shevchishin considered Sobolev manifolds of maps from Riemann surfaces to almost complex manifolds ([22], [23]). Manifolds of maps between Riemannian manifolds (in particular, Sobolev loop spaces) have been studies by several authors, see e.g. [1], [8], [12].

Proof. Let $L(D,Y)$ denote any one of the above spaces. We need to construct holomorphic charts in $L(D,Y)$.

Given a holomorphic vector bundle $\pi\colon E \to U$ over an open set $U \subset S$ containing $\bar D$, we denote by $\Gamma_L(D,E)$ the space of all section $\bar D \to E_{\bar D}$ over $\bar D$ which belong to $L(D,E)$. This is a locally convex topological vector space; Banach for $\mathcal{A}^{k,\alpha}$ or $L^{k,p}_{\mathcal{O}}$, Fréchet for $\mathcal{A}^\infty$, and Hilbert for $L^{k,2}_{\mathcal{O}}$. In each of these cases we have a continuous linear inclusion $L(D,Y) \hookrightarrow \mathcal{A}(D,Y)$.

Fix a map $f \in L(D,Y)$. Theorem 1.1 furnishes an open Stein neighbourhood $\Omega \subset S \times Y$ of the graph $G_f = \{(z,f(z))\colon z \in \bar D\}$ and a biholomorphic map $\Theta\colon \Omega \to \widetilde\Omega \subset E$ onto an open set $\widetilde\Omega$ in the total space of a holomorphic vector bundle $\pi\colon E \to U$ such that $\bar D \subset U \subset S$ and $\pi\circ\Theta\colon \Omega \to S$ is the restriction to Ω of the base projection $(z,y) \to z$.

Since Θ is holomorphic in a neighbourhood of the graph G_f, the map $\bar D \ni z \to \widetilde f(z) := \Theta(z,f(z)) \in E_z$ is a section of the restricted bundle $E_{\bar D} \to \bar D$ which belongs to the space $\Gamma_L(D,E)$.

The graph G_g of any $g \in L(D,Y)$ sufficiently near f is also contained in Ω, and the composition of g with Θ defines an isomorphism $g \to \widetilde g = \Theta(\cdot\,,g)$ between an open neighbourhood of f in $L(D,Y)$ and an open neighbourhood of $\widetilde f$ in $\Gamma_L(D,E)$; i.e., we get a local chart on $L(D,Y)$. It is easily verified that the transition map between any such pair of charts is holomorphic (the argument given in [10] for $\mathcal{A}(D,Y)$ applies in all cases); hence the collection of all such charts defines a holomorphic manifold structure on $L(D,Y)$.

The above construction also shows that the tangent space to the manifold $L(D,Y)$ at a point $f \in L(D,Y)$ is $\Gamma_L(D,f^*TY)$, the space of sections of class $L(D)$ of the vector bundle $f^*TY \to \bar D$. □

The complex structures introduced above enjoy the following functorial property; compare with [27, Proposition 2.3] and observe that the proof given there carries over to our situation as well.

Proposition 2.2. *Let $D \Subset S$ be a strongly pseudoconvex domain in a Stein manifold S, let Y and Y' be complex manifolds, and let $\Phi\colon S\times Y \to Y'$ be a holomorphic map. Then the induced map $\Phi_*\colon L(D,Y) \to L(D,Y')$ defined by $\Phi_*(f)(z) = \Phi(z,f(z))$ $(z \in \bar D)$ is holomorphic.*

The discussion following Proposition 2.3 in [27] also shows that the functorial property described in the above proposition characterizes the complex structures under consideration.

These complex structures are functorial also with respect to maps of the source domains: Given strongly pseudoconvex domains $D \Subset S$, $D' \Subset S'$ in Stein manifolds S resp. S', and given a holomorphic map $\Psi \colon S \to S'$ satisfying $\Psi(D) \subset D'$, the pull-back map $\Psi^* \colon L(D', Y) \to L(D, Y)$ defined by $f \to f \circ \Psi$ is holomorphic (see [27, Proposition 2.4]). In particular, the evaluation map $e \colon \bar{D} \times L(D, Y) \to Y$, $e(z, f) = f(z)$, is continuous, and is holomorphic in f for a fixed $z \in \bar{D}$ (see the proof of Proposition 2.5 in [27]). Weaker hypothesis on Ψ may suffice in individual cases. For example, if $\Psi \colon \bar{D} \to \bar{D}'$ is of class $\mathcal{C}^r$ and holomorphic in D then $\Psi^* \colon \mathcal{A}^{k,\alpha}(D, Y') \to \mathcal{A}^{k,\alpha}(D, Y)$ is holomorphic provided that $k + \alpha \le r$.

Remark 2.3. The above proof of Theorem 2.1 is simpler than the one which was given in [10] for the spaces $\mathcal{A}^k(D, Y)$. Indeed, here we do not require a new splitting lemma for each of the spaces – we only need it for the space $\Gamma_{\mathcal{A}}(D, E)$ (Lemma 3.2 below); for this reason we only need $\mathcal{C}^2$ boundary regularity of D, as opposed to $\mathcal{C}^k$ regularity which was assumed in [10] to conclude that $\mathcal{A}^k(D, Y)$ is a Banach manifold.

The constructions in this section apply to many other mapping spaces consisting of continuous maps $\bar{D} \to Y$ which are holomorphic in the interior D; the essential required properties are localization (to allow passage to local coordinates) and stability under compositions with holomorphic maps (to get a complex manifold structure).

