University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 61 111 Ljubljana, Slovenia

Preprint series, Vol. 45 (2007), 1028

CLOSED HOLOMORPHIC 1-FORMS WITHOUT ZEROS ON STEIN MANIFOLDS

Irena Majcen

ISSN 1318-4865

January 15, 2007

Ljubljana, January 15, 2007

CLOSED HOLOMORPHIC 1-FORMS WITHOUT ZEROS ON STEIN MANIFOLDS

by

IRENA MAJCEN

University of Ljubljana
Ljubljana, Slovenia

1 Introduction

A *Stein* manifold is a complex manifold biholomorphic to a closed complex submanifold of a complex Euclidean space $\mathbb{C}^N$. Stein manifolds of complex dimension one are precisely open Riemann surfaces.

Suppose that X is a Stein manifold. In every class of the cohomology group $H^1(X, \mathbb{C})$ there is a closed holomorphic 1-form (Theorem 1 in [16], see also [1, p. 160] and Theorem 2 in [17, p. 208]). The existence of a holomorphic function $f : X \to \mathbb{C}$ without critical points, proved by Forstnerič in [4], implies that in the zero class there is a closed holomorphic 1-form without zeros, namely df. (For the case when X is an open Riemann surface see also [10].) Our goal in this paper is to show that a closed holomorphic 1-form without zeros can be chosen in every cohomology class. (See also [14] for open Riemann surfaces.)

Theorem 1. *Let X be a Stein manifold. Every cohomology class in the cohomology group $H^1(X, \mathbb{C})$ is represented by a closed holomorphic 1-form without zeros.*

Note that Theorem 1 is not true for an arbitrary complex manifold X. For example, if X is a compact Riemann surface of genus g then by the Riemann-Roch theorem each closed holomorphic 1-form has precisely $2g-2$ zeros.

Denote by $\mathcal{O}(X)$ the algebra of all holomorphic functions on X. A compact set $K \subset X$ is said to be $\mathcal{O}(X)$-*convex* if for any point $x \in X \setminus K$ there exists $f \in \mathcal{O}(X)$ such that $|f(x)| > \max_K |f|$.

Theorem 1 is a consequence of the following result.

Theorem 2. *Let $K \subset X$ be a compact, $\mathcal{O}(X)$-convex subset and θ a closed 1-form on X. Let ω be a closed holomorphic 1-form on a neighborhood U of K, with $\omega|_x \neq 0$ for all $x \in K$, such that $\int_C \omega = \int_C \theta$ for each closed curve $C \subset U$. Then there exists a closed holomorphic 1-form $\widetilde{\omega}$ without zeros on X such that $[\widetilde{\omega}] = [\theta] \in H^1(X, \mathbb{C})$, and there exists a holomorphic injective mapping h in a neighborhood of K, close to the identity, such that $\widetilde{\omega} = h^*\omega$ near K.*

A closed nowhere vanishing holomorphic 1-form ω on X defines a holomorphic foliation $\mathcal{F}$ of codimension 1 with the tangent bundle $T\mathcal{F} = \ker \omega$. Theorem 2 then gives an approximation theorem for holomorphic nonsingular hypersurface foliations, defined by nonvanishing closed holomorphic 1-forms. The approximation is done by a global foliation $\widetilde{\mathcal{F}}$ on X such that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are conjugate to each other on a neighborhood of K.

Necessary and sufficient conditions for a compact real manifold without boundary to admit a representative without zeros in each class of the de Rham cohomology group have been studied in [15].

2 Proof of the main theorems

We only need to prove Theorem 2. It can be assumed that θ in Theorem 2 is a holomorphic 1-form on X (Theorem 1 in [16]). We show that there exists a closed holomorphic 1-form ω such that $\omega|_x \neq 0$ for all $x \in X$ and

$$\int_{C_i} \theta = \int_{C_i} \omega$$

for every closed curve $C_i \subset X$ where $\{C_1, C_2, \ldots\}$ is a basis of $H_1(X, \mathbb{R})$.

Choose a number $\widehat{c} \in \mathbb{R}$. Fix a smooth strongly plurisubharmonic Morse exhaustion function $\rho \colon X \to \mathbb{R}$ such that $\rho < \widehat{c}$ in K but $\rho > \widehat{c}$ in U^c ([13], Theorem 5.1.6.). The sublevel sets $\{\rho \leq c\}$ are compact and $\mathcal{O}(X)$-convex for all $c \in \mathbb{R}$, and for every regular value c of ρ, $\{\rho < c\}$ is a smooth strongly pseudoconvex domain. We can replace K by $\{\rho \leq \widehat{c}\}$ and in the following subsections the notations K and U will be used for other purposes.

Suppose that a closed holomorphic nonvanishing 1-form ω is given in a neighborhood of $\{\rho \leq c\}$, where c is a regular value of ρ, and suppose that ω has the same periods as θ on each closed curve $C_i \subset \{\rho \leq c\}$. The main step in the proof is a construction of a closed holomorphic nowhere vanishing 1-form $\widetilde{\omega}$ on a neighborhood of $\{\rho \leq \widetilde{c}\}$, which has the same periods as θ, where $\widetilde{c} > c$ is also a regular value of ρ. We will obtain a global solution using a limit process.

Our construction requires two distinct arguments depending on whether the interval $[c, \widetilde{c}]$ contains a critical value of ρ or not and is similar to the scheme in Section 6. in [4].

We choose a distance function d on X induced by a smooth Riemannian metric on TX. Suppose that $\alpha\colon V \to X$ is a holomorphic mapping on an open set V in X. We denote $||\alpha - id||_V = \sup_{x\in V} d(\alpha(x), x)$.

We first cite a result which will be needed several times in the proof.

Theorem 3 (Theorem 4.1 in [4]). *Let A and B be compact sets in a complex manifold X such that $A \cup B$ has a basis of Stein neighborhoods in X and $\overline{A\backslash B} \cap \overline{B\backslash A} = \emptyset$. Given an open set $\widetilde{C} \subset X$ containing $C := A \cap B$ there exist open sets $A' \supset A$, $B' \supset B$, $C' \supset C$ with $C' \subset A' \cap B' \subset \widetilde{C}$, satisfying the following. For every $\eta > 0$ there is $\varepsilon_\eta > 0$ such that for each injective holomorphic map $\gamma\colon \widetilde{C} \to X$ with $||\gamma - id||_{\widetilde{C}} < \varepsilon_\eta$ there exist injective holomorphic maps $\alpha\colon A' \to X$, $\beta\colon B' \to X$ satisfying*

$$\gamma = \beta \circ \alpha^{-1} \text{ on } C', \quad ||\alpha - id||_{A'} < \eta, \quad ||\beta - id||_{B'} < \eta.$$

