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Preprint series, Vol. 45 (2007), 1033

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ISSN 1318-4865

March 23, 2007

Ljubljana, March 23, 2007

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Abstract

An *H-system* is a system of first-order linear homogeneous difference equations for a single unknown function T, with coefficients which are polynomials with complex coefficients. We consider solutions of H-systems which are of the form $T: \operatorname{dom}(T) \to \mathbb{C}$ where either $\operatorname{dom}(T) = \mathbb{Z}^d$, or $\operatorname{dom}(T) = \mathbb{Z}^d \setminus S$ and S is the set of integer singularities of the system. It is shown that any natural number is the dimension of the solution space of some consistent H-system, and that in the case $d \geq 2$ there are H-systems whose solution space is infinite-dimensional. The relationship between dimensions of solution spaces in the two cases $\operatorname{dom}(T) = \mathbb{Z}^d$ and $\operatorname{dom}(T) = \mathbb{Z}^d \setminus S$ is investigated. Finally we give an appropriate corollary to the Ore-Sato theorem on possible forms of solutions of H-systems in this setting.

1 Introduction

Linear homogeneous recurrence equations with polynomial coefficients, and systems of such equations, play a significant role in combinatorics and in the theory of hypergeometric functions; the question of the dimension of the space of solutions of such systems is of importance for many problems.

*The work is partially supported by the ECO-NET program of the French Foreign Affairs Ministry.

†Partially supported by RFBR grant 04-01-00757.

‡Partially supported by MŠZŠ RS under grant P1-0294.

Let $n_1, \ldots, n_d$ be variables ranging over the integers. We consider *H*-systems, i.e., systems of equations of the form $f_i(n_1, \ldots, n_d)\, T(n_1, \ldots, n_i + 1, \ldots, n_d) = g_i(n_1, \ldots, n_d)\, T(n_1, \ldots, n_d)$, where $f_i, g_i \in \mathbb{C}[x_1, \ldots, x_d] \setminus \{0\}$ are relatively prime polynomials for $i = 1, \ldots, d$. The notion of singular points (singularities) of such systems can be defined in the usual way. Such singularities make obstacles (sometimes insuperable) for continuation of partial solutions of the system on all of $\mathbb{Z}^d$.

In this paper we consider two spaces of solutions of *H*-systems: the space V_1 of solutions defined everywhere on $\mathbb{Z}^d$, and the space V_2 of solutions that are defined at all nonsingular points of $\mathbb{Z}^d$. The precise definitions are given in Section 2 where it is also shown that the dimension of V_2 equals the number of components induced on $\mathbb{Z}^d$ by the singularities of the system.

In Sections 3, 4 and 5 we investigate the dimensions of the spaces V_1, V_2 and their relationship. It is well known [7] that if (in the case $d = 1$) one considers the germs of sequences at infinity (i.e., classes of functions $T : \mathbb{N}_0 \to \mathbb{C}$ which agree from some point on), the dimension of the solution space is 1. However, the situation is different with $\dim V_1$ and $\dim V_2$. In Section 3 we prove for the case $d = 1$ that if the equation has singularities then $1 \leq \dim V_1 < \dim V_2 < \infty$, and that for any integers s, t such that $1 \leq s < t$ there exists an equation with $\dim V_1 = s$ and $\dim V_2 = t$ (the case where there is no singularity is trivial: $\dim V_1 = \dim V_2 = 1$). In Section 4 we show that in the case $d > 1$ the possibilities are even richer: for any $s, t \in \mathbb{N} \cup \{\infty\}$ there exists an *H*-system with $\dim V_1 = s$ and $\dim V_2 = t$.

The central part of the paper is Section 5 where we prove that $\dim V_1 > 0$ for every consistent *H*-system. Our proof of this deceptively simple fact is based on the well-known Ore-Sato theorem [5, 6, 8]. In Section 6 we show that, contrary to some interpretations in the literature (e.g., [3, 4]), the Ore-Sato theorem does *not* imply that every solution of an *H*-system is of the form

$$R(n_1, \ldots, n_d) \frac{\prod_{i=1}^{p} \Gamma(a_{i,1} n_1 + \ldots + a_{i,d} n_d + \alpha_i)}{\prod_{j=1}^{q} \Gamma(b_{j,1} n_1 + \ldots + b_{j,d} n_d + \beta_j)}\, u_1^{n_1} \cdots u_d^{n_d} \tag{1}$$

where $R \in \mathbb{C}(x_1, \ldots, x_d)$, $a_{ik}, b_{jk} \in \mathbb{Z}$, and $\alpha_i, \beta_j \in \mathbb{C}$ (for the case when the solution of the system is holonomic, and R is required to be a polynomial, we have already noted this in [2]). We conclude by giving an appropriate corollary to the Ore-Sato theorem on possible forms of solutions of systems under consideration.

We write $p \perp q$ to indicate that polynomials $p, q \in \mathbb{C}[x_1, \ldots, x_d]$ are relatively prime. We call a set $A \subseteq \mathbb{Z}^d$ *algebraic* if there is a polynomial $p \in \mathbb{C}[x_1, \ldots, x_d] \setminus \{0\}$ which vanishes on A. We write $\mathbf{u} = (u_1, \ldots, u_d)$ for d-tuples of numbers or indeterminates, and $\mathbf{u}^{\mathrm{T}}\mathbf{v} = \sum_{i=1}^{d} u_i v_i$ for their inner product. We denote by $\mathbf{e}_i$ the d-tuple whose components are zero except the i-th one which is 1. The monomial $x_1^{u_1} \cdots x_d^{u_d}$ is denoted by $\mathbf{x}^{\mathbf{u}}$. A polynomial $p \in \mathbb{C}[\mathbf{x}]$ is *integer-linear* if $p(\mathbf{x}) = \mathbf{a}^{\mathrm{T}}\mathbf{x} + \beta$ where $\mathbf{a} \in \mathbb{Z}^d$ and $\beta \in \mathbb{C}$. The set of positive integers is denoted by $\mathbb{N}$, and the set of non-negative integers by $\mathbb{N}_0$.

2 *H*-systems and their solution spaces

Definition 1 *An H-*system[1] *of dimension d is a system of equations of the form*

$$f_i(\mathbf{n})T(\mathbf{n}+\mathbf{e}_i) = g_i(\mathbf{n})T(\mathbf{n}) \qquad (2)$$

for $i = 1, \ldots, d$, where $f_i, g_i \in \mathbb{C}[\mathbf{x}] \setminus \{0\}$ and $f_i \perp g_i$. The rational functions $g_i/f_i \in \mathbb{C}(\mathbf{x}) \setminus \{0\}$, $i = 1, \ldots, d$, are called the certificates *of (2), and a function $T : \mathrm{dom}(T) \to \mathbb{C}$ is a* solution *of (2) if (2) is satisfied for all $\mathbf{n} \in \mathrm{dom}(T)$ such that $\mathbf{n}+\mathbf{e}_i \in \mathrm{dom}(T)$ for $i = 1, \ldots, d$. A solution of an H-*system *is called a* hypergeometric term.

Definition 2 *Rational functions $F_1, \ldots, F_d \in \mathbb{C}(\mathbf{x}) \setminus \{0\}$ are* compatible *if*

$$F_i(\mathbf{x})F_j(\mathbf{x}+\mathbf{e}_i) \;=\; F_j(\mathbf{x})F_i(\mathbf{x}+\mathbf{e}_j)$$

for all $1 \le i < j \le d$. We call an H-system of the form (2) consistent *if its certificates are compatible.*

If an *H*-system has a solution with non-algebraic support, then it is consistent, and its certificates are uniquely determined by this solution (see [2]). Note that in the case $d = 1$, every *H*-system (containing a single equation) is consistent.