3. A splitting lemma

In this section we prove a splitting lemma for fiberwise injective holomorphic maps on holomorphic vector bundles with continuous boundary values. Lemma 3.2 below, which generalizes Theorem 3.2 in [9] where the bundle in question was trivial, is the main ingredient in the proof of Theorem 1.1.

We begin by recalling the notion of a *Cartan pair* and of a *convex bump* [10, Def. 2.3].

Definition 3.1. A pair of open subsets $D_0, D_1 \Subset S$ in a Stein manifold S is said to be a *Cartan pair* of class $\mathcal{C}^\ell$ ($\ell \ge 2$) if

(i) D_0, D_1, $D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$ are strongly pseudoconvex domains with $\mathcal{C}^\ell$ boundaries, and
(ii) $\overline{D_0 \backslash D_1} \cap \overline{D_1 \backslash D_0} = \emptyset$ (the separation property).

We say that D_1 *is a convex bump on* D_0 if, in addition to the above, there is a biholomorphic map from an open neighbourhood of $\bar{D}_1$ in S onto an open subset of $\mathbb{C}^n$ ($n = \dim S$) which maps D_1 and $D_{0,1}$ onto strongly convex domains in $\mathbb{C}^n$.

Suppose that $D = D_0 \cup D_1$ is a Cartan pair in a Stein manifold S. Let $\pi\colon E \to \bar D$ be a continuous complex vector bundle which is holomorphic over D (an $\mathcal{A}(D)$-vector bundle). Such bundle E embeds as an $\mathcal{A}(D)$-subbundle E' of a trivial bundle $\mathbb{T}^N = \bar D \times \mathbb{C}^N$ for a sufficiently large integer N, and for every such embedding there is a Whitney direct sum decomposition $\mathbb{T}^N = E' \oplus E''$ of class $\mathcal{A}(D)$. (These facts follows from Cartan's Theorem B for $\mathcal{A}(D)$-vector bundles; see Leiterer [24], [25] and Heunemann [19], [20].) Fix such a decomposition and identify E with the subbundle E'. We shall denote the variable in $\bar D$ by z and the variable in $\mathbb{C}^N$ by t. On the fiber $\{z\} \times \mathbb{C}^N$ we have a unique decomposition $t = t' \oplus t'' \in E'_z \oplus E''_z$ which is of class $\mathcal{A}(D)$ with respect to the base variable $z \in \bar D$. Let $|\cdot|$ denote the standard Euclidean norm on $\mathbb{C}^N$, and let $\mathbb{B} = \{t \in \mathbb{C}^N : |t| < 1\}$. For every $r > 0$ and $z \in \bar D$ we set

$$E_{z,r} = \{t = t' \oplus 0'' \in E_z : |t| < r\} = E_z \cap r\mathbb{B}.$$

Given a subset $K \subset \bar D$ and $r > 0$ we shall write

$$E_K = \cup_{z\in K} E_z, \quad E_{K,r} = \cup_{z\in K} E_{z,r} = E_K \cap (K \times r\mathbb{B}). \tag{3.1}$$

Every continuous fiber-preserving map $E_{K,r} \to E_K$ is of the form $\gamma(z,t') = (z, \psi(z,t'))$ for $z \in K$ and $|t'| = |t' \oplus 0''| < r$; we shall say that γ is of class $\mathcal{A}$ if it is holomorphic in the interior of its domain. Let $id(z,t') = (z,t')$ denote the identity map on E. Set

$$||\gamma - id||_{K,r} = \sup\{|\psi(z,t') - t'| : z \in K,\ |t'| < r\}.$$

Lemma 3.2. *Let $D = D_0 \cup D_1$ be a Cartan pair of class $\mathcal{C}^2$ in a Stein manifold S, and let $\pi\colon E \to \bar D$ be an $\mathcal{A}(D)$-bundle. Set $K = \bar D_{0,1} = \bar D_0 \cap \bar D_1$. Given numbers $0 < r' < r$ and $\epsilon > 0$, there is a number $\delta > 0$ satisfying the following. For every fiber preserving map $\gamma\colon E_{K,r} \to E_K$ of class $\mathcal{A}(E_{K,r})$ with $||\gamma - id||_{K,r} < \delta$ there exist injective fiber preserving maps $\alpha\colon E_{\bar D_0,r'} \to E_{\bar D_0}$, $\beta\colon E_{\bar D_1,r'} \to E_{\bar D_1}$, of class $\mathcal{A}$ on their respective domains, satisfying $||\alpha - id||_{\bar D_0,r} < \epsilon$, $||\beta - id||_{\bar D_1,r} < \epsilon$ and*

$$\gamma \circ \alpha = \beta \quad \text{on } E_{K,r'}.$$

If in addition γ preserves the zero section (i.e., $\gamma(z,0) = (z,0)$ for $z \in K$) then α and β can be chosen to satisfy the same property.

If the Cartan pair $D = D_0 \cup D_1$ is of class $\mathcal{C}^\ell$, $\ell \geq 2$, then for any integer $l \in \{0,1,\ldots,\ell\}$ the analogous result holds with the bundle E and the maps α, β, γ of class $\mathcal{A}^l$ on their respective domains (i.e., holomorphic inside and of class $\mathcal{C}^l$ up to the boundary), with $\mathcal{C}^l$ estimates.

Proof. We begin with the special case when the bundle E is trivial, $E = \mathbb{T}^N = \bar D \times \mathbb{C}^N$ for some $N \in \mathbb{N}$. In this case the result coincides with Theorem 3.2 in [9]; as that paper is a preprint at the time of this writing, we offer here a simpler proof in order to make the paper self-contained. It is similar to the proof of Proposition 5.2 in [11, p. 141].