2.1 The noncritical case

Let $A \subset \widetilde{A} \subset X$ be compact sets in X. If there exists a smooth strongly plurisubharmonic function ρ in an open set $\Omega \supset \overline{\widetilde{A}\backslash A}$ which has no critical points on Ω and satisfies

$$A \cap \Omega = \{x \in \Omega\colon \rho(x) \le 0\}, \quad \widetilde{A} \cap \Omega = \{x \in \Omega\colon \rho(x) \le 1\},$$

we call $\widetilde{A}$ a *noncritical strongly pseudoconvex extension* of A. The set $A_t = A \cup \{\rho \le t\} \subset X$ is a smooth strongly pseudoconvex domain in X for each $t \in [0, 1]$ and the family smoothly increases from $A = A_0$ to $\widetilde{A} = A_1$. A manifold X is said to be a *noncritical strongly pseudoconvex extension* of A if there is a smooth exhaustion function $\rho\colon X \to \mathbb{R}$ such that $A = \{\rho \le 0\}$ and ρ is strongly plurisubharmonic and without critical points on $\{\rho \ge 0\} = X\backslash \mathrm{intA}$.

With this notations the noncritical case can be rephrased as follows.

Proposition 4. *Let θ be a closed 1-form on a Stein manifold X. Suppose that $\widetilde{A} \subset X$ is a noncritical strongly pseudoconvex extension of $A \subset \widetilde{A}$. Let ω be a closed holomorphic 1-form on a neighborhood of A with $\omega|_x \ne 0$ for all $x \in A$ such that $\int_C \omega = \int_C \theta$ for all closed curves $C \subset A$. Choose $\varepsilon > 0$.*

There exists a closed holomorphic 1-form $\widetilde{\omega}$ on a neighborhood of $\widetilde{A}$, with $\widetilde{\omega}|_x \neq 0$ for all $x \in \widetilde{A}$, such that $\int_C \widetilde{\omega} = \int_C \theta$ for all closed curves $C \subset \widetilde{A}$, and there exists a holomorphic injective mapping α in a neighborhood V of A such that $\widetilde{\omega} = \alpha^*\omega$ near A and $||\alpha - id||_V < \varepsilon$.

Proof. We first introduce the notion of a convex bump. We shall use the same kind of bumps as in [4] and [5]. Denote the coordinates on $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$ where $z_j = x_j + iy_j$ and let

$$P = \{z \in \mathbb{C}^n \colon |x_j| < 1,\ |y_j| < 1,\ j = 1, 2, \ldots, n\}$$

denote the open unit cube. Set $P' = \{z \in P \colon y_n = 0\}$. We say that a compact set $B \subset X$ is a *convex bump* on a compact set $A \subset X$ if there exist an open set $U \subset X$ containing B, a biholomorphic map $\varphi \colon U \to P$ onto $P \subset \mathbb{C}^n$ and smooth strongly concave functions $h, \widetilde{h} \colon P' \to [-a, a]$ for some $a < 1$ such that $h \leq \widetilde{h}$ and $h = \widetilde{h}$ near the boundary of P' and

$$\varphi(A \cap U) = \{z \in P \colon y_n \leq h(z_1, \ldots, z_{n-1}, x_n)\},$$

$$\varphi((A \cup B) \cap U) = \{z \in P \colon y_n \leq \widetilde{h}(z_1, \ldots, z_{n-1}, x_n)\}.$$

Assume that $A \subset \widetilde{A}$ is a strongly pseudoconvex extension. Applying Lemma 12.3. in [11] we find finitely many compact strongly pseudoconvex domains $A = A_0 \subset A_1 \subset \ldots \subset A_{k_0} = \widetilde{A}$ in X such that $A_{k+1} = A_k \cup B_k$ for every $k = 0, 1, \ldots, k_0 - 1$ where B_k denotes a convex bump on A_k as defined above. (For similar constructions see also [4], [6], [7], [8], [9], [12].)

The construction breaks into k_0 steps of the same kind. In the k-th step we show how to obtain a holomorphic 1-form ω_{k+1} with prescribed periods on a neighborhood of A_{k+1} if we are given ω_k on a neighborhood of A_k which satisfies

$$\omega_k = \alpha_{k-1}^* \omega_{k-1} \text{ in } A'_{k-1} \quad \text{and} \quad ||\alpha_{k-1} - Id||_{A'_{k-1}} \leq \frac{\varepsilon}{2^k},$$

where A'_{k-1} is a neighborhood of A_{k-1}.

Choose a neighborhood U of B_k as in the definition of a convex bump. With the notations as in the definition choose $c \in (a, 1)$ sufficiently close to 1 such that the compact support of $\widetilde{h} - h$ is contained in cP'. Let $L := c\overline{P} \subset \mathbb{C}^n$ and $\widetilde{L} := \varphi^{-1}(L) \subset U$. By letting c tend to 1, we may assume that $B_k \subset \widetilde{L}$. Set $\widetilde{K} = A_k \cap \widetilde{L}$ and $K = \varphi(\widetilde{K}) \subset P$. Note that K is convex and hence $\widetilde{K}$ has a simply connected neighborhood V.

There is a noncritical holomorphic function f in a neighborhood V of $\widetilde{K}$ such that $\omega_k = df$ on V. Choose $\varepsilon' > 0$. By Proposition 3.3. in [4] we

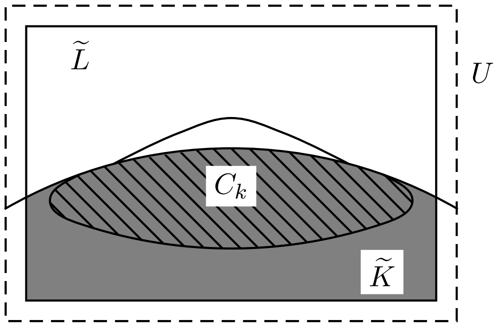

Figure 1: Convex bump

obtain a holomorphic submersion $g\colon V' \to \mathbb{C}$ in an open set $V' \supset \widetilde{L}$ which approximates f uniformly in a neighborhood of $\widetilde{K}$.

Denote $C_k = A_k \cap B_k$. By Lemma 5.1. in [4] there exist constants $\varepsilon_0 > 0$, $M > 0$ and an open set $W \subset X$ with $C_k \subset W \subset V$ satisfying the following property. Given $\varepsilon' \in (0, \varepsilon_0)$ and a holomorphic submersion $g\colon W \to \mathbb{C}$ with $\sup_{x\in W} |f(x) - g(x)| < \varepsilon'$ there is an injective holomorphic map $\gamma\colon W \to X$ satisfying $f = g \circ \gamma$ on W and $||\gamma - id||_W < M \cdot \varepsilon'$. By the above argument such a holomorphic submersion g exists, hence we have the decomposition $f = g \circ \gamma$ on W. But γ can also be decomposed.