Definition 3 *Let $\mathcal{H}$ be an H-system of the form (2). A point $\mathbf{n} \in \mathbb{Z}^d$ is*

- *a* trailing integer singularity *of $\mathcal{H}$ if there exists i, $1 \le i \le d$, such that $g_i(\mathbf{n}) = 0$;*
- *a* leading integer singularity *of $\mathcal{H}$ if there exists i, $1 \le i \le d$, such that $f_i(\mathbf{n}-\mathbf{e}_i) = 0$;*
- *an* integer singularity *of $\mathcal{H}$ if it is a leading or a trailing integer singularity of $\mathcal{H}$.*

Definition 4 *Let $S(\mathcal{H})$ denote the set of all integer singularities of $\mathcal{H}$. Denote*

- *by $V_1(\mathcal{H})$ the $\mathbb{C}$-linear space of all solutions of $\mathcal{H}$ which are defined at all elements of $\mathbb{Z}^d$, and*
- *by $V_2(\mathcal{H})$ the $\mathbb{C}$-linear space of all solutions of $\mathcal{H}$ which are defined at all elements of $\mathbb{Z}^d \setminus S(\mathcal{H})$.*

[1] The prefix "*H*" refers to Jakob Horn and to the adjective "hypergeometric" as well.

We consider only integer singularities here, therefore we will drop the adjective "integer" in the sequel. Sometimes we will also drop the name of the H-system, and will write V_1, V_2 instead of $V_1(\mathcal{H})$, $V_2(\mathcal{H})$.

Definition 5 *Two points* $\mathbf{p}, \mathbf{p}' \in \mathbb{Z}^d$ *are* adjacent *if* $\mathbf{p} - \mathbf{p}' = \pm\, \mathbf{e}_i$ *for some* $i \in \{1, \ldots, d\}$. *A finite sequence* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k \in \mathbb{Z}^d$ *is a* path from $\mathbf{p}_1$ to $\mathbf{p}_k$ of length $k-1$ *if* $\mathbf{p}_i$ *is adjacent to* $\mathbf{p}_{i+1}$ *for* $i = 1, 2, \ldots, k-1$. *Given an H-system* $\mathcal{H}$*, the* components induced by $\mathcal{H}$ on $\mathbb{Z}^d$ *are the equivalence classes of the following equivalence relation* $\sim$ *in* $\mathbb{Z}^d$*:* $\mathbf{p}' \sim \mathbf{p}''$ *iff there exists a path from* $\mathbf{p}'$ *to* $\mathbf{p}''$ *which contains no singularity of* $\mathcal{H}$. *If* T *is a solution of* $\mathcal{H}$*, then for each component* C *induced by* $\mathcal{H}$ *on* $\mathbb{Z}^d$*, the restriction of* T *to* C *is called a* constituent *of* T.

Proposition 1 *Let* $\mathcal{H}$ *be a consistent H-system. Then* $\dim V_2$ *is equal to the number of components induced by* $\mathcal{H}$ *on* $\mathbb{Z}^d$.

Proof: To each component C_i induced by $\mathcal{H}$ on $\mathbb{Z}^d$ we assign a solution T_i of (2) which is 1 at a selected point $\mathbf{p}_i \in C_i$, and 0 on all the remaining components. The values of T_i on the remaining points of C_i are uniquely determined by (2). It is clear that the set of all T_i is a basis for V_2. □

3 The univariate case

When $d = 1$ the system (2) is of the form

$$f(n)T(n+1) = g(n)T(n) \tag{3}$$

where $f(n), g(n) \in \mathbb{C}[n] \setminus \{0\}$ and $f(n) \perp g(n)$.

Example 1 ($\dim V_1 = 1$, $\dim V_2 = k$) Consider the recurrence

$$T(n+1) = p_k(n)\ T(n) \tag{4}$$

where $k \geq 1$ and $p_k(n) = \prod_{i=0}^{k-2}(n - 2i + 1)$. Here we use the convention that a product is 1 if its lower limit exceeds its upper limit. Clearly the set of singularities of (4) is $\{2i - 1 \mid i = 0, 1, \ldots, k-2\}$, so $\dim V_2 = k$. To compute $\dim V_1$, note that any solution $T(n)$ of (4) defined for all $n \in \mathbb{Z}$ is a constant multiple of

$$F_k(n) = \begin{cases} (-1)^{(k-1)n} / \prod_{i=0}^{k-2}(2i - n - 1)!, & n < 0, \\ 0, & n \geq 0. \end{cases}$$

Therefore $\dim V_1 = 1$. ■

Example 2 ($\dim V_1 = m$, $\dim V_2 = m+1$) Now consider the recurrence

$$q_m(n+1)\ T(n+1) = q_m(n)\ T(n) \tag{5}$$

where $m \geq 1$ and $q_m(n) = \prod_{i=1}^{m}(n+2i+1)$. The set of singularities is $\{-(2i+1) \mid i = 1,2,\ldots,m\}$, so $\dim V_2 = m+1$. Let $T(n)$ be a solution of (5) defined for all $n \in \mathbb{Z}$. By substituting $n = -2(i+1)$ for $i = 1,2,\ldots,m$ into (5), we see that $T(n) = 0$ for these values of n. Likewise, by substituting $n = -3$ into (5), we find that $T(-2) = 0$. Using (5) it follows by induction on n that $T(n) = 0$ for all $n \leq -2(m+1)$ and for all $n \geq -2$ as well. On the other hand, it is easy to check that

$$G_m^{(i)}(n) = \delta_{n,-(2i+1)}$$

(where δ is the Kronecker delta) is a solution of (5) for $i = 1,2,\ldots,m$. Therefore $\dim V_1 = m$. ■

Before describing the general situation we need a definition and a lemma.

Definition 6 *Let $\mathcal{H}$ be an H-system of the form (3). An interval of integers*

$$I = \{k, k+1, \ldots, k+m\}, \quad m \geq 0, \tag{6}$$

is a segment of singularities *of $\mathcal{H}$ if $I \subseteq S(\mathcal{H})$ while $k-1, k+m+1 \notin S(\mathcal{H})$.*

Lemma 1 *Each segment of singularities (6) of equation (3) is of (at least) one of the following types:*

(i) all elements of the segment are trailing singularities;

(ii) all elements of the segment are leading singularities;

(iii) there exists j, $0 \leq j < m$, such that $k, k+1, \ldots, k+j$ are leading singularities, while $k+j+1, \ldots, k+m$ are trailing singularities.

Proof: If $u \in \mathbb{Z}$ is a trailing singularity and $u+1$ a leading singularity of (3) then $f(u) = g(u) = 0$, contrary to the assumption $f \perp g$. So any segment of singularities of (3) consists of a (possibly empty) interval of leading singularities followed by a (possibly empty) interval of trailing singularities. □

Theorem 1 *Let S denote the set of singularities of equation (3).*
a) If $S = \emptyset$ then $\dim V_1 = \dim V_2 = 1$.
b) If $S \neq \emptyset$ then $1 \leq \dim V_1 < \dim V_2 < \infty$.

Proof: a) This is clear.
b) There is only a finite set of components induced on $\mathbb{Z}$ by (3), therefore $\dim V_2 < \infty$.

Next we prove that $\dim V_1 < \dim V_2$. First we show that if (6) is a segment of singularities of (3), then the restriction of V_1 to

$$\hat{I} = \{k-1, k, , \ldots, k+m, k+m+1\}$$

has dimension ≤ 1, while the analogous restriction of V_2 obviously has dimension 2. Indeed, if u is a trailing singularity, then any element of V_1 vanishes at $u+1$; and if u is a leading singularity, then any element of V_1 vanishes at $u-1$. By Lemma 1 we have three possibilities (i), (ii), (iii) for (6). In case (i) we have $T(k+1) = T(k+2) = \ldots = T(k+m+1) = 0$, in case (ii) $T(k-1) = T(k) = \ldots = T(k+m-1) = 0$, in case (iii) $T(k-1) = T(k) = \ldots T(k+j-1) = 0$ and $T(k+j+2) = T(k+j+3) = \ldots = T(k+m+1) = 0$; in each case $T(n)$ can be non-zero at most in two points of $\hat{I}$, however the value at one of them is uniquely determined by the value at the other one. Therefore the dimension of the restricted V_1 is ≤ 1. The same holds for dimension of the restriction of V_1 to the set

$$\{k-v, k-v+1, \ldots, k, k+1, \ldots, k+m, k+m+1, \ldots, k+w\},$$

where $k, k+1, \ldots, k+m$ are singularities, while $k-v, \ldots, k-1$ and $k+m+1, \ldots, k+w$ are not. Gluing together two such restrictions with coinciding, say, $k+m+1, \ldots, k+w$, and non-intersecting singular parts, we get the dimension ≤ 2, while the dimension of the corresponding restriction of V_2 is 3 and so on. This proves that $\dim V_1 < \dim V_2$.