Recall that $(\gamma(z,t) = (z,\psi(z,t))$, where $\psi\colon K\times r\mathbb{B}\to\mathbb{C}^N$ is close to the map $\psi_0(z,t)=t$. We denote by C_r (resp. by Γ_r) the Banach space consisting of all continuous maps $K\times r\mathbb{B}\ni(z,t)\to\psi(z,t)\in\mathbb{C}^N$ which are holomorphic in the interior $D_{0,1}\times r\mathbb{B}$ of $K\times r\mathbb{B}$ and satisfy

$$\begin{aligned} ||\psi||_{C_r} &= \sup_{(z,t)\in K\times r\mathbb{B}} |\psi(z,t)| < +\infty, \\ ||\psi||_{\Gamma_r} &= \sup_{(z,t)\in K\times r\mathbb{B}} \big(|\psi(z,t)| + |\partial_t\psi(z,t)|\big) < +\infty. \end{aligned}$$

Here, ∂_t denotes the partial differential with respect to the variable $t\in\mathbb{C}^N$, and $|\partial_t\psi(z,t)|$ is the Euclidean operator norm.

Replacing the number $r>0$ in Lemma 3.2 with a slightly smaller number we can assume (in view of the Cauchy estimates) that ψ belongs to Γ_r and that $||\psi-\psi_0||_{\Gamma_r}$ is as small as desired, where $\psi_0(z,t)=t$. Fix such r and choose a number r' with $0<r'<r$. Let $A_{r'}$ (resp. by $B_{r'}$) denote the Banach space of all continuous maps $\bar D_0\times r'\mathbb{B}\to\mathbb{C}^N$ (resp. $\bar D_1\times r'\mathbb{B}\to\mathbb{C}^N$) which are holomorphic in the interior of the respective set, endowed with the sup-norm. By [9, Lemma 3.4] there exist continuous linear operators $\mathcal{A}\colon C_{r'}\to A_{r'}$, $\mathcal{B}\colon C_{r'}\to B_{r'}$ satisfying

$$c=\mathcal{A}(c)-\mathcal{B}(c), \quad c\in C_{r'}. \tag{3.2}$$

The proof in [9] uses a linear solution operator for the $\bar\partial$-equation on the level of $(0,1)$-forms on D satisfying the sup-norm estimates, and the variables t are treated as parameters.

Given $\psi\in\Gamma_r$ sufficiently near ψ_0 and $c\in C_{r'}$ near 0, we define

$$\Phi(\psi,c)(z,t)=\psi(z,t+\mathcal{A}(c)(z,t))-(t+\mathcal{B}(c)(z,t)), \quad (z,t)\in K\times r'\mathbb{B}.$$

It is easily verified that $(\psi,c)\to\Phi(\psi,c)$ is a $\mathcal{C}^1$ (even smooth) map from an open neighbourhood of the point $(\psi_0,0)$ in the Banach space $\Gamma_r\times C_{r'}$ to the Banach space $C_{r'}$. (Although we are composing maps which are only continuous up to the boundary of their respective domains in the z-variable, we are inserting $\mathcal{A}(c)$ in the second variable of ψ, and ψ is holomorphic with respect to that variable on a larger domain.)

Since $\Phi(\psi_0,c)=\mathcal{A}(c)-\mathcal{B}(c)=c$ by (3.2), the implicit function theorem shows that in a neighbourhood of $(\psi_0,0)$ in $\Gamma_r\times C_{r'}$ we can solve the equation $\Phi(\psi,c)=0$ on c: There is a $\mathcal{C}^1$ map $\psi\to\mathcal{C}(\psi)\in C_{r'}$, defined in an open neighbourhood of ψ_0 in Γ_r and satisfying

$$\Phi(\psi,\mathcal{C}(\psi))=0, \quad \mathcal{C}(\psi_0)=0. \tag{3.3}$$

Consider the functions

$$a_\psi=t+\mathcal{A}\circ\mathcal{C}(\psi)\in A_{r'}, \quad b_\psi=t+\mathcal{B}\circ\mathcal{C}(\psi)\in B_{r'}.$$

From (3.3) and the definition of Φ we obtain

$$\psi(z,a_\psi(z,t))=b_\psi(z,t), \quad (z,t)\in K\times r'\mathbb{B}. \tag{3.4}$$

Setting $\alpha(z,t) = (z, a_\psi(z,t))$, $\beta(z,t) = (z, b_\psi(z,t))$, we see that (3.4) gives $\gamma \circ \alpha = \beta$. We also get sup-norm estimates on $a_\psi - t$ (resp. on $b_\psi - t$) on $\bar D_0 \times r'\mathbb{B}$ (resp. on $\bar D_1 \times r'\mathbb{B}$) in terms of $||\psi - \psi_0||_{\Gamma_r}$. If the latter number is sufficiently small, the maps a_ψ and b_ψ are as close as desired to the map $(z,t) \to t$ and hence, after we shrink r' slightly and apply again the Cauchy estimates in the t-variable, we can assume that they are fiberwise injective holomorphic. This proves Lemma 3.2 when $E = \bar D \times \mathbb{C}^N$.

The case with $\mathcal{C}^l$ boundary values is obtained in the same way by using the appropriate Banach spaces; compare with Theorem 3.2 in [9].