Set $\eta = \frac{\varepsilon}{2^{k+1}}$ and $\varepsilon' = \min\{\frac{\varepsilon_\eta}{M}, \frac{\varepsilon_0}{2}\}$. By the above construction the mapping $\gamma\colon W \to X$ can be chosen to satisfy $||\gamma - id||_W < \varepsilon_\eta$ and can be then split using Theorem 3 as

$$\gamma = \beta_k \circ \alpha_k^{-1} \quad \text{with} \quad ||\alpha_k - id||_{A'_k} < \frac{\varepsilon}{2^{k+1}}, \quad ||\beta_k - id||_{B'_k} < \frac{\varepsilon}{2^{k+1}}.$$

Define a holomorphic 1-form ω_{k+1} by

$$\omega_{k+1} = \begin{cases} \alpha_k^* \omega_k & \text{on } A'_k, \\ d(g \circ \beta_k) & \text{on } B'_k. \end{cases}$$

Clearly ω_{k+1} is well defined and without zeros. The mapping α_k provides a homotopy of curves C and $\alpha_k(C)$ for any closed curve $C \subset A_0$, thus the relation $\omega_{k+1} = \alpha_k^* \omega_k$ implies that 1-forms ω_{k+1} and ω_k have the same periods (by Stokes' theorem).

Repeating this step k_0 times we obtain $\widetilde{\omega} = \omega_{k_0}$ with the required properties in a neighborhood of $\widetilde{A} = A_{k_0}$ satisfying $\widetilde{\omega} = h^*\omega$ on A_0 where

$h = \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_{k_0}$. By construction we have

$$\begin{aligned} ||h - id||_{A_0} &= ||\alpha_1 \circ \ldots \circ \alpha_{k_0} - \alpha_2 \circ \ldots \circ \alpha_{k_0} + \ldots + \alpha_{k_0} - id||_{A_0} \\ &\leq \sum_{k=1}^{k_0 - 1} ||\alpha_k \circ \ldots \circ \alpha_{k_0} - \alpha_{k+1} \circ \ldots \circ \alpha_{k_0}||_{A_0} + ||\alpha_{k_0} - id||_{A_0} \\ &< \frac{\varepsilon}{2} + \ldots + \frac{\varepsilon}{2^{k_0 - 1}} + \frac{\varepsilon}{2^{k_0}} < \varepsilon, \end{aligned}$$

since we may assume that $\alpha_2 \circ \ldots \circ \alpha_{k_0}(A_0) \subset A_0'$, where $A_0' \supset A_0$ is a domain of α_1. □ □

With this we have finished the proof of the noncritical case. In the following subsection we treat the critical case.

2.2 The critical case

Let p be a critical point of ρ and let k denote the Morse index of p. If $k = 0$, ρ has a local minimum at p; as c passes $\rho(p)$, a new connected component appears in $\{\rho < c\}$ (see Lemma 2.3. in [12]). Hence ω can be trivially extended by taking a differential of any noncritical holomorphic function near p.

From now on we assume $k \geq 1$. There is no loss of generality in assuming $\rho(p) = 0$. Choose $c_0 \in (0, 1)$ such that p is the only critical point of ρ in $[-c_0, 3c_0]$. In what follows we explain how to obtain a closed nowhere vanishing holomorphic 1-form $\widetilde{\omega}$ with prescribed periods in a neighborhood of $\{\rho \leq +c_0\}$ provided that a required 1-form ω has already been constructed on a neighborhood of $\{\rho \leq -c_0\}$.

Denote by $P \subset \mathbb{C}^n$ the open unit polydisc. Using Lemma 2.5 in [12] we may assume that all the critical points of ρ are *nice*, meaning that there is a neighborhood $U \subset X$ of p and a biholomorphic coordinate map $\varphi\colon U \to P$, with $\varphi(p) = 0$, such that the function $\widetilde{\rho}(z) := \rho(\varphi^{-1}(z))$ is given by

$$\widetilde{\rho}(z) = \sum_{j=1}^{n} \mu_j y_j^2 - \sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{n} x_j^2,$$

where $\mu_j \geq 1$ for all j and $\mu_j > 1$ when $1 \leq j \leq k$. Denote the smallest of the numbers $\mu_1, \ldots, \mu_k$ by μ.

Write $z = (z', z'') = (x' + iy', x'' + iy'')$, where $z' \in \mathbb{C}^k$ and $z'' \in \mathbb{C}^{n-k}$. The set $E_0 \subset U$ defined by

$$\varphi(E_0) = \{(x' + iy', z'')\colon y' = 0,\ z'' = 0,\ |x'|^2 \leq c_0\}$$

is a k-dimensional core of a handle attached from the outside to $\{\rho \le -c_0\}$. Such cores were used in [4]. The handles introduced in [12] could have been used as well, but the first aproach has been chosen for it will be used in the proof of a more general result in Section 3.

Let $c = (1 - \frac{1}{\mu})^2 c_0$. By the noncritical case we may assume that ω is given on $\{\rho < -c/2\}$. Define a smaller k-dimensional handle $E \subset E_0$ by the condition

$$\varphi(E) = \{(x' + iy', z'')\colon y' = 0,\ z'' = 0,\ |x'|^2 \le c\}.$$

Note that bE is a $(k-1)$-sphere contained in $\{\rho = -c\}$.

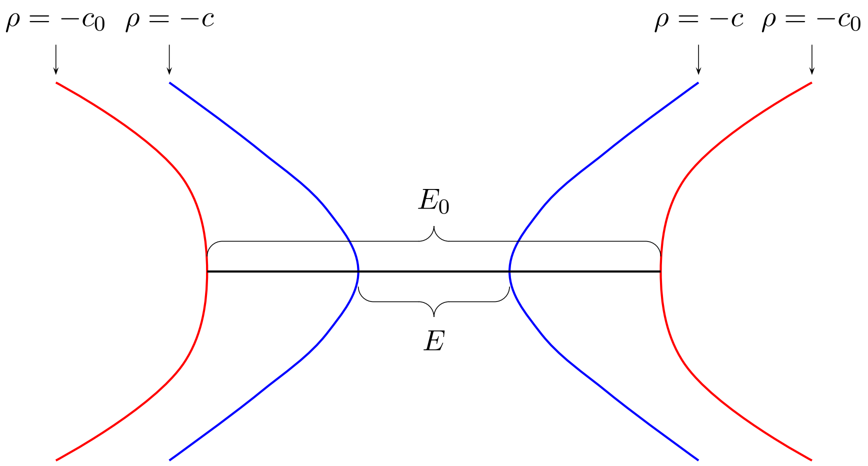

Figure 2: Handles E and E_0

When $k \ge 3$ we have $\pi_1(bE) = 0$, hence $\int_\Gamma \theta = 0$ for any closed curve $\Gamma \subset bE$. When $k = 2$, bE is a circle bounding the 2-disc E and hence $\int_{bE} \omega = \int_{bE} \theta = \int_E d\theta = 0$. In both cases we conclude that there exist a neighborhood V of bE in $\{\rho \le -\frac{c}{2}\}$ and a holomorphic submersion $f\colon V \to \mathbb{C}$ such that $\omega = df$ on V.