Finally we prove that $\dim V_1 \geq 1$. If there are leading singularities, let n_0 be the largest leading singularity. Set $T(n_0) = 1$ and $T(n) = 0$ for $n < n_0$. None of the points $n > n_0$ is a leading singularity, hence the value of T at $n > n_0$ is uniquely determined by the recurrence (3) and the initial condition $T(n_0) = 1$. If there are no leading singularities, let n_0 be the least trailing singularity. Set $T(n_0) = 1$ and $T(n) = 0$ for $n > n_0$. None of the points $n < n_0$ is a trailing singularity, hence the value of T at $n < n_0$ is uniquely determined by the recurrence (3) and the initial condition $T(n_0) = 1$. In either case V_1 contains a non-zero solution. □

Theorem 2 *For any integers s, t such that $1 \leq s < t$ there exists an equation of the form (3) such that $\dim V_1 = s$ and $\dim V_2 = t$.*

Proof: Consider the recurrence

$$q_m(n+1)\ T(n+1) = p_k(n) q_m(n)\ T(n) \tag{7}$$

where $k, m \geq 1$, $p_k(n)$ is as in Example 1, and $q_m(n)$ is as in Example 2. Here the set of singularities is $\{2i-1 \mid i = 0, 1, \ldots, k-2\} \cup \{-(2i+1);\ i = 1, 2, \ldots, m\}$, so $\dim V_2 = k+m$. Let $T(n)$ be a solution of (7) defined for all $n \in \mathbb{Z}$. In exactly the same way as in Example 2 we can see that $T(n) = 0$ for $n =$

$-2, -4, \ldots, -2(m+1)$, $n \leq -2(m+1)$ or $n \geq -2$, and that $G_m^{(i)}(n) = \delta_{n,-(2i+1)}$ is a solution of (7) for $i = 1, 2, \ldots, m$. Therefore $\dim V_1 = m$.

If $1 \leq s < t$, let $m = s$ and $k = t - s$. Then for equation (7), $\dim V_1 = m = s$ and $\dim V_2 = k + m = t$. □

We conclude this section by some remarks on computation of $\dim V_1$ and $\dim V_2$. Let $\mathcal{H}$ denote equation (3). According to Proposition 1, $\dim V_2(\mathcal{H})$ is the number of components induced on $\mathbb{Z}$ by $\mathcal{H}$ and is thus easy to compute. We claim that $\dim V_1(\mathcal{H})$ equals the dimension of the kernel of a bidiagonal matrix B defined as follows. Let α be the maximum and β the minimum of the integer roots of $f(x)g(x)$; if $\mathcal{H}$ has no integer singularities then we can take $\alpha = \beta = 1$. Let B be the $(\alpha - \beta + 1) \times (\alpha - \beta + 2)$ matrix with entries

$$b_{i,j} = \begin{cases} f(\alpha - i + 1), & j = i, \\ -g(\alpha - i + 1), & j = i + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $1 \leq i \leq \alpha - \beta + 1$ and $1 \leq j \leq \alpha - \beta + 2$. Indeed, any vector v such that $Bv = 0$ can be extended to a solution of $\mathcal{H}$ in a unique way. This mapping is an isomorphism between the kernel of B and $V_1(\mathcal{H})$.

Incidentally, this gives an alternative proof of the inequality $\dim V_1 \geq 1$: B has more columns than rows, hence its kernel is nontrivial.

4 The relation between dimensions of V_1 and V_2 in the multivariate case

If $d \geq 2$ in (2) then the dimensions of V_1 and/or V_2 can be infinite as shown by the following examples.

Example 3 ($\dim V_1 = \infty$, $\dim V_2 = 1$) Let $\mathcal{H}$ be the system

$$\begin{aligned} (n_1 - 4n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - 4n_2)T(n_1, n_2), \\ (n_1 - 4n_2 - 4)T(n_1, n_2 + 1) &= (n_1 - 4n_2)T(n_1, n_2). \end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1,4i}\delta_{n_2,i}, \qquad \text{for } i \in \mathbb{Z},$$

are linearly independent solutions of $\mathcal{H}$ on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. On the other hand, $S(\mathcal{H}) = \{(n_1, n_2) \mid n_1 = 4n_2\}$, so $\mathcal{H}$ induces a single component on $\mathbb{Z}^2$, and $\dim V_2 = 1$. ■

Example 4 ($\dim V_1 = 1$, $\dim V_2 = \infty$) Let B be the system

$$\begin{aligned} (n_1 - 4n_2)T(n_1 + 1, n_2) &= (n_1 - 4n_2 + 1)T(n_1, n_2), \\ (n_1 - 4n_2)T(n_1, n_2 + 1) &= (n_1 - 4n_2 - 4)T(n_1, n_2). \end{aligned}$$

It can be shown that any solution of B defined on all $\mathbb{Z}^2$ is a constant multiple of $n_1 - 4n_2$, so $\dim V_1 = 1$. On the other hand, $S(B) = \{(n_1, n_2) \mid n_1 - 4n_2 \in \{-4, -1, 1, 4\}\}$, so each of the points $(4i, i)$ for $i \in \mathbb{Z}$ is a separate component of $\mathbb{Z}^2$ induced by B, hence $\dim V_2 = \infty$. ■

Example 5 ($\dim V_1 = \dim V_2 = \infty$) Let C be the system

$$\begin{aligned}(n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - n_2)(n_1 - n_2 + 2)T(n_1, n_2),\\ (n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1, n_2 + 1) &= (n_1 - n_2)(n_1 - n_2 - 2)T(n_1, n_2).\end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1,i}\delta_{n_2,i}, \qquad \text{for } i \in \mathbb{Z}, \tag{8}$$

are linearly independent solutions of C on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. As $S(C) = \{(n_1, n_2) \mid n_1 - n_2 \in \{-2, 0, 2\}\}$, each of the points $(i, i-1)$ and $(i, i+1)$ for $i \in \mathbb{Z}$ is a separate component of $\mathbb{Z}^2$ induced by C, so $\dim V_2 = \infty$ as well. ■

The following theorem describes the general situation.

Theorem 3 *Let* $1 \leq s, t \leq \infty$. *Then there exists an H-system such that* $\dim V_1 = s$ *and* $\dim V_2 = t$.

Proof: Let $t \geq 2$ and $p_t(n_1, n_2) = \prod_{i=0}^{t-2}(n_1 - n_2 + 3i)$. Then the set of singularities of

$$\begin{aligned}p_t(n_1 + 1, n_2)T(n_1 + 1, n_2) &= p_t(n_1, n_2)T(n_1, n_2),\\ p_t(n_1, n_2 + 1)T(n_1, n_2 + 1) &= p_t(n_1, n_2)T(n_1, n_2)\end{aligned}$$

is $S = \{(n_1, n_2) \mid n_1 - n_2 \in \{-3i \mid 0 \leq i \leq t - 2\}\}$. As in Example 5, the functions (8) are linearly independent solutions of this system on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. On the other hand, the number of components induced on $\mathbb{Z}^2$ is t, so $\dim V_2 = t$.

Let $s \geq 2$ and

$$q_s(n_1, n_2) = \prod_{i=1}^{s-1}((n_1 - 2i)^2 + n_2^2). \tag{9}$$

Then the set of singularities of

$$\begin{aligned}(n_1 - 4n_2)q_{s+1}(n_1 + 1, n_2)T(n_1 + 1, n_2) &= (n_1 - 4n_2 + 1)q_{s+1}(n_1, n_2)T(n_1, n_2),\\ (n_1 - 4n_2)q_{s+1}(n_1, n_2 + 1)T(n_1, n_2 + 1) &= (n_1 - 4n_2 - 4)q_{s+1}(n_1, n_2)T(n_1, n_2)\end{aligned}$$

is $S = \{(n_1, n_2) \mid n_1 - 4n_2 \in \{-4, -1, 1, 4\}\} \cup \{(2i, 0) \mid 1 \leq i \leq s\}$. Each of the points $(4i, i)$ for $i \in \mathbb{Z}$ is a separate component, so $\dim V_2 = \infty$. It can be shown that any solution $T(n_1, n_2)$ defined on all $\mathbb{Z}^2$ vanishes everywhere except at the points $(2i, 0)$ where $1 \le i \le s$, and that

$$T_i(n_1, n_2) = \delta_{n_1,2i}\delta_{n_2,0}, \tag{10}$$

for $i = 1, 2, \ldots, s$, are linearly independent solutions of this system defined on all $\mathbb{Z}^2$. Hence $\dim V_1 = \infty$.