It remain to consider the general case when E is an $\mathcal{A}(D)$-vector subbundle of $\mathbb{T}^N = \bar D \times \mathbb{C}^N$. As before we identify E with its image $E' \subset \mathbb{T}^N$ and write $\mathbb{T}^N = E' \oplus E''$ (an $\mathcal{A}(D)$-decomposition). Let $t = t' \oplus t'' \in E'_z \oplus E''_z$ denote the corresponding splitting of the fiber variable. We associate to each self-map $\gamma(z,t') = (z, \psi'(z,t'))$ of E' the self-map

$$\widetilde\gamma(z,t) = (z, \psi(z,t)), \quad \psi(z,t) = \psi'(z,t') \oplus t''$$

of $\mathbb{T}^N$ (we added the identity map on the second summand E''). If ψ' is sufficiently close to the map $(z,t') \to t'$ then ψ is close to the map $\psi_0(z,t) = t$, and the first part of the Lemma (for the trivial bundle) furnishes $\mathbb{C}^N$-valued maps

$$\begin{aligned} a(z,t) &= a'(z,t) \oplus a''(z,t), \quad (z,t) \in \bar D_0 \times r'\mathbb{B}, \\ b(z,t) &= b'(z,t) \oplus b''(z,t), \quad (z,t) \in \bar D_1 \times r'\mathbb{B} \end{aligned}$$

satisfying $\psi(z, a(z,t)) = b(z,t)$ for $(z,t) \in K \times r'\mathbb{B}$. Comparing the E' and the E'' components of this identity at $t'' = 0$ we get

$$\psi'(z, a'(z,t')) = b'(z,t'), \quad a''(z,t') = b''(z,t').$$

Hence the maps $\alpha(z,t') = (z, a'(z,t'))$ and $\beta(z,t') = (z, b'(z,t'))$ satisfy the conclusion of Lemma 3.2. □

4. Proof of Theorem 1.1

We assume the notation and hypotheses of Theorem 1.1. Thus, let $f\colon \bar D \to X$ be a continuous section of a holomorphic submersion $h\colon X \to S$ (i.e., $h(f(z)) = z$ for $z \in \bar D$) which is holomorphic in D. Set $\Sigma = f(\bar D)$. Recall that $VT(X) = \ker dh$ is the vertical tangent space of X.

By the Oka-Grauert theory there is an integer $N \in \mathbb{N}$ and a surjective complex vector bundle map $L\colon \mathbb{T}^N = \bar D \times \mathbb{C}^N \to VT(X)|_\Sigma$ of class $\mathcal{A}(D)$, i.e., holomorphic over D and continuous up to the boundary [19], [25]. Let $\mathbb{B}^N$ denote the open unit ball in $\mathbb{C}^N$. Proposition 4.1 in [10] furnishes a map $F\colon \bar D \times r\mathbb{B}^N \to X$ of class $\mathcal{A}(D \times r\mathbb{B}^N)$ for some $r > 0$ such that for all $z \in \bar D$ we have $F(\{z\} \times r\mathbb{B}^N) \subset X_z$ and

$$F(z,0) = f(z), \quad \partial_t|_{t=0} F(z,t) = L_z\colon \mathbb{C}^N \to VT_{f(z)}X. \tag{4.1}$$

Such F is said to be a *dominating (holomorphic) spray with the core section* $F_0 = F(\cdot\,, 0) = f$. The proof (which we shall not reproduce here in detail) is essentially an application of Lemma 3.2; here is an outline. Over any compact subset $K \subset D$ one can find such F by using a Stein neighbourhood of $f(D)$ in X, provided by Siu's theorem [30], and composing small complex-time flows of suitably chosen vertical (tangential to the fibers of the submersion $h\colon X \to S$) holomorphic vector fields in a neighbourhood of $f(K)$ in X. This gives a desired spray over a domain $\bar{D}_0 \subset D$ obtained by pushing the bondary of D in just a little. To extend this spray up to the boundary of D one proceeds by successively attaching to the base domain small convex bumps, each of which is mapped by f into a local chart of X. At every step one approximates the given spray by another one defined over the bump which is being attached at this step, with the same central section; then one glues the two sprays into a single spray over the union of their base sets by appealing to Lemma 3.2. The parameter set in $\mathbb{C}^N$ is allowed to shrink at each step.

The same procedure can also be applied to continue past the boundary of D like it was done in the proof of Theorem 5.1 in [10], but with the essential difference that the central section can no longer be preserved (since it is only defined over $\bar{D}$). In this way one obtains a dominating spray $\widetilde{F}\colon U \times r'\mathbb{B}^N \to X$ over an open neighbourhood $U \subset S$ of $\bar{D}$; the trouble is that the number $r' > 0$ is possibly much smaller than r, and the construction in [10] is not sufficiently precise to insure that the image of $\widetilde{F}$ contains the graph $\Sigma = f(\bar{D})$. We shall now show how to insure this inclusion by using the more precise Lemma 3.2 in this paper.

We begin by a dominating spray $F\colon \bar{D} \times r\mathbb{B}^N \to X$ satisfying (4.1), furnished by Proposition 4.1 in [10]. Set $E''_z = \ker \partial_t|_{t=0} F(z,t) \subset \mathbb{C}^N$ for $z \in \bar{D}$; this defines an $\mathcal{A}(D)$-subbundle $E'' \subset \mathbb{T}^N = \bar{D} \times \mathbb{C}^N$. By [19] and [25] there exists a complementary $\mathcal{A}(D)$-subbundle $E' \subset \mathbb{T}^N$ such that $\mathbb{T}^N = E' \oplus E''$. By [18] we can approximate E' sufficiently well over $\bar{D}$ by a holomorphic vector subbundle $E \subset U \times \mathbb{C}^N$ over an open set $U \subset S$ containing $\bar{D}$ such that $\mathbb{T}^N = E_{\bar{D}} \oplus E''$.