By Lemma 6.4. in [4] there is a constant $c' \in (\frac{1}{2}c, c)$ such that f and its partial derivatives $\frac{\partial f}{\partial z_l}$ extend smoothly to $(\{\rho \le -c'\} \cap V) \cup E$ (without changing their values on $\{\rho \le -c'\}$) and the Jacobian matrix $J(\widetilde{f}) = (\frac{\partial f_j}{\partial z_l})$ of the extension $\widetilde{f}$ has complex rank q at each point of E. Inspection of the proof shows that c' can be chosen arbitrarily close to c, hence we may assume that

$$\varphi^{-1}\Big(\{(x' + iy', z'')\colon y' = 0,\ z'' = 0,\ c' \le |x'|^2 \le c\}\Big) \subset V.$$

Let $d(z,E) = \inf_{w\in E} d(z,w)$. The function $d^2(z,E)$ is strictly plurisubharmonic in a neighborhood of E and provides a family of pseudoconvex neighborhoods of E. Fix a compact pseudoconvex set L such that $E \subset L \subset U$ and $L \cap \{\rho \le -c'\} \subset V$. Choose $c'' \in (c', c)$ and denote $K = (L \cap \{\rho \le -c''\}) \cup E$. Lemma 6.6. in [4] gives for every $\delta > 0$ an open neighborhood $V' \subset X$ of the set K and a holomorphic submersion $g\colon V' \to \mathbb{C}$ such that $|\widetilde{f} - g|_K < \delta$, $|d\widetilde{f} - dg|_E < \delta$. Here $|f|_K$ is the uniform norm of f on K and $|df|_E$ is the norm of its differential on E, measured in a fixed Hermitean metric on TX.

As in the noncritical case we shall use g to define a holomorphic 1-form on a larger domain. The main difference is that this larger domain is not a sublevel set with some convex bumps but a sublevel set of a new strongly plurisubharmonic function τ.

Lemma 5 (Lemma 6.7. in [4]). *There exists a smooth strongly plurisubharmonic function τ on $\{\rho < 3c_0\} \subset X$ which has no critical values in $(0, 3c_0) \subset \mathbb{R}$ and satisfies*

(i) $\{\rho \le -c_0\} \cup E \subset \{\tau \le 0\} \subset \{\rho \le -c\} \cup E$ *and*

(ii) $\{\rho \le c_0\} \subset \{\tau \le 2c_0\} \subset \{\rho < 3c_0\}$.

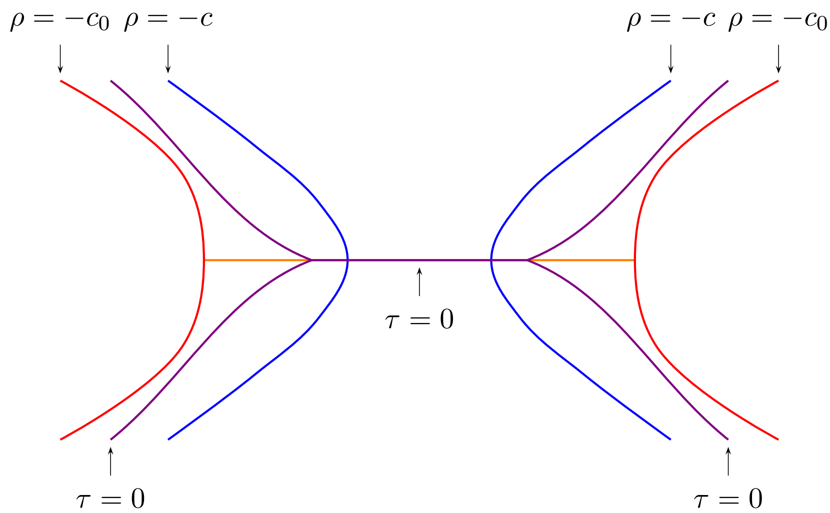

Figure 3: The level set $\{\tau = 0\}$

Choose $c''' \in (c'', c)$. The set $\Omega = \{\rho < c'''\} \cup V'$ is a neighborhood of $\{\rho \le c\} \cup E$. Consider a family of sublevel sets $\{\tau \le t\}$ as t increases from

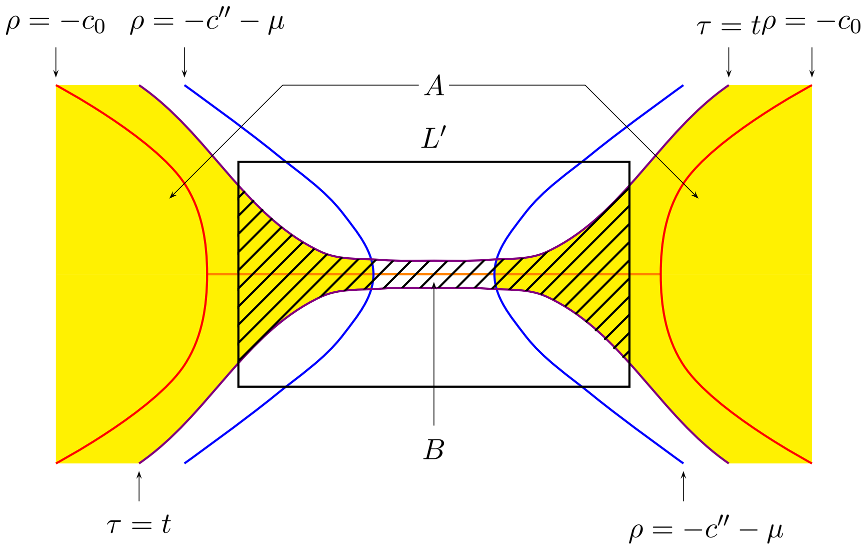

Figure 4: Compact sets A and B

0 to $2c_0$. The property (i) of τ implies that for sufficiently small $t > 0$ we have $\{\tau \leq t\} \subset \Omega$. Fix a number t with such property.