Together with Examples 3 – 5 this proves the assertion in the case when at least one of s, t is infinite.

Now assume that s, t are natural numbers, and let $r_t(n_1, n_2) = \prod_{i=1}^{t-1}(n_1 + 2i + 1)$. Consider the system

$$\begin{aligned} q_s(n_1 + 1, n_2)T(n_1 + 1, n_2) &= q_s(n_1, n_2)r_t(n_1, n_2)T(n_1, n_2), \\ q_s(n_1, n_2 + 1)T(n_1, n_2 + 1) &= q_s(n_1, n_2)T(n_1, n_2), \end{aligned}$$

where q_s is as in (9). It can be shown that any solution $T(n_1, n_2)$ defined on all $\mathbb{Z}^2$ vanishes for all (n_1, n_2) such that $n_1 > -(2t - 1)$ and (n_1, n_2) is not of the form $(2i, 0)$ with $1 \le i \le s-1$. Further, a basis of V_1 is given by the s functions $T_i(n_1, n_2)$ for $i = 0, 1, \ldots, s - 1$ where

$$T_0(n_1, n_2) = \begin{cases} \frac{(-1)^{(t-1)n_1}}{\prod_{i=1}^{s-1}((n_1-2i)^2+n_2^2)\prod_{i=1}^{t-1}(-n_1-2i-1)!}, & n_1 \le -(2t-1), \\ 0, & \text{otherwise}, \end{cases}$$

and $T_i(n_1, n_2)$ are as in (10) for $i = 1, 2, \ldots, s - 1$. It follows that $\dim V_1 = s$. The set of singularities of this system is $S = \{(2i, 0) \mid 1 \le i \le s-1\} \cup \{(-(2i + 1), j) \mid 1 \le i \le t-1, j \in \mathbb{Z}\}$, and the number of components induced on $\mathbb{Z}^2$ is t, so $\dim V_2 = t$ as desired. □

We considered the case $d = 2$ here. The corresponding H-systems for the case of an arbitrary $d > 1$ can be obtained by adding equations $T(\mathbf{n} + \mathbf{e}_i) = T(\mathbf{n})$, $i = 3, \ldots, d$, to the systems with $d = 2$.

5 Existence of solutions in the multivariate case

In this section we assume that $\mathcal{H}$ is a consistent H-system of the form (2), and show that $\dim V_1(\mathcal{H}) > 0$. Our proof of this fact is based on the Ore-Sato theorem (Theorem 4).

Let K be a field. For $k \in \mathbb{Z}$ and $\alpha \in K$, denote by $(\alpha)_k$ the *Pochhammer symbol*

$$(\alpha)_k = \begin{cases} \displaystyle\prod_{j=0}^{k-1}(\alpha + j), & k \ge 0, \\ \displaystyle\prod_{j=1}^{|k|}\frac{1}{\alpha - j}, & k < 0, \quad \alpha \ne 1, 2, \ldots, |k|. \end{cases}$$

Theorem 4 (Ore-Sato) *Let* $\{G_{\mathbf{n}}(\mathbf{x}) \in \mathbb{C}(\mathbf{x}) \mid \mathbf{n} \in \mathbb{Z}^d\}$ *be a family of rational functions satisfying the* cocycle condition

$$\forall\, \mathbf{n}, \mathbf{m} \in \mathbb{Z}^d : \quad G_{\mathbf{n}+\mathbf{m}}(\mathbf{x}) = G_{\mathbf{n}}(\mathbf{x}) \cdot G_{\mathbf{m}}(\mathbf{x}+\mathbf{n}). \tag{11}$$

Then we can write

$$G_{\mathbf{n}}(\mathbf{x}) = C(\mathbf{n}) \cdot \prod_{j=1}^{p} \left((\mathbf{a}^{(j)\mathrm{T}}\mathbf{x} + \beta_j)_{\mathbf{a}^{(j)\mathrm{T}}\mathbf{n}} \right)^{s_j} \cdot \frac{R(\mathbf{x}+\mathbf{n})}{R(\mathbf{x})} \tag{12}$$

where $C : \mathbb{Z}^d \to \mathbb{C}$ *satisfies* $C(\mathbf{n}+\mathbf{m}) = C(\mathbf{n})C(\mathbf{m})$, $p \in \mathbb{N}_0$, $\mathbf{a}^{(j)} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\beta_j \in \mathbb{C}$, $s_j \in \mathbb{Z} \setminus \{0\}$, *and* $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$.

For a proof, see [8, pp. 26–33][2].

Corollary 1 *Let* $F_1(\mathbf{x}), \ldots, F_d(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$ *be compatible rational functions (see Def. 2). Then for* $i = 1, 2, \ldots, d$ *we can write*

$$F_i(\mathbf{x}) = c_i \cdot \prod_{j=1}^{p} \left(\left(\mathbf{a}^{(j)\mathrm{T}}\mathbf{x} + \beta_j \right)_{a_i^{(j)}} \right)^{s_j} \cdot \frac{R(\mathbf{x}+\mathbf{e}_i)}{R(\mathbf{x})} \tag{13}$$

where $c_i \in \mathbb{C}$, $p \in \mathbb{N}_0$, $\mathbf{a}^{(j)} = (a_1^{(j)}, a_2^{(j)}, \ldots, a_d^{(j)}) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\beta_j \in \mathbb{C}$, $s_j \in \mathbb{Z} \setminus \{0\}$, $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x}) \setminus \{0\}$, *the complete factorization of* $R(\mathbf{x})$ *contains no integer-linear factors,* $\gcd(a_1^{(j)}, a_2^{(j)}, \ldots, a_d^{(j)}) = 1$, *and the first non-zero component of* $\mathbf{a}^{(j)}$ *is positive, for* $j = 1, 2, \ldots, p$.

Proof: Write $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$. To each sequence of unit vectors $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ from $B \cup (-B)$ assign a rational function

$$\tilde{G}_{\mathbf{d}_1,\ldots,\mathbf{d}_r}(\mathbf{x}) = \prod_{j=1}^{r} \tilde{F}_j(\mathbf{x} + \mathbf{n}_{j-1})$$

where

$$\tilde{F}_j(\mathbf{x}) = \begin{cases} F_i(\mathbf{x}), & \mathbf{d}_j = \mathbf{e}_i, \\ F_i(\mathbf{x}-\mathbf{e}_i)^{-1}, & \mathbf{d}_j = -\mathbf{e}_i, \end{cases}$$

and $\mathbf{n}_j = \sum_{i=1}^{j} \mathbf{d}_i$, for $0 \le j \le r$. As $F_1, \ldots, F_d$ are compatible, $\tilde{G}_{\mathbf{d}_1,\ldots,\mathbf{d}_r}(\mathbf{x})$ does not change if two consecutive terms in the sequence $\mathbf{d}_1, \ldots, \mathbf{d}_r$ are transposed. Hence $\tilde{G}_{\mathbf{d}_1,\ldots,\mathbf{d}_r}(\mathbf{x}) = \tilde{G}_{\mathbf{d}_{\pi(1)},\ldots,\mathbf{d}_{\pi(r)}}(\mathbf{x})$ for any permutation π of

[2] In fact, a more general version of the Ore-Sato theorem is proved in [8], with $\mathbb{Z}^d$ replaced by an arbitrary abelian group Ξ generated by d elements, and with $\mathbb{C}$ replaced by an arbitrary algebraically closed field Ω of characteristic zero. Note however that in the statement and proof of this theorem in [8], $\prod_{l \le k \le -1} \psi_i(x+k)^{-1}$ should be replaced by $\prod_{l \le k \le -1} \psi_i(x-k)^{-1}$.