Recall the notation E_K and $E_{K,r}$ (3.1). Let G denote the restriction of the spray F to the set $E_{\bar{D},r}$. Our choice of E implies that the partial differential $\partial_{t'}|_{t'=0} G(z,t')\colon E_z \to VT_{f(z)}X$ is a linear isomorphism for every $z \in \bar{D}$, and hence we can insure by decreasing $r > 0$ if necessary that G is fiberwise biholomorphic. In particular, the restricted bundle $E_{\bar{D}}$ is isomorphic to the bundle $VT(X)|_\Sigma$; in the special case when $X = S \times Y$ and Σ is the graph of an $\mathcal{A}(D,Y)$-map $f\colon \bar{D} \to Y$, the latter bundle is isomorphic to f^*TY. Note that G maps the zero section of $E_{\bar{D}}$ onto $\Sigma = f(\bar{D})$.

Theorem 1.1 now follows from the following lemma.

Lemma 4.1. *Let $G\colon E_{\bar{D},r} \to X$ be an injective spray of class $\mathcal{A}$ as above. There exist a number r' with $0 < r' < r$, a decreasing sequence of open* *sets* $O_1 \supset O_2 \supset \cdots$ *in* S *with* $\cap_{k=1}^{\infty} O_k = \bar{D}$*, and a sequence of injective holomorphic sprays* $G_k\colon E_{O_k,r'} \to X$ *such that* G_k *converges to* G *uniformly on* $E_{\bar{D},r'}$ *as* $k \to \infty$ *and*

$$\Sigma \subset G_k(E_{O_k,r'}), \quad k = 1, 2, \ldots.$$

Proof. We proceed as in the proof of Theorem 5.1 in [10], but paying very close attention to the ranges of the approximating sprays in order to insure that each of them contains the graph $\Sigma = f(\bar{D})$.

Since $h\colon X \to S$ is a holomorphic submersion, there exist for each point $x_0 \in X$ open neighbourhoods $x_0 \in W \subset X$, $h(x_0) \in V \subset S$, and biholomorphic maps $\phi\colon V \to \mathbb{B}^n \subset \mathbb{C}^n$, $\Phi\colon W \to \mathbb{B}^n \times \mathbb{B}^m \subset \mathbb{C}^n \times \mathbb{C}^m$, such that $\phi(h(x)) = pr_1(\Phi(x))$ for every $x \in W$. Note that Φ is of the form

$$\Phi(x) = \big(\phi(h(x)), \phi'(x)\big) \in \mathbb{B}^n \times \mathbb{B}^m, \quad x \in W,$$

where $\phi' = pr_2 \circ \Phi$. We call such (W, V, Φ) a *special coordinate chart* on X.

Narasimhan's lemma on local convexification of a strongly pseudoconvex hypersurface gives finitely many special coordinate charts (W_j, V_j, Φ_j) on X, with $\Phi_j = (\phi_j \circ h, \phi'_j)$, such that $bD \subset \cup_{j=1}^{j_0} V_j$ and the following hold for all $j = 1, \ldots, j_0$ (for (ii) and (iii) we may have to decrease $r > 0$):

(i) $\phi_j(bD \cap V_j)$ is a strongly convex hypersurface in the ball $\mathbb{B}^n$,
(ii) the spray G maps the set $E_{\bar{D}\cap V_j,r}$ into W_j, and
(iii) $\phi'_j \circ G(E_{\bar{D}\cap V_j,r}) \Subset \mathbb{B}^m$.

Fix an $r > 0$ such that the above properties hold and choose a number $r' \in (0, r)$; we will show that the conclusion Lemma 4.1 holds for this r'. Also choose a number $c < 1$ sufficiently close to 1 such that the sets $U_j = \phi_j^{-1}(c\mathbb{B}^n) \Subset V_j$ $(j = 1, \ldots, j_0)$ still cover bD.

By a finite induction on j we shall find strongly pseudoconvex domains $D = D_0 \subset D_1 \subset \cdots \subset D_{j_0} \Subset U$ in S, numbers $r = r_0 > r_1 > \cdots > r_{j_0} = r'$ and injective sprays $G_j\colon E_{\bar{D}_j,r_j} \to X$ of class $\mathcal{A}$ $(j = 0, 1, \ldots, j_0)$, with $G_0 = G$, such that for every $j = 1, \ldots, j_0$ the restriction of G_j to $E_{\bar{D}_{j-1},r_j}$ will be as close as desired to G_{j-1} in the sup-norm topology. The domain D_j will depend on the desired rate of approximation of G_{j-1} by G_j (since we shall use Runge approximation), and it will be chosen such that

$$D_{j-1} \subset D_j \subset D_{j-1} \cup V_j, \quad bD_{j-1} \cap U_j \subset D_j$$

for $j = 1, \ldots, j_0$. That is, we enlarge (bump out) D_{j-1} inside V_j so that the part of bD_{j-1} which lies in the smaller set U_j is contained in the next domain D_j. As the U_j's cover bD, the final domain D_{j_0} will contain $\bar{D}$ in its interior, and the spray $\widetilde{G} = G_{j_0}\colon E_{\bar{D}_{j_0},r'} \to X$ will approximate G as close as desired on $E_{\bar{D},r'}$; in particular, we shall arrange that Σ is contained in $\widetilde{G}(E_{\bar{D}_{j_0},r'})$. To keep the induction going we will also insure at every step that the properties (ii) and (iii) above remain valid with (D, G) replaced by

(D_j, G_j) for all $j = 1, \ldots, j_0$. The restriction of $\widetilde{G}$ to the interior $E_{D_{j_0},r'}$ can be taken as one of the sprays in the conclusion of the lemma.