Choose a small number $\mu > 0$. Let L' be a compact pseudoconvex set such that $E \subset L' \subset \text{int}L$ and let $K' = (L' \cap \{\rho \leq -c'' - \mu\}) \cup E$. Finally, define

$$A = \{\tau \leq t\} \cap \{\rho \leq -c'' - \mu\}, \quad B = L' \cap \{\tau \leq t\}.$$

Note that $A \cup B = \{\tau \leq t\}$ and $A \cap B = \{\rho \leq -c'' - \mu\} \cap \{\tau \leq t\} \cap L'$, hence $W = \text{int } K$ is a neighborhood of $A \cap B$. Taking μ small enough, we have $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$. By Lemma 5.1. in [4] we obtain positive constants ε_0, M and an open neighborhood $W' \subset W$ of $A \cap B$ such that the following holds. For any $\varepsilon' \in (0, \varepsilon_0)$ and a holomorphic submersion $g \colon W \to \mathbb{C}$ with $\sup_{x \in W} |f(x) - g(x)| < \varepsilon'$ there is an injective holomorphic map $\gamma \colon W' \to X$ satisfying

$$f = g \circ \gamma \text{ on } W' \quad \text{and} \quad ||\gamma - id||_{W'} < M \cdot \varepsilon'.$$

The submersion g is provided by the argument above for $\delta < \varepsilon_0$. Additionally, to split γ using Theorem 3 and obtain an estimate $||\alpha - id||_{A'} < \frac{\varepsilon}{2}$, we take $\eta = \frac{\varepsilon}{2}$ and choose $\delta > 0$ so small that $\delta \cdot 2M < \varepsilon_\eta$. Note that $\widetilde{C}$ coresponds to W'. Thus Theorem 3 gives injective holomorphic maps α and β mapping from neighborhoods A' of A, respectively B' of B, into X such that

$$\gamma = \beta \circ \alpha^{-1} \text{ on } C', \quad ||\alpha - id||_{A'} < \frac{\varepsilon}{2}, \quad ||\beta - id||_{B'} < \frac{\varepsilon}{2}.$$

Finally define

$$\omega' = \begin{cases} \alpha^*\omega & \text{on } A' \\ d(g\circ\beta) & \text{on } B'. \end{cases}$$

Note that ω' is given on a neighborhood of $\{\tau \le t\}$ and by the noncritical case (Proposition 4) we further obtain $\widetilde{\omega}$ on a neighborhood of $\{\tau \le 2c_0\} \supset \{\rho \le c_0\}$ as desired.

It remains to construct $\widetilde{\omega}$ when $k = 1$. There are two possibilities:

(1) the core of a handle E_0 connects two components of $\{\rho \le -c_0\}$,

(2) the core of a handle E_0 is a part of a new closed curve in the group $H_1(\{\rho \le c_0\}, \mathbb{R})$.

When the situation is as in (1), $\widetilde{\omega}$ can be constructed as above. The possibility (2) requires some additional effort. We can choose a new curve C_0 in such a way that $E_0 \subset C_0$ and $C_0 \backslash E_0 \subset \{\rho \le -c_0\}$. The construction of $\widetilde{\omega}$ is similar as in the case $k \ge 2$, so to obtain $\widetilde{\omega} = h^*\omega$, the additional requirement is that $\int_{C_0} \widetilde{\omega} = \int_{C_0} \theta$.

Since bE consists of two points a and b, we can choose disjoint neighborhoods V_a and V_b of a and b such that $\omega|_{V_a} = df_a$ and $\omega|_{V_b} = df_b$, where $f_a\colon V_a \to \mathbb{C}$ and $f_b\colon V_b \to \mathbb{C}$ are noncritical functions satisfying $f_b(b) - f_a(a) = \int_{C_0} \theta - \int_{C_0\backslash E} \omega$. Denote $V = V_a \cup V_b$.

Applying the construction when $k \ge 2$ we get $\widetilde{\omega} = d(g\circ\beta)$ on a neighborhood of E for some noncritical function g. The integral condition implies that $f\circ\alpha(b) - f\circ\alpha(a) + \int_{C_0\backslash E} \alpha^*\omega = \int_{C_0} \theta$. In order to find such α and choose the appropriate f, we define a family of maps.

For each $t \in \Delta \subset \mathbb{C}$ let f_t be a noncritical function such that

$$f_t\colon V \to \mathbb{C}, \quad f_t|_{V_a} = f_a \quad \text{and} \quad f_t|_{V_b} = f_b + t.$$

We proceed as in the case $k \ge 2$. Using Lemma 6.4. in [4] for a family of functions f_t that depend holomorphically on parameter t, we obtain functions $\widetilde{f_t}$ on $(\{\rho \le c'\} \cap V) \cup E$ which smoothly extend f_t and its partial derivatives $\frac{\partial f_t}{\partial z_l}$, and the Jacobian matrices $J(\widetilde{f_t})$ have complex rank 1 at each point of E. We define L and K as before and use Lemma 6.6. in [4] to obtain a family $g_t\colon V' \to \mathbb{C}$ such that

$$|\widetilde{f_t} - g_t|_K < \delta \quad \text{and} \quad |d\widetilde{f_t} - dg|_E < \delta.$$

Let A and B be defined as above. Lemma 5.1. in [4] gives maps γ_t, holomorphic in t, such that $f_t = g_t \circ \gamma_t$ on a neighborhood of K. Define

$\gamma(x,t) := (\gamma_t(x), t)$. By Theorem 4.1. in [4] for sets $A \times \Delta$ and $B \times \Delta$ we obtain a splitting $\gamma = \beta \circ \alpha^{-1}$ where the mappings α and β are of the same form as γ, that is $\alpha(x,t) = (\alpha_t(x), t)$ and $\beta(x,t) = (\beta_t(x), t)$. Thus we get a family of 1-forms defined by

$$\omega_t = \begin{cases} \alpha_t^* \omega & \text{on } A', \\ d(g_t \circ \beta_t) & \text{on } B'. \end{cases}$$

We now show that there is a number $t' \in \Delta$ such that $\int_{C_0} \omega_{t'} = \int_{C_0} \theta$. Define

$$\varphi(t) := t, \quad \psi(t) := f_t(\alpha_t(b)) - f_t(\alpha_t(a)) + \int_{C_0 \setminus E} \alpha^* \omega - \int_{C_0} \theta.$$

Obviously ψ and φ are holomorphic and for a fixed $s < 1$, s close to 1, we have $|\varphi(t) - \psi(t)|$ small for $t \in b(s\Delta)$ and $|\varphi(t)| = |t| = s$ for $t \in b(s\Delta)$, hence

$$|\varphi(t) - \psi(t)| < |\varphi(t)|$$

on $b(s\Delta)$. By Rouche's theorem φ and ψ have the same number of zeros on $s\Delta$. Since $\varphi(0) = 0$, there is a $t' \in s\Delta$ such that $\varphi(t') = 0$. The form $\omega_{t'}$ satisfies all the requirements.