$\{1, 2, \ldots, r\}$. In particular, we can sort the sequence $\mathbf{d}_1, \ldots, \mathbf{d}_r$ into a sequence of the form $\mathbf{e}_1, \ldots, \mathbf{e}_1, -\mathbf{e}_1, \ldots, -\mathbf{e}_1, \ldots, \mathbf{e}_d, \ldots, \mathbf{e}_d, -\mathbf{e}_d, \ldots, -\mathbf{e}_d$. Since $\tilde{G}_{\mathbf{d}_j,\mathbf{d}_{j+1}}(\mathbf{x}) = 1$ if $\mathbf{d}_j = -\mathbf{d}_{j+1}$, by definition of $\tilde{F}_j$, and

$$\tilde{G}_{\mathbf{d}_1,\ldots,\mathbf{d}_r,\mathbf{d}_{r+1},\ldots,\mathbf{d}_s}(\mathbf{x}) = \tilde{G}_{\mathbf{d}_1,\ldots,\mathbf{d}_r}(\mathbf{x}) \cdot \tilde{G}_{\mathbf{d}_{r+1},\ldots,\mathbf{d}_s}(\mathbf{x} + \mathbf{n}_r), \tag{14}$$

by definition of $\tilde{G}$, this sequence can be reduced by omitting each consecutive pair of $\mathbf{e}_i$ and $-\mathbf{e}_i$. It follows that $\tilde{G}_{\mathbf{d}_1,\ldots,\mathbf{d}_r}$ depends only on $\mathbf{d}_1 + \cdots + \mathbf{d}_r = \mathbf{n}_r$.

Thus we can define a family of rational functions $\{G_{\mathbf{n}}(\mathbf{x}) \in \mathbb{C}(\mathbf{x}) \mid \mathbf{n} \in \mathbb{Z}^d\}$ by setting

$$G_{\mathbf{n}}(\mathbf{x}) = \tilde{G}_{\mathbf{d}_1,\ldots,\mathbf{d}_r}(\mathbf{x})$$

where $\mathbf{d}_1, \ldots, \mathbf{d}_r$ is any sequence of vectors from $B \cup (-B)$ summing to $\mathbf{n}$. Because of (14), the family $\{G_{\mathbf{n}}(\mathbf{x}) \mid \mathbf{n} \in \mathbb{Z}^d\}$ satisfies the cocycle condition (11), hence by Theorem 4, $G_{\mathbf{n}}(\mathbf{x})$ has the form (12). Notice that $G_{\mathbf{e}_i}(\mathbf{x}) = F_i(\mathbf{x})$ and $\mathbf{a}^{(j)\mathrm{T}}\mathbf{e}_i = a_i^{(j)}$, so with $\mathbf{n} = \mathbf{e}_i$ and $C(\mathbf{e}_i) = c_i$, (12) turns into (13).

If $R(\mathbf{x}) = \tilde{R}(\mathbf{x}) \left(\mathbf{a}^{\mathrm{T}}\mathbf{x} + \beta\right)^s$ where $\tilde{R}(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$, $\mathbf{a} \in \mathbb{Z}^d$, $\beta \in \mathbb{C}$ and $s \in \mathbb{Z}$, then $R(\mathbf{x} + \mathbf{e}_i)/R(\mathbf{x}) = \tilde{R}(\mathbf{x} + \mathbf{e}_i)/\tilde{R}(\mathbf{x}) \cdot \left(\mathbf{a}^{\mathrm{T}}\mathbf{x} + \beta + 1\right)_{a_i}^s / \left(\mathbf{a}^{\mathrm{T}}\mathbf{x} + \beta\right)_{a_i}^s$. Thus we can extract all integer-linear factors from R and replace them by appropriate Pochhammer symbols in the product in (13).

The last two claims follow from the formulæ

$$\left(\mathbf{a}^{\mathrm{T}}\mathbf{x} + \beta\right)_{a_i} = \delta^{a_i} \prod_{k=0}^{\delta-1} \left(\left(\frac{\mathbf{a}}{\delta}\right)^{\mathrm{T}} \mathbf{x} + \frac{\beta + k}{\delta} \right)_{a_i/\delta}$$

where $\delta = \gcd(a_1, a_2, \ldots, a_d)$, and

$$\left(\mathbf{a}^{\mathrm{T}}\mathbf{x} + \beta\right)_{a_i} = (-1)^{a_i} \left(-\mathbf{a}^{\mathrm{T}}\mathbf{x} - \beta + 1\right)_{-a_i}^{-1},$$

both easily verifiable by direct computation. □

To each rational function $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x})$ we associate a sequence of rational functions $\hat{R} : \mathbb{Z}^d \to \mathbb{C}(\mathbf{x})$ by setting $\hat{R}(\mathbf{n}) = R(\mathbf{n} + \mathbf{x})$. Obviously we have

Proposition 2 *If $R(\mathbf{x})$ is not identically zero, then for all $\mathbf{n} \in \mathbb{Z}^d$, $\hat{R}(\mathbf{n})$ is not identically zero.*

Define a *valuation* $\mathrm{val} : \mathbb{C}(\mathbf{x}) \to \mathbb{Z}$ in the following way: For $p(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ let

$$\mathrm{val}\, p = \min\{e_1 + \cdots + e_d \mid x_1^{e_1} \ldots x_d^{e_d} \text{ is a monomial of } p\}.$$

If $R(\mathbf{x}) \in \mathbb{C}(\mathbf{x}) \setminus \mathbb{C}[\mathbf{x}]$ and $R = p/q$ where $p, q \in \mathbb{C}[\mathbf{x}]$ and $p \perp q$, let

$$\mathrm{val}\, R = \mathrm{val}\, p - \mathrm{val}\, q.$$

Proposition 3 *(i) If $p \in \mathbb{C}[\mathbf{x}]$ then $\deg_{x_1} p + \cdots + \deg_{x_d} p \geq \mathrm{val}\, p \geq 0$, and*

$$\mathrm{val}\, p > 0 \iff p(\mathbf{0}) = 0.$$

(ii) If $R_1, R_2 \in \mathbb{C}(\mathbf{x})$ then $\operatorname{val} R_1 R_2 = \operatorname{val} R_1 + \operatorname{val} R_2$.

Proof: Assertion (i) is obvious, and so is (ii) when $R_1, R_2 \in \mathbb{C}[\mathbf{x}]$. To prove (ii) in general, write $R_i = p_i/q_i$ where $p_i, q_i \in \mathbb{C}[\mathbf{x}]$ and $p_i \perp q_i$, for $i = 1, 2$. Denote $r = \gcd(p_1, q_2)$, $s = \gcd(p_2, q_1)$, $p_1' = p_1/r$, $q_2' = q_2/r$, $p_2' = p_2/s$, $q_1' = q_1/s$. Then $R_1 R_2 = p_1 p_2/(q_1 q_2) = p_1' p_2'/(q_1' q_2')$ where $p_1' p_2' \perp q_1' q_2'$. Hence $\operatorname{val} R_1 R_2 = \operatorname{val} p_1' p_2' - \operatorname{val} q_1' q_2' = \operatorname{val} p_1' + \operatorname{val} p_2' - \operatorname{val} q_1' - \operatorname{val} q_2' = \operatorname{val} p_1' + \operatorname{val} r + \operatorname{val} p_2' + \operatorname{val} s - \operatorname{val} q_1' - \operatorname{val} s - \operatorname{val} q_2' - \operatorname{val} r = \operatorname{val} p_1' r + \operatorname{val} p_2' s - \operatorname{val} q_1' s - \operatorname{val} q_2' r = \operatorname{val} p_1 + \operatorname{val} p_2 - \operatorname{val} q_1 - \operatorname{val} q_2 = \operatorname{val} R_1 + \operatorname{val} R_2$, as claimed. □

Proposition 4 *The sequence* $\operatorname{val} \hat{R}(\mathbf{n})$ *is bounded everywhere on* $\mathbb{Z}^d$ *for any* $R \in \mathbb{C}(\mathbf{x})$.

Proof: Let $R = p/q$ where $p, q \in \mathbb{C}[\mathbf{x}]$ and $p \perp q$. By Proposition 3(i) we have

$$\deg_{x_1} p + \cdots + \deg_{x_d} p \;\geq\; \operatorname{val} \hat{R}(\mathbf{n}) \;\geq\; -(\deg_{x_1} q + \cdots + \deg_{x_d} q)$$

for any $\mathbf{n} \in \mathbb{Z}^d$. □

The following proposition is evident.