The underlying geometric scheme is precisely as in the proof of Theorem 5.1 in [10]; we recall it briefly to make the proof comprehensible. Since all steps will be exactly of the same kind, it suffices to explain how to get the pair (D_1, G_1) from $(D, G) = (D_0, G_0)$.

We begin by finding a domain $D_1' \subset S$ with $\mathcal{C}^2$ boundary which is a convex bump on $D = D_0$ (Definition 3.1) and such that $\overline{U}_1 \cap \bar{D} \subset \bar{D}_1' \subset V_1$. To do this, we shall first find a set $\widetilde{D}_1' \subset \mathbb{B}^n$ with suitable properties and then take $D_1' = \phi_1^{-1}(\widetilde{D}_1')$. Choose a smooth function $\chi \geq 0$ with compact support on $\mathbb{B}^n$ such that $\chi = 1$ on $c\mathbb{B}^n$. Recall that $U_1 = \phi_1^{-1}(c\mathbb{B}^n)$. Let $\tau \colon \mathbb{B}^n \to \mathbb{R}$ be a strongly convex defining function for the domain $\phi_1(D \cap V_1) \subset \mathbb{B}^n$. Choose a number $c' \in (c, 1)$ close to 1 such that the hypersurface $\phi_1(bD \cap V_1) = \{\tau = 0\}$ intersects the sphere $\{\zeta \in \mathbb{C}^n \colon |\zeta| = c'\}$ transversely. If $\delta > 0$ is chosen sufficiently small then the set

$$\{\zeta \in \mathbb{C}^n \colon |\zeta| < c',\ \tau(\zeta) + \delta\chi(\zeta) < 0\}$$

could serve our purpose, except that it is not smooth along the intersection of the (convex) hypersurfaces $\{|\zeta| = c'\}$ and $\{\tau + \delta\chi = 0\}$. By rounding off the corners we get a strongly convex set $\widetilde{D}_1' \subset \mathbb{B}^n$ such that $D_1' = \phi_1^{-1}(\widetilde{D}_1') \subset V_1$ satisfies the desired properties. (See fig. 1 which is taken from [10].)

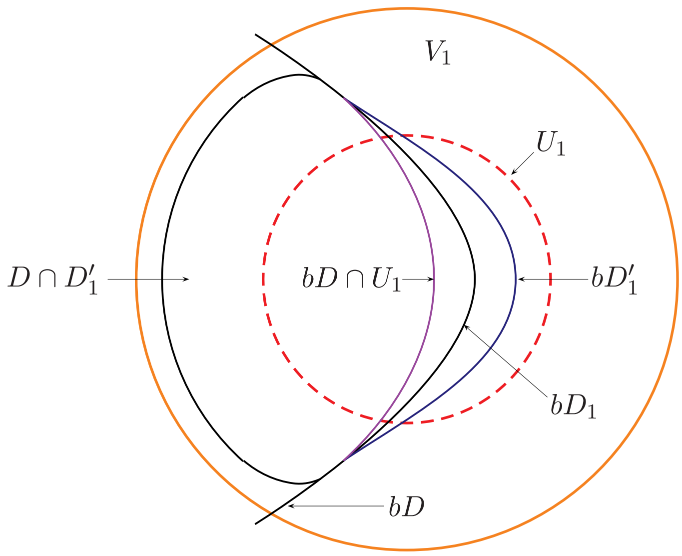

FIGURE 1. The domains D_1' and D_1

Recall that $G = G_0 \colon E_{\bar{D},r} \to X$ is an injective spray of class $\mathcal{A}$ satisfying properties (i)–(iii) above. Choose numbers r_1, r_1', r_1'' with $r' < r_1 < r_1' < r_1'' < r$. By using the special coordinate chart (W_1, V_1, Φ_1) we can find an open set $V_1' \subset V_1$ containing $\bar D \cap V_1$ and an injective spray $G' \colon E_{\bar V_1', r_1''} \to X$, with range contained in W_1, whose restriction to $E_{\bar D \cap \bar V_1, r_1''}$ approximates G as close as desired in the uniform topology. (The set V_1' will depend on the choice of G' since we are using Runge approximation; the only important thing is that $\bar D \cap V_1 \subset V_1'$. For further details see [10].)

Assuming that the approximation of G by G' is sufficiently close $E_{\bar V_1', r_1''}$, there exists a (unique) fiberwise biholomorphic map $\gamma \colon E_{\bar D \cap \bar V_1, r_1'} \to E$ of class $\mathcal{A}$ satisfying the equation

$$G(z,t) = G'(\gamma(z,t)) = G'(z, \psi(z,t)), \quad z \in \bar D \cap \bar V_1,\ t \in E_{z, r_1'}.$$

(Here is a small but important difference from [10]: there we used dominating – submersive – sprays, and a transition map γ between a pair of sprays was obtained by applying the implicit function theorem along the core section, resulting in a possibly large shrinking of its domain in the fiber direction. Here we use injective holomorphic sprays, and hence the domain of γ shrinks arbitrarily little provided that the two sprays are sufficiently close.)