2.3 Conclusion

We now finish the proof of Theorem 2 by explaining the global scheme. Denote by $p_1, p_2, \ldots$ the critical points of ρ in $\{\rho > \widehat{c}\}$. Since each critical level set contains a unique critical point we may assume that

$$\rho(p_1) < \rho(p_2) < \rho(p_3) < \ldots .$$

We inductively choose a sequence of real numbers $\widetilde{c}_0 < \widetilde{c}_1 < \widetilde{c}_2 < \ldots$ such that

$$\widetilde{c}_0 < \rho(p_1) < \widetilde{c}_1 < \rho(p_2) < \ldots .$$

Let $d_j = \rho(p_j) - \widetilde{c}_{j-1}$. By choosing $\widetilde{c}_{j-1}$ sufficiently close to $\rho(p_j)$ it can be achieved for each $j = 1, 2, \ldots$ that $\rho(p_j) + 3d_j < \rho(p_{j+1})$. We may also require $\widetilde{c}_j > \rho(p_j) + 3d_j$ and $\widetilde{c}_0 > \widehat{c}$. Define a new sequence

$$\begin{aligned} c_{2k} &= \widetilde{c}_k, \\ c_{2k+1} &= \rho(p_{k+1}) + d_{k+1}, \end{aligned}$$

where $k = 0, 1, 2, \ldots$. If there are only finitely many critical points, choose the remainder of this sequence arbitrarily with $\lim_{j \to \infty} c_j = \infty$.

In the j-th stage of construction we assume inductively that we have a nonvanishing holomorphic 1-form ω_j in a neighborhood of $\{\rho \le c_j\}$, a holomorphic injective mapping α_j on a neighborhood of $\{\rho \le c_{j-1}\}$ and a number $\varepsilon_j < \varepsilon_{j-1}$ which satisfy the following:

(1) $\int_C \omega_j = \int_C \theta$ for each closed curve $C \subset \{\rho \le c_j\}$,

(2) $\omega_j = \alpha_j^* \omega_{j-1}$ and $||\alpha_j - id|| < \varepsilon_{j-1}$ on a domain of α_j,

(3) for all $x, y \in \{\rho \le c_{j-1}\}$ the condition $d(x,y) < \varepsilon_j$ implies

$$||\alpha_k \circ \alpha_{k+1} \circ \ldots \circ \alpha_j(x) - \alpha_k \circ \alpha_{k+1} \circ \ldots \circ \alpha_j(y)|| < \frac{\varepsilon}{2^{j+1}}$$

for each number $k \in \{1, 2, \ldots, j\}$.

Since $\{\rho \le c_0\}$ is a noncritical strongly pseudoconvex extension of $\{\rho \le \widehat{c}\}$, Proposition 4 provides a closed holomorphic 1-form ω_0 on a neighborhood of $\{\rho \le c_0\}$ such that $\omega_0 = \alpha_0^* \omega$ on a neighborhood of $\{\rho \le \widehat{c}\}$, where $||\alpha_0 - id|| < \frac{\varepsilon}{2}$. Choose ε_0 such that for every $x, y \in \{\rho \le \widehat{c}\}$ with $d(x,y) < \varepsilon_0$ it follows that $||\alpha_0(x) - \alpha_0(y)|| < \frac{\varepsilon}{2}$. We now explain how to obtain ω_{j+1}.

If there is no critical value on the interval $[c_j, c_{j+1}]$, we use Proposition 4 to get ω_{j+1} and α_{j+1} which satisfy

$$\int_C \omega_{j+1} = \int_C \theta \text{ for each closed curve } C \subset \{\rho \le c_{j+1}\}, \qquad (1)$$

and

$$\omega_{j+1} = \alpha_{j+1}^* \omega_j, \ ||\alpha_{j+1} - id|| < \varepsilon_j \text{ on the domain of } \alpha_{j+1}. \qquad (2)$$

Now suppose that the interval $[c_j, c_{j+1}]$ contains a critical value. The construction in subsection 2.2 provides ω_{j+1} and α_{j+1} for which statements (1) and (2) hold. In both cases, we may (by uniform continuity) choose $\varepsilon_{j+1} < \varepsilon_j$ in such a way that $d(x,y) < \varepsilon_{j+1}$ implies

$$||\alpha_k \circ \alpha_{k+1} \circ \ldots \circ \alpha_{j+1}(x) - \alpha_k \circ \alpha_{k+1} \circ \ldots \circ \alpha_{j+1}(y)|| < \frac{\varepsilon}{2^{j+2}} \qquad (3)$$

for all $x, y \in \{\rho \le c_j\}$ and for each number $k \in \{1, 2, \ldots, j+1\}$.

Define a global 1-form $\widetilde{\omega}$ as a limit $\widetilde{\omega} = \lim_{j\to\infty} \omega_j$. It remains to show that the sequence converges and ω satisfies the stated properties. Let K be an arbitrary compact set in X. There is an index j such that $K \subset \{\rho \le c_{j-1}\}$. Since $\omega|_K = \lim_{k\ge j} \omega_j|_K$, we have

$$\omega|_K = \lim_{k\to\infty} (\alpha_j \circ \ldots \circ \alpha_{j+k})^* \omega_j|_K.$$

Set $h_j := \lim_{k\to\infty}(\alpha_j \circ \ldots \circ \alpha_{j+k})$. To see that h_j is holomorphic, first choose any $\delta > 0$. There is an index k such that $2^{j+k-1} \cdot \delta > \varepsilon$. For any positive integer l we have

$$||\alpha_j \circ \ldots \circ \alpha_{j+k+l} - \alpha_j \circ \ldots \circ \alpha_{j+k}||_K =$$

$$\left|\left| \sum_{s=0}^{l-1} (\alpha_j \circ \ldots \circ \alpha_{j+k+s+1} - \alpha_j \circ \ldots \circ \alpha_{j+k+s}) \right|\right|_K \leq$$

$$\leq \sum_{s=0}^{l-1} ||\alpha_j \circ \ldots \circ \alpha_{j+k+s} \circ \alpha_{j+k+s+1} - \alpha_j \circ \ldots \circ \alpha_{j+k+s}||_K.$$

Since by (2) we have $||\alpha_{j+k+s+1} - id||_K < \varepsilon_{j+k+s}$, the condition (3) implies that

$$||\alpha_j \circ \ldots \circ \alpha_{j+k+l} - \alpha_j \circ \ldots \circ \alpha_{j+k}||_K \leq \sum_{s=0}^{l-1} \frac{\varepsilon}{2^{j+k+s}} < \sum_{s=0}^{l-1} \frac{2^{j+k-1}\delta}{2^{j+k+s}} < \delta$$

for any positive integer l. Since each α_k is injective, it follows by chosing the norms precise enough that h_j is holomorphic and injective. Therefore ω is a closed holomorphic 1-form on X that is without zeros. For any closed curve $C \subset K$ we have

$$\int_C \omega = \int_C h^*\omega_j = \int_{h(C)} \omega_j = \int_C \theta$$

by Stokes theorem. Since K was chosen arbitrarily, the conclusion holds for any closed curve $C \subset X$, which implies that ω is in the same cohomology class as θ.