Proposition 5 *If rational functions $F_1, \ldots, F_d \in \mathbb{C}(\mathbf{x})$ are compatible (see Def. 2) then the sequences $\hat{F}_1, \ldots, \hat{F}_d$ are also compatible in the sense that for any $\mathbf{n} \in \mathbb{Z}^d$, the rational functions $\hat{F}_i(\mathbf{n})\hat{F}_j(\mathbf{n} + \mathbf{e}_i)$ and $\hat{F}_j(\mathbf{n})\hat{F}_i(\mathbf{n} + \mathbf{e}_j)$ are equal in $\mathbb{C}(\mathbf{x})$, for $1 \leq i \leq j \leq d$.*

Let $\mathcal{H}$ be a consistent H-system of the form (2). By Corollary 1 we can write its certificates $F_i = g_i/f_i$ in the form (13). For $i = 1, 2, \ldots, d$, define

$$F_i'(\mathbf{x}) = c_i \cdot \prod_{j=1}^{p} \left(\left(\mathbf{a}^{(j)\mathrm{T}} \mathbf{x} + \beta_j \right)_{a_i^{(j)}} \right)^{s_j}, \tag{15}$$

$$F_i''(\mathbf{x}) = \frac{R(\mathbf{x} + \mathbf{e}_i)}{R(\mathbf{x})}. \tag{16}$$

Evidently rational functions $F_1', \ldots, F_d'$ as well as $F_1'', \ldots, F_d''$ are compatible.

We will associate with $\mathcal{H}$ three sequences $\xi, \eta, \varphi : \mathbb{Z}^d \to \mathbb{C}(\mathbf{x})$ with rational-function values, defined by the following requirements:

- $\xi(\mathbf{0}) = 1$, $\xi(\mathbf{n} + \mathbf{e}_i) = \xi(\mathbf{n})\hat{F}_i'(\mathbf{n})$, $i = 1, \ldots, d$,
- $\eta(\mathbf{n}) = \hat{R}(\mathbf{n})$,
- $\varphi(\mathbf{n}) = \xi(\mathbf{n})\eta(\mathbf{n})$.

Notice that existence and uniqueness of ξ follow from Proposition 2 and from compatibility of $\hat{F}'_1, \ldots, \hat{F}'_d$. For $i = 1, \ldots, d$ set

$$F'_i = \frac{g'_i}{f'_i},\ F''_i = \frac{g''_i}{f''_i},$$

where $g'_i, f'_i, g''_i, f''_i \in \mathbb{C}[\mathbf{x}]$, $g'_i \perp f'_i$, $g''_i \perp f''_i$. Then ξ, η satisfy the systems

$$\hat{f}'_i(\mathbf{n})\xi(\mathbf{n}+\mathbf{e}_i) = \hat{g}'_i(\mathbf{n})\xi(\mathbf{n}),\ \ i = 1, 2, \ldots, d, \tag{17}$$

$$\hat{f}''_i(\mathbf{n})\eta(\mathbf{n}+\mathbf{e}_i) = \hat{g}''_i(\mathbf{n})\eta(\mathbf{n}),\ \ i = 1, 2, \ldots, d. \tag{18}$$

As a consequence of equalities (17), (18) and of the fact that $R(\mathbf{x})$ contains no integer linear factors we get

$$\hat{f}_i(\mathbf{n})\varphi(\mathbf{n}+\mathbf{e}_i) = \hat{g}_i(\mathbf{n})\varphi(\mathbf{n}),\ \ i = 1, 2, \ldots, d. \tag{19}$$

Our nearest goal is to show that the sequence $\operatorname{val}\varphi(\mathbf{n})$ is bounded.

With any factor $\left(\mathbf{a}^{(j)\mathrm{T}}\mathbf{x} + \beta_j\right)_{a_i^{(j)}}$ in (15), we associate $|a_i^{(j)}|$ hyperplanes in $\mathbb{C}^d$: those hyperplanes are defined by the equations

$$\mathbf{a}^{(j)\mathrm{T}}\mathbf{x} + \beta_j + l = 0,\ l = 0, 1, \ldots, a_i^{(j)} - 1$$

if $a_i^{(j)} > 0$, and by

$$\mathbf{a}^{(j)\mathrm{T}}\mathbf{x} + \beta_j + l = 0,\ l = -1, -2, \ldots, a_i^{(j)}$$

if $a_i^{(j)} < 0$. All the factors from (15) generate a finite set of hyperplanes which we will denote by P. The number of elements of P we will denote by N. We call a point $\mathbf{n} \in \mathbb{Z}^d$ *special* if it belongs to at least one hyperplane from P.

The following proposition is a consequence of the definition of the sequence $\xi(\mathbf{n})$:

Proposition 6 *If two points $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^d$ are adjacent and $\operatorname{val}\xi(\mathbf{n}) \neq \operatorname{val}\xi(\mathbf{n}')$, then at least one of these points is special. In this case $|\operatorname{val}\xi(\mathbf{n}) - \operatorname{val}\xi(\mathbf{n}')| \leq |s_1| + \cdots + |s_d|$.*

In order to show that $\operatorname{val}\varphi(\mathbf{n})$ is bounded, we prove three lemmas.

Lemma 2 *Let $\mathbf{n}', \mathbf{n}'' \in \mathbb{Z}^d$ both be non-special. Then there exists a path between them which contains no more than $2Nd$ special points.*

Proof: By induction on d. If $d = 1$, the claim is evident. Assume that $d > 1$ and $\mathbf{n}' = (n'_1, \ldots, n'_d)$, $\mathbf{n}'' = (n''_1, \ldots, n''_d)$. Consider the sets

$$L' = \{(n'_1, \ldots, n'_{d-1}, t) \mid t \in \mathbb{Z}\},\ \ L'' = \{(n''_1, \ldots, n''_{d-1}, t) \mid t \in \mathbb{Z}\}.$$

Since $\mathbf{n}', \mathbf{n}''$ are not special, each of L', L'' contains a finite number of special points, and there exists $t_0 \in \mathbb{Z}$ such that both $\mathbf{n}_0' = (n_1', \ldots, n_{d-1}', t_0)$ and $\mathbf{n}_0'' = (n_1'', \ldots, n_{d-1}'', t_0)$ are not special. The straight path from $\mathbf{n}'$ to $\mathbf{n}_0'$ contains no more than N special points, as well as the straight path from $\mathbf{n}_0''$ to $\mathbf{n}''$. By induction hypothesis, there is a path in the set $\{(n_1, \ldots, n_{d-1}, t_0) \mid (n_1, \ldots, n_{d-1}) \in \mathbb{Z}^{d-1}\}$ from $\mathbf{n}_0'$ to $\mathbf{n}_0''$ that contains no more than $2N(d-1)$ special points. So there is a path from $\mathbf{n}'$ to $\mathbf{n}''$ that contains no more than $2N + 2N(d-1) = 2Nd$ special points. $\square$

Lemma 3 *Let* $\mathbf{a} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $\beta \in \mathbb{C}$, $\mathbf{q} \in \mathbb{Z}^d$, *and* $r \in \mathbb{N}_0$. *Denote*

$$A = \{\mathbf{n} \in \mathbb{Z}^d \mid \mathbf{a}^{\mathrm{T}}\mathbf{n} = \beta, |n_i - q_i| \le r \text{ for } i = 1, \ldots, d\}.$$

Then $|A| \le (2r+1)^{d-1}$.

Proof: Since $\mathbf{a} \neq \mathbf{0}$, there exists $k \in \{1, \ldots, d\}$ such that $a_k \neq 0$. Denote $B = \{\mathbf{n} \in \mathbb{Z}^d \mid n_k = q_k, |n_i - q_i| \le r \text{ for } i = 1, \ldots, d\}$. The orthogonal projection $p: A \to B$, $\mathbf{n} \mapsto \mathbf{n} - (n_k - q_k)\mathbf{e}_k$ is injective, hence $|A| \le |B| = (2r+1)^{d-1}$. $\square$

Lemma 4 *Let* $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ *be a special point. Then there exists a non-special point* $\mathbf{n}^*$, *and a path from* $\mathbf{n}$ *to* $\mathbf{n}^*$ *which contains at most* $\frac{(N+1)d}{2}$ *special points.*

Proof: The set P of hyperplanes is finite, so not all points in $\mathbb{Z}^d$ are special. Let $r+1$ be the length of a shortest path from $\mathbf{n}$ to a non-special point $\mathbf{n}^*$. This path contains $r+2$ points, out of which at most $r+1$ are special. Notice that by the definition of r, the set

$$C_r = \{(\bar{n}_1, \ldots, \bar{n}_d) \in \mathbb{Z}^d \mid |\bar{n}_i - n_i| \le r/d, \, i = 1, \ldots, d\}$$

contains only special points. By Lemma 3, a hyperplane from P contains at most $(2\lfloor r/d \rfloor + 1)^{d-1}$ points from C_r, hence $|C_r| \le N(2\lfloor r/d \rfloor + 1)^{d-1}$. But $|C_r| = (2\lfloor r/d \rfloor + 1)^d$, so $N \ge 2\lfloor r/d \rfloor + 1$, and consequently $r + 1 \le (N+1)d/2$, which proves the assertion. $\square$

As a consequence we have

Proposition 7 *The sequence* $\operatorname{val} \xi(\mathbf{n})$ *is bounded on* $\mathbb{Z}^d$.