Assuming as we may that γ is sufficiently close to the identity map on its domain, we apply Lemma 3.2 on the Cartan pair (D, D_1') to obtain $\gamma \circ \alpha = \beta$ on $E_{\bar D \cap \bar V_1, r_1}$, where $\alpha \colon E_{\bar D, r_1} \to E$ and $\beta \colon E_{\bar D_1', r_1} \to E$ are injective holomorphic maps which are close to the identity on their respective domains. It follows that the injective sprays

$$G \circ \alpha \colon E_{\bar D, r_1} \to X, \quad G' \circ \beta \colon E_{\bar D_1' \cap \bar V_1', r_1} \to X$$

agree on the intersection of their domains and hence define a fiberwise biholomorphic spray $G_1 \colon E_{\bar D \cup (\bar D_1' \cap \bar V_1'), r_1} \to X$ of class $\mathcal{A}$. By the construction, G_1 approximates G uniformly on $E_{\bar D, r_1}$ as close as desired.

It remains to restrict G_1 to a suitably chosen strongly pseudoconvex domain $D_1 \Subset S$ contained in $D \cup (D_1' \cap V_1')$ and satisfying the other required properties. We choose D_1 such that it agrees with D outside of V_1, while

$$D \cap V_1 = \phi_1^{-1}(\{\zeta \in \mathbb{B}^n \colon \tau(\zeta) + \epsilon \chi(\zeta) < 0\})$$

for a small $\epsilon > 0$ (fig. 1). By choosing ϵ sufficiently small (depending on G_1) we can insure that properties (i)–(iii) are satisfied by the pair (D_1, G_1), thus completing the first step of the induction.

Applying the same procedure to (D_1, G_1) and the second special coordinate chart (W_2, V_2, Φ_2) we get the next pair (D_2, G_2). After j_0 steps we find a domain $D_{j_0} \subset S$ containing $\bar D$ and a spray $\widetilde G = G_{j_0} \colon E_{\bar D_{j_0}, r'} \to X$ which approximates G as close as desired on $E_{\bar D, r'}$. If the approximation is close enough then the range of $\widetilde G$ contains Σ as required.

The sequence G_k in Lemma 4.1 is chosen to consist of sprays $\widetilde G$ obtained as above, approximating G ever more closely on $E_{\bar D, r'}$. □

We now conclude the proof of Theorem 1.1. Let $\Omega_0 \subset X$ be an open neighbourhood of $f(\bar D)$. Choose the initial spray $G \colon E_{\bar D, r} \to X$ in Lemma

4.1 such that $G(E_{\bar{D},r}) \subset \Omega_0$ (this is achieved by decreasing the number $r > 0$ if necessary). Let $G_k \colon E_{O_k,r'} \to X$ $(k = 1, 2, \ldots)$ be a sequence of sprays furnished by Lemma 4.1. Set

$$\widetilde{\Omega}_k = E_{O_k,r'} \subset E, \quad \Omega_k = G_k(E_{O_k,r'}) \subset X, \quad \Theta_k = G_k^{-1} \colon \Omega_k \to \widetilde{\Omega}_k.$$

If k is sufficiently large then $\Omega_k \subset \Omega_0$, and for such k the map $\Theta_k \colon \Omega_k \to \widetilde{\Omega}_k$ satisfies the conclusion of Theorem 1.1.

5. Examples and problems

We give some examples and open problems related to Theorem 1.1. Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$.

Our first example shows that Theorem 1.1 fails in general if the map is not continuous at the boundary.

Example 5.1. *There exists a bounded holomorphic function on $\mathbb{D}$ such that the closure of its graph does not have a Stein neighbourhood basis in $\mathbb{C}^2$.*

Indeed, let $f \in H^\infty(\mathbb{D})$ be a bounded holomorphic function on the unit disc such that $\sup_{z\in\mathbb{D}} |f(z)| = 1$ and the cluster set of f at every boundary point $e^{i\theta} \in b\mathbb{D}$ equals $\overline{\mathbb{D}}$. (Such functions are easily found by using interpolation theorems for $H^\infty(\mathbb{D})$, see [13, Ch. VII].) The closure K of the graph of f in $\mathbb{C}^2$ is the union of the graph with all vertical discs $\{e^{i\theta}\} \times \bar{\mathbb{D}}$, $\theta \in \mathbb{R}$. By the classical argument of Hartogs [17] any open Stein neighbourhood of K in $\mathbb{C}^2$ also contains the unit bidisc.

Problem 5.2. Characterize the bounded holomorphic functions on the disc $\mathbb{D}$ for which the closure of the graph admits a basis of open Stein neighbourhoods in $\mathbb{C}^2$.

The next two examples illustrate that Corollary 1.2 fails in general for *images* of maps, as opposed to their graphs. However, see [9, Theorem 2.1].

Example 5.3. This example was communicated to me by E. L. Stout (private communication, September 19, 2006):

For every $N > 1$ there is an $\mathcal{A}(\mathbb{D})$-map $\overline{\mathbb{D}} \to \mathbb{C}^N$ whose image has no Stein neighbourhood basis.