3 Linearly independent 1-forms

In this section we prove the following generalization of Theorem 1 and Theorem II in [4].

Theorem 6. *Let X^n be a Stein manifold whose holomorphic cotangent bundle T^*X admits a trivial complex subbundle of rank q for some $1 \leq q < n$. Given closed 1-forms $\theta_1, \ldots, \theta_q$ on X there exist closed holomorphic 1-forms $\omega_1, \ldots, \omega_q$ satisfying*

$$[\omega_j] = [\theta_j] \in H^1(X, \mathbb{C}) \quad \textit{for each } j = 1, \ldots, q$$

and

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_q|_x \neq 0 \quad \textit{for all } x \in X.$$

Remark 7. *T^*X always contains a trivial subbundle of rank $q = [\frac{n+1}{2}]$ and this value is optimal in general (see [2, p. 714] and [3, Proposition 3]).*

Proof. We choose continuous $(1,0)$-forms $\omega'_1, \dots, \omega'_q$ such that

$$\omega'_1 \wedge \dots \wedge \omega'_q|_x \neq 0$$

for all $x \in X$. These forms span a trivial subbundle of T^*X. We construct pointwise linearly independent holomorphic 1-forms $\omega_1, \dots, \omega_q$ such that $[\omega_j] = [\theta_j] \in H^1(X, \mathbb{C})$ for each $j = 1, \dots, q$, where the collection $(\omega_1, \dots, \omega_q)$ is homotopic to $(\omega'_1, \dots, \omega'_q)$ through the homotopy of q-tuples of independent sections of T^*X. The construction is similar to the proof of Theorem 2 and we briefly illustrate the necessary changes in the noncritical and the critical case.

(I.) The noncritical case. Choose a local representation of a bump as in subsection 2.1 and represent each ω_j by a noncritical holomorphic function f_j on a neighborhood of $\widetilde{K}$. Approximate a submersion $f = (f_1, \dots, f_q)$ by a holomorphic submersion $g\colon V' \to \mathbb{C}^q$, $V' \supset \widetilde{L}$, using Proposition 3.3. in [4]. By Lemma 5.1. in [4] there is an injective holomorphic map $\gamma\colon W \to X$, $W \subset V'$, satisfying $f = g \circ \gamma$ on W and $||\gamma - id||_W < \varepsilon$. Using Theorem 3 we obtain $\gamma = \beta \circ \alpha^{-1}$ with α and β close to identity. Finally define $\widetilde{\omega}_j$ as

$$\widetilde{\omega}_j = \begin{cases} \alpha^*\omega_j, \\ d(g_j \circ \beta). \end{cases}$$

By construction $\widetilde{\omega}_j$ has the same periods as θ_j and no zeros. Since

$$\widetilde{\omega}_1 \wedge \dots \wedge \widetilde{\omega}_q = \begin{cases} \alpha^*(\omega_1 \wedge \dots \wedge \omega_q) & \text{on } A', \\ \beta^*(dg_1 \wedge \dots \wedge dg_q) & \text{on } B', \end{cases}$$

the 1-forms $\widetilde{\omega}_1, \dots, \widetilde{\omega}_q$ are pointwise linearly independent.

(II.) The critical case. The pointwise independency of 1-forms $\omega_1, \dots,$ ω_q is invariant under small perturbations, hence there is an $\varepsilon_\alpha > 0$ such that the conditions $||\alpha_j - id||_{A'} < \varepsilon_\alpha$ for $j = 1, \dots, q$ imply that

$$\alpha_1^*\omega_1 \wedge \dots \wedge \alpha_q^*\omega_q|_x \neq 0 \quad \text{for all } x \in A'.$$

(II.1) The Morse index is at least 2. We use the notations as in subsection 2.2. There is a neighborhood V of bE in $\{\rho \leq -\frac{c}{2}\}$ and a holomorphic submersion $f\colon V \to \mathbb{C}^q$ such that on V we have local representations $\omega_j = df_j$ for $j = 1, \ldots, q$.

By Lemma 6.4. in [4] there is a constant $c' > c$ close to c such that f and it's partial derivatives restricted to $\{\rho \leq -c'\}$ extend smoothly to $\widetilde{f}$ on $(\{\rho \leq -c'\} \cap V) \cup E$ where the Jacobian matrix $J(\widetilde{f}) = (\partial \widetilde{f}_j / \partial z_l)$ has complex rank q at each point of E.

By Lemma 6.6. in [4] we get for every $\delta > 0$ a holomorphic submersion $g\colon V' \to \mathbb{C}^q$ such that $|\widetilde{f} - g|_K < \delta$, $|d\widetilde{f} - dg|_E < \delta$. A small perturbation of each element of the set $\{dg_1, \ldots, dg_q\}$ leaves pointwise independency invariant, thus there is $\varepsilon_\beta > 0$ such that $d(g_1 \circ \beta_1), \ldots, d(g_q \circ \beta_q)$ are pointwise independent if $||\beta_j - id|| < \varepsilon_\beta$.

We use Lemma 5.1. in [4] and Theorem 3 to get $f_j = g_j \circ \gamma_j$ and split $\gamma_j = \beta_j \circ \alpha_j^{-1}$. This can be done in such a way that $||\alpha_j - id|| < \varepsilon_\alpha$ and $||\beta_j - id|| < \varepsilon_\beta$ for each j. The obtained 1-forms

$$\widetilde{\omega}_j = \begin{cases} \alpha_j^* \omega_j & \text{on } A', \\ d(g_j \circ \beta_j) & \text{on } B', \end{cases}$$

are then pointwise independent with prescribed periods and no zeros.

(II.2) Morse index of a critical point is 1. In this case the neighborhood V of bE, where $df_j = \omega_j$, can be chosen as a disjoint union of two open sets V_a and V_b. We may further assume that

$$f_j(b) - f_j(a) = \int_{C_0} \theta_j - \int_{C_0 \setminus E} \omega_j.$$

Let $f_j^{t_j}\colon V \to \mathbb{C}$ be defined as $f_j^{t_j}|_{V_a} = f_j$ and $f_j^{t_j}|_{V_b} = f_j^{t_j} + t_j$. Denote $f_{t_1 \ldots t_q} = (f_1^{t_1}, \ldots, f_q^{t_q})\colon V \to \mathbb{C}^q$. Lemma 6.4. and Lemma 6.6. in [4] can be used for a family of submersions $\{f_{t_1 \ldots t_q}\}$ holomorphically depending on parameters $(t_1, \ldots, t_q) \in \Delta \times \ldots \times \Delta$ to get families $\widetilde{f}_{t_1 \ldots t_q}$ and $g_{t_1 \ldots t_q}$. Again we may find such number $\varepsilon_\beta > 0$ that the condition $||\beta_j^{t_j} - id|| < \varepsilon_\beta$ implies the pointwise independency of $d(g_1^{t_1} \circ \beta_1^{t_1}), \ldots, d(g_q^{t_q} \circ \beta_q^{t_q})$.