Proof: By Lemmas 2 and 4 there exists a path from $\mathbf{0}$ to $\mathbf{n}$ that contains no more than $(N+1)d + 2Nd = (3N+1)d$ special points. By Proposition 6 we have then $|\operatorname{val} \xi(\mathbf{n}) - \operatorname{val} \xi(\mathbf{0})| \le d(3N+1)(|s_1| + \cdots + |s_d|)$. $\square$

Finally we get the following result.

Proposition 8 *The sequence* $\operatorname{val} \varphi(\mathbf{n})$ *is bounded on* $\mathbb{Z}^d$.

Proof: By Proposition 3(ii), $\operatorname{val} \varphi(\mathbf{n}) = \operatorname{val} \xi(\mathbf{n}) + \operatorname{val} \eta(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{Z}^d$. So the sequence $\operatorname{val} \varphi(\mathbf{n})$ is bounded by Propositions 7 and 4. $\square$

Definition 7 *Let $\mathcal{H}$ be an H-system of the form (2).*

We say that a point $\mathbf{n}' \in \mathbb{Z}^d$ is accessible *from a point $\mathbf{n} \in \mathbb{Z}^d$* w.r.t. $\mathcal{H}$ *if there exists a path $\mathbf{n}_1, \ldots, \mathbf{n}_k$ such that $\mathbf{n}_1 = \mathbf{n}$, $\mathbf{n}_k = \mathbf{n}'$ and for each $j \in \{1, \ldots, k-1\}$ there is $i \in \{1, \ldots, d\}$ such that either $\mathbf{n}_{j+1} = \mathbf{n}_j + \mathbf{e}_j$ and $f_i(\mathbf{n}_j) \neq 0$, or $\mathbf{n}_{j+1} = \mathbf{n}_j - \mathbf{e}_j$ and $g_i(\mathbf{n}_{j+1}) \neq 0$.*

Otherwise $\mathbf{n}'$ is inaccessible *from $\mathbf{n}$* w.r.t. $\mathcal{H}$. *If $M \subseteq \mathbb{Z}^d$, then M is* inaccessible w.r.t. $\mathcal{H}$ *if for any $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^d$ such that $\mathbf{n} \notin M$ and $\mathbf{n}' \in M$, $\mathbf{n}'$ is inaccessible from $\mathbf{n}$ w.r.t. $\mathcal{H}$.*

We will omit the qualification "w.r.t. $\mathcal{H}$" when the system $\mathcal{H}$ is clear from the context.

Since the sequence $\operatorname{val}\varphi(\mathbf{n})$ is bounded on $\mathbb{Z}^d$, we can define

$$m = \min_{\mathbf{n} \in \mathbb{Z}^d} \operatorname{val}\varphi(\mathbf{n})$$

and associate with $\mathcal{H}$ the non-empty set

$$M_{\mathcal{H}} = \{\mathbf{n} \in \mathbb{Z}^d \mid \operatorname{val}\varphi(\mathbf{n}) = m\}.$$

Lemma 5 *$M_{\mathcal{H}}$ is inaccessible.*

Proof: It is sufficient to prove that if $\mathbf{a}$ is adjacent to $\mathbf{b}$, $\mathbf{a} \notin M_{\mathcal{H}}$ and $\mathbf{b} \in M_{\mathcal{H}}$, then $\mathbf{b}$ is inaccessible from $\mathbf{a}$. W.l.g. assume that $\mathbf{b} = \mathbf{a} + \mathbf{e}_1$. By (19) we have

$$\hat{f}_1(\mathbf{a})\varphi(\mathbf{b}) = \hat{g}_1(\mathbf{a})\varphi(\mathbf{a}). \tag{20}$$

By Proposition 3(ii), $\operatorname{val}\hat{f}_1(\mathbf{a}) + \operatorname{val}\varphi(\mathbf{b}) = \operatorname{val}\hat{g}_1(\mathbf{a}) + \operatorname{val}\varphi(\mathbf{a})$. As $\mathbf{a} \notin M_{\mathcal{H}}$ and $\mathbf{b} \in M_{\mathcal{H}}$, we have $\operatorname{val}\varphi(\mathbf{a}) > \operatorname{val}\varphi(\mathbf{b})$, therefore

$$\operatorname{val}\hat{f}_1(\mathbf{a}) \;>\; \operatorname{val}\hat{g}_1(\mathbf{a}).$$

By Proposition 3(i), $\operatorname{val}\hat{g}_1(\mathbf{a}) \geq 0$, implying that $\operatorname{val} f_1(\mathbf{a} + \mathbf{x}) = \operatorname{val}\hat{f}_1(\mathbf{a}) > 0$. So by Proposition 3(i), $f_1(\mathbf{a}) = 0$. This proves that $\mathbf{b}$ is inaccessible from $\mathbf{a}$. □

Lemma 6 *Let $\mathcal{H}$ be an H-system of the form (2). If $\mathbf{a}, \mathbf{b} \in M_{\mathcal{H}}$ are such that $\mathbf{b}$ is inaccessible from $\mathbf{a}$, then $\mathbf{a}$ is inaccessible from $\mathbf{b}$ as well.*

Proof: It suffices to prove the statement for the case where $\mathbf{a}$ is adjacent to $\mathbf{b}$. W.l.g. assume that $\mathbf{b} = \mathbf{a} + \mathbf{e}_1$. From (20) we find that $\operatorname{val}\hat{f}_1(\mathbf{a}) + \operatorname{val}\varphi(\mathbf{b}) = \operatorname{val}\hat{g}_1(\mathbf{a}) + \operatorname{val}\varphi(\mathbf{a})$ as before, but this time $\operatorname{val}\varphi(\mathbf{b}) = \operatorname{val}\varphi(\mathbf{a})$, so $\operatorname{val}\hat{f}_1(\mathbf{a}) = \operatorname{val}\hat{g}_1(\mathbf{a})$. Since $\mathbf{b}$ is inaccessible from $\mathbf{a}$, $f_1(\mathbf{a}) = 0$, which implies that $\operatorname{val}\hat{f}_1(\mathbf{a}) > 0$. Hence $\operatorname{val}\hat{g}_1(\mathbf{a}) > 0$ as well. Therefore $g_1(\mathbf{a}) = 0$, and the claim follows. □

Theorem 5 *Let $\mathcal{H}$ be a consistent H-system. Then $\dim V_1(\mathcal{H}) > 0$.*

Proof: Pick any $\mathbf{a} \in M_{\mathcal{H}}$ and let $S(\mathbf{a}) = \{\mathbf{p} \in M_{\mathcal{H}} \mid \mathbf{p}$ is accessible from $\mathbf{a}\}$. We claim that $S(\mathbf{a})$ is inaccessible. Indeed, take $\mathbf{p} \in S(\mathbf{a})$ and $\mathbf{q} \notin S(\mathbf{a})$. Then either $\mathbf{q} \in M_{\mathcal{H}} \setminus S(\mathbf{a})$ or $\mathbf{q} \notin M_{\mathcal{H}}$. In the former case, $\mathbf{p}$ is inaccessible from $\mathbf{q}$ because otherwise, by Lemma 6, $\mathbf{q}$ is accessible from $\mathbf{p}$ and hence from $\mathbf{a}$, which is impossible since $\mathbf{q} \notin S(\mathbf{a})$. In the latter case, $\mathbf{p}$ is inaccessible from $\mathbf{q}$ because $\mathbf{p} \in M_{\mathcal{H}}$, $\mathbf{q} \notin M_{\mathcal{H}}$, and $M_{\mathcal{H}}$ is inaccessible by Lemma 5. This proves the claim.