Let E be a Cantor set of length zero contained in $b\mathbb{D}$; then E is a peak-interpolation set for the disc algebra $\mathcal{A}(\mathbb{D})$. Let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^N$ for some $N > 1$. Choose a continuous map f from E onto the sphere $b\mathbb{B}$. There is a map $F = (F_1, \ldots, F_N) \colon \bar{\mathbb{D}} \to \mathbb{C}^N$, satisfying $F_j \in \mathcal{A}(\mathbb{D})$ for each j, such that $F|_E = f$ and $|F(z)|^2 = \sum |F_j(z)|^2 < 1$ for all $z \in \overline{\mathbb{D}}\backslash E$ (Stout [31] and Globevnik [14]). Now let D' be a domain in $\mathbb{D}$ obtained by moving each of the open arcs of $b\mathbb{D}\backslash E$ in just a little, leaving the end points fixed; so D' is conformally a disc and $bD' \cap b\mathbb{D} = E$. Then $F(\bar{D}')$ is a compact set consisting of the sphere $b\mathbb{B}$ together with a proper subset of $\mathbb{B}$, and hence it has no Stein neighbourhood basis (any Stein neighbourhood also contains the ball $\mathbb{B}$). It is necessary to pass to a smaller domain $D' \subset \mathbb{D}$ because F might take $\overline{\mathbb{D}}$ onto the ball $\overline{\mathbb{B}}$ which has a basis of Stein neighbourhoods.

Example 5.4. This example is a minor modification of the one which was communicated to me by J.-P. Rosay on April 6, 2004:

There exists a smooth ($\mathcal{C}^\infty$) injective map Φ from the closed unit ball $\overline{\mathbb{B}}$ in $\mathbb{C}^5$ into $\mathbb{C}^8$, that is a holomorphic embedding of the open unit ball, such that $\Phi(\overline{\mathbb{B}})$ has no basis of Stein neighbourhoods.

We proceed as follows. The set

$$M = \{(z_1,\ldots,z_5) \in \overline{\mathbb{B}}\colon z_1 z_2 \cdots z_5 = \sqrt{5}^{\,-5}\}$$

is a real four dimensional submanifold of the boundary of $\mathbb{B}$ which is complex tangential to the sphere $b\mathbb{B}$ at each point. By Chaumat and Chollet ([3], [4]) every compact subset of M is a peak interpolation set for $\mathcal{A}^\infty(\mathbb{B})$, the Fréchet algebra of functions holomorphic on the ball and smooth up to the boundary. Let H be a closed Hartogs figure in $\mathbb{C}^2$:

$$(\zeta_1,\zeta_2) \in H \iff (|\zeta_1| \le 1\ ,\ |\zeta_2| \le \frac{1}{2}) \text{ or } (\frac{1}{2} \le |\zeta_1| \le 1\ ,\ |\zeta_2| \le 1).$$

Let H_0 be a diffeomorphic copy of H in M (such exists by dimension reasons). Consider a smooth map $\varphi\colon \overline{\mathbb{B}} \to \mathbb{C}^2$ that is holomorphic on the open ball and whose restriction to H_0 is a diffeomeorphism from H_0 onto H. Also choose a function $h \in \mathcal{A}^\infty(\mathbb{B})$ that is zero on H_0 and that vanishes nowhere else on the closed unit ball. (Such ϕ and h are obtained by appealing to [3], [4].) Define a map $\Phi\colon \overline{\mathbb{B}} \to \mathbb{C}^8$ by

$$\Phi(z) = \Phi(z_1,\ldots,z_5) = (\phi(z), h(z), z_1 h(z),\ldots, z_5 h(z)).$$

It is easy to check that Φ is injective on $\overline{\mathbb{B}}$, it is of maximum rank (immersion) at every point of $\overline{\mathbb{B}}\backslash H_0$, it maps H_0 onto $H \subset \mathbb{C}^2 \times \{0\}^6$, and $\Phi(z) \notin \mathbb{C}^2 \times \{0\}^6$ if $z \in \overline{\mathbb{B}} \setminus H_0$.

Since $\Phi(\overline{\mathbb{B}})$ contains the Hartogs figure $H \times \{0\}^6$, any open Stein neighbourhood of it will also contain the unit bidisc in $\mathbb{C}^2 \times \{0\}^6$; as this bidisc is not included in $\Phi(\overline{\mathbb{B}})$, the latter set has no basis of Stein neighbourhoods.

Problem 5.5. Let D be a strongly pseudoconvex domain in a Stein manifold S. Let $f\colon \bar{D} \to Y$ be a $\mathcal{C}^2$ map to a complex manifold Y such that f is holomorphic in D and is an embedding near the boundary bD. Does $f(\bar{D})$ admit a Stein neighborhood basis in Y ? (For bordered Riemann surfaces D the affirmative answer was given in [9, Theorem 2.1].)

Problem 5.6. Let K be a compact set with a Stein neighborhood basis in a complex manifold S. Assume that $f\colon K \to Y$ is a continuous map to a complex manifold Y which is a uniform limit on K of a sequence of holomorphic maps $f_j\colon V_j \to Y$ defined in small open neighbourhoods of K.

Does the graph $G_f = \{(z, f(z))\colon z \in K\}$ admit a basis of open Stein neighboorhoods in $S \times Y$?

The answer is easily seen to be affirmative if $Y = \mathbb{C}^N$ and, by the embedding theorem, also if Y is a Stein manifold. When K is the closure of a strongly pseudoconvex domain the answer is given by Corollary 1.2. Another case of interest is the closure of a *weakly pseudoconvex domain* D such that $K = \bar{D}$ admits a Stein neighborhood basis. (The latter condition is nontrivial as is shown by the worm domain of Diederich and Fornæss [6].) One should perhaps first look at the case when D is a pseudoconvex domain of finite type in $\mathbb{C}^2$ and f is smooth at the boundary of D.

Problem 5.7. Does Corollary 1.2 still hold if Y is a complex space with singularities? Is the answer different if f is more regular at the boundary of D? (For partial results when D is a bordered Riemann surface see [9, §2].)

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