By Lemma 5.1. in [4] we get for each $j = 1, \ldots, q$ maps $\gamma_j^{t_j}$ such that $f_j^{t_j} = g_j^{t_j} \circ \gamma_j^{t_j}$. Splitting $\gamma_j(x, t) = (\gamma_j^t(x), t)$ with respect to $A \times \Delta$ and $B \times \Delta$ we get $\gamma_j = \beta_j \circ \alpha_j^{-1}$ where

$$\alpha_j(x, t) = (\alpha_j^t(x), t), \ \beta_j(x, t) = (\beta_j^t(x), t), \ ||\alpha_j - id|| < \varepsilon_\alpha, \ ||\beta_j - id|| < \varepsilon_\beta.$$

We finally define

$$\widetilde{\omega}_j^{t_j} = \begin{cases} (\alpha_j^{t_j})^* \omega_j, \\ d(g_j^{t_j} \circ \beta_j^{t_j}). \end{cases}$$

Using Rouché's theorem as at the end of subsection 2.2, we find $\widetilde{t}_j \in \Delta$ in such a way that

$$\int_{C_0} \widetilde{\omega}_j^{\widetilde{t}_j} = \int_{C_0} \theta_j.$$

By construction the 1-forms $\widetilde{\omega}_1^{\widetilde{t}_1}, \ldots, \widetilde{\omega}_q^{\widetilde{t}_q}$ have no zeros and are pointwise independent. □ □

4 Algebraic example

Due to Forstnerič [4] each Stein manifold X admits a noncritical holomorphic function (for an open Riemann surface see also [10]). When $X \subset \mathbb{C}^N$ is an affine algebraic manifold, a natural question is whether there exists a holomorphic polynomial P on $\mathbb{C}^N$ whose restriction to X is noncritical to X. The following counter example was provided by R. Narasimhan (private communication).

Let $t \in \mathbb{C}$ and let $\Gamma \subset \mathbb{C}$ be a lattice of rank two. Let $x(t)$, $y(t)$ be Γ-periodic meromorphic functions (elliptic functions) satisfying an equation

$$y^2 = x^3 + Ax + B.$$

Denote by C the corresponding curve in $\mathbb{C}^2$.

Proposition 8. *For every polynomial $P \in \mathbb{C}[x, y]$ the restriction*

$$P|_C \colon C \to \mathbb{C}$$

has at least one critical point on C.

Proof. Let $p(t) = P(x(t), y(t))$ for $t \in \mathbb{C} \backslash \Gamma$. Clearly p and its derivative $\dot{p}$ are Γ-periodic functions, and all points of the lattice are singularities of the same kind due to periodicity. Since P is a polynomial and $x(t)$, $y(t)$ are meromorphic, they cannot be essential singularities.

If $\dot{p}$ is regular at points of Γ, it follows that it is bounded on $\mathbb{C}$ and hence constant, thus $\dot{p} = c$. Hence $p(t) = ct + b$ for some $b \in \mathbb{C}$ but this function fails to be periodic. Thus $\dot{p}$ must have a pole at every point of Γ and hence $1/\dot{p}$ has zeros at these points.

If $\dot{p}$ has no zeros on $\mathbb{C}$, it follows that $1/\dot{p}$ is a bounded function on $\mathbb{C}$ which is a contradiction. This means that $\dot{p}(t_0) = 0$ for some $t_0 \in \mathbb{C}\backslash\Gamma$, which means that $P|_C$ has a critical point at $(x(t_0), y(t_0)) \in C$. □ □

Acknowledgements. I am very grateful to F. Forstnerič for helpful discussions. I wish to thank R. Narasimhan for suggesting the example in section 4 and for proposing that Theorem 1 may hold on open Riemann surfaces (private communication, 2004).

Research on this work was supported by a fellowship from the Agency for Research and Development (ARRS) of the Republic of Slovenia (contract no. 1000-05-310002).

References

[1] Cartan, H.: Problèmes globaux dans la théorie des fonctions analytiques de plusieurs variables complexes, Proc. Int. Cong. Math. **1**, 152–164 (1950)

[2] Forster, O.: Some remarks on parallelizable Stein manifold, Bull. Amer. Math. Soc., **73**, 712–716 (1967)

[3] Forster, O.: Plongements des variétés de Stein, Comment. Math. Helv., **45**, 170–184 (1970)

[4] Forstnerič, F.: Noncritical holomorphic functions on Stein manifolds, Acta Math., **191**, 143–189 (2003)

[5] Forstnerič, F: Holomorphic submersions from Stein manifolds, Ann. Inst. Fourier, **54**, 1913–1942 (2004)

[6] Forstnerič F., Prezelj J.: Oka's principle for holomorphic fiber bundles with sprays, Math. Ann., **317**, 117–154 (2000)

[7] Forstnerič, F., Prezelj J.: Oka's principle for holomorphic submersions with sprays, Math. Ann., 322, 633–666 (2002)

[8] Forstnerič, F., Prezelj J.: Extending holomorphic sections from complex subvarieties, Math. Z., 236, 43–68 (2001)

[9] Gromov, M.: Oka's principle for holomorphic sections of elliptic bundles, J. Amer. Math. Soc., **2**, 851–897 (1989)

[10] Gunning, R. C., Narasimhan, R.: Immersion of open Riemann surfaces, Math. Ann., **174**, 103–108 (1967)

[11] Henkin, G. M., Leiterer, J.: Andreotti-Grauert Theory by Integral Formulas. Progress in Math., 74, Birkhäuser, Boston (1988)

[12] Henkin, G. M., Leiterer, J.: The Oka-Grauert principle without induction over the basis dimension. Math. Ann., **311**, 71–93 (1998)

[13] Hörmander, L.: An Introduction to Complex Analysis in Several Variables. North-Holland Publishing Company, Amsterdam (1973)

[14] Kusunoki, Y., Sainouchi, Y.: Holomorphic differentials on open Riemann surfaces. J. Math. Kyoto Univ., **11**, 181–194 (1971)

[15] Latour, F.: Existence de 1-formes fermées non singulières dans une classe de cohomologie de Rham. Inst. Hautes Etudes Sci Publ. Math., **80**, 135–194 (1995).

[16] Serre, J.-P.: Quelques problèmes globaux relatifs aux variétés de Stein. (French) in *Colloque sur les fonctions de plusieurs variables*, *Bruxelles*, 1953, pp. 57–68. Georges Thone, Liège; Masson & Cie, Paris, 1953.

[17] Stein, K: Analytische Funktionen mehrerer komplexer Verändlicher zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem. Math. Ann., **123**, 201–222 (1951)