Now define $T : \mathbb{Z}^d \to \mathbb{C}$ as follows. Set $T(\mathbf{a}) = 1$ and define T on $S(\mathbf{a}) \setminus \{\mathbf{a}\}$ recursively, using the system $\mathcal{H}$. This is possible because if $\mathbf{p} \in M_{\mathcal{H}}$ is accessible from $\mathbf{a}$ along some path $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_k$ where $\mathbf{t}_1 = \mathbf{a}$ and $\mathbf{t}_k = \mathbf{p}$, then the entire path belongs to $M_{\mathcal{H}}$ (otherwise there is a j, $1 \le j \le k-1$, such that $\mathbf{t}_j \notin M_{\mathcal{H}}$, $\mathbf{t}_{j+1} \in M_{\mathcal{H}}$, and $\mathbf{t}_{j+1}$ is accessible from $\mathbf{t}_j$, which contradicts Lemma 5). Finally, set $T(\mathbf{p}) = 0$ for all $\mathbf{p} \notin S(\mathbf{a})$.

We claim that T satisfies (2) for all $\mathbf{n} \in \mathbb{Z}^d$ and all $i \in \{1, \ldots, d\}$. Indeed, if $\mathbf{n}, \mathbf{n} + \mathbf{e}_i \in S(\mathbf{a})$ then (2) is satisfied by definition of T and by consistency of $\mathcal{H}$. If $\mathbf{n}, \mathbf{n} + \mathbf{e}_i \notin S(\mathbf{a})$ then both sides of (2) are zero by definition of T. If $\mathbf{n} \in S(\mathbf{a})$ and $\mathbf{n} + \mathbf{e}_i \notin S(\mathbf{a})$ (or vice versa) then again both sides of (2) are zero by definition of T and because $S(\mathbf{a})$ is inaccessible. Hence T is a non-zero solution of $\mathcal{H}$. □

Example 6 Let $\mathcal{H}$ be the system

$$
\begin{aligned}
(n_1+n_2+2)T(n_1+1,n_2) &= (n_1+n_2)(n_1-n_2)T(n_1,n_2),\\
(n_1+n_2+2)(n_1-n_2-1)T(n_1,n_2+1) &= (n_1+n_2)T(n_1,n_2).
\end{aligned}
$$

It is easy to check that $\mathcal{H}$ is a consistent H-system with certificates

$$F_1(n_1,n_2) = \frac{(n_1+n_2)(n_1-n_2)}{n_1+n_2+2} = (n_1-n_2)_1 \frac{R(n_1+1,n_2)}{R(n_1,n_2)},$$

$$F_2(n_1,n_2) = \frac{n_1+n_2}{(n_1+n_2+2)(n_1-n_2-1)} = (n_1-n_2)_{-1} \frac{R(n_1,n_2+1)}{R(n_1,n_2)}$$

(cf. (13)), where

$$R(n_1,n_2) = \frac{1}{(n_1+n_2)(n_1+n_2+1)}.$$

Note that for $(n_1+n_2)(n_1+n_2+1)(n_1+n_2+2) \ne 0$, $\mathcal{H}$ is satisfied by

$$T(n_1,n_2) = \frac{(-1)^{n_1+n_2}}{\Gamma(1-n_1+n_2)} R(n_1,n_2),$$

but this solution does not belong to $V_1(\mathcal{H})$.

In this case $\xi(n_1,n_2)$ satisfies

$$
\begin{aligned}
\xi(0,0) &= 1,\\
\xi(n_1+1,n_2) &= (n_1-n_2+x_1-x_2)\xi(n_1,n_2),\\
\xi(n_1,n_2+1) &= \frac{\xi(n_1,n_2)}{n_1-n_2-1+x_1-x_2}.
\end{aligned}
$$

It is straightforward to verify that $\xi(n_1,n_2) = (x_1 - x_2)_{n_1-n_2}$ and

$$\operatorname{val}\xi(n_1,n_2) = \begin{cases} 0, & n_1 \le n_2, \\ 1, & \text{otherwise.} \end{cases}$$

Next, $\eta(n_1,n_2) = 1/((n_1+n_2+x_1+x_2)(n_1+n_2+1+x_1+x_2))$, and

$$\operatorname{val}\eta(n_1,n_2) = \begin{cases} -1, & (n_1+n_2)(n_1+n_2+1) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $m = \min_{(n_1,n_2)\in\mathbb{Z}^2} \operatorname{val}\varphi(n_1,n_2) = -1$, and

$$M_{\mathcal{H}} = \{(n_1,n_2) \in \mathbb{Z}^2 \mid (n_1+n_2)(n_1+n_2+1) = 0 \ \wedge \ n_1 \le n_2\}.$$

By taking $\mathbf{a} = (0,0)$ in the proof of Theorem 5, we have $S(\mathbf{a}) = M_{\mathcal{H}}$, and the corresponding non-zero solution belonging to $V_1(\mathcal{H})$ is

$$T(n_1,n_2) = \begin{cases} \frac{1}{\Gamma(2n_2+1)}, & n_1+n_2 = 0 \ \wedge \ n_1 \le n_2, \\ \frac{1}{\Gamma(2n_2+2)}, & n_1+n_2+1 = 0 \ \wedge \ n_1 \le n_2, \\ 0, & \text{otherwise.} \end{cases}$$

■

6 The Ore-Sato theorem and its consequences

The Ore-Sato theorem (see Theorem 4) is commonly believed to imply that every hypergeometric term is of the form (1). For example, in [3, p. 223] one reads: "*From Ore's result it can be deduced that the most general form of A_{mn} is of the form*

$$A_{mn} = R(m,n)\gamma_{mn}a^m b^n$$

where R is a fixed rational function of m and n, a and b are constants, and γ_{mn} is a gamma product (...) that is to say it is of the form

$$\gamma_{mn} = \prod_i \Gamma(a_i + u_i m + v_i n)/\Gamma(a_i)$$

where the a_i are arbitrary (real or complex) constants, and the u_i and v_i are arbitrary integers which may be positive, negative, or zero." A similar quote can be found in [4, p. 5].

It may be the case that in the literature referred to above, A_{mn} is implicitly assumed to be non-zero for all $m,n \in \mathbb{Z}$. This possibility is supported by the fact that, e.g., in [3] the corresponding *H*-system is given in terms of the two quotients $A_{m+1,n}/A_{mn}$ and $A_{m,n+1}/A_{mn}$. But such a severe restriction would exclude from consideration many important functions, such as the binomial coefficient $A_{mn} = \binom{m}{n}$, and all polynomials with integer roots.

However if we do not adopt this restriction, then there are hypergeometric terms which cannot be written in the form (1), as illustrated by the following example.

Example 7 Take the H-system

$$
\begin{aligned}
p(n_1,n_2)T(n_1+1,n_2) &= p(n_1+1,n_2)T(n_1,n_2), \\
p(n_1,n_2)T(n_1,n_2+1) &= p(n_1,n_2+1)T(n_1,n_2),
\end{aligned} \tag{21}
$$

where $p(n_1,n_2) = (n_1 - n_2 - 1)(n_1 - n_2 + 1)$. It can be checked that any $T : \mathbb{Z}^2 \to \mathbb{C}$ which satisfies $T(n_1,n_2) = 0$ unless $n_1 = n_2$ is a solution of (21). In particular,

$$
T(n_1,n_2) = \begin{cases} 2^{n_1^2}, & n_1 = n_2, \\ 0, & \text{otherwise} \end{cases}
$$

is a solution of (21), even though it does not have the form (1) because it grows too fast along the diagonal. ∎

There are examples which look less artificial and where the solution has a non-algebraic support, such as $T(n_1,n_2) = |n_1 - n_2|$. In [1, Example 6] it is shown that this hypergeometric term cannot be written in the form (1) if R is assumed to be a polynomial. In a similar way it can be shown that the same is true even if R is allowed to be a rational function.

The following statement does follow from the Ore-Sato theorem.

Corollary 2 *Let T be a hypergeometric term. If T has non-algebraic support, then any constituent*[3] *of T is of the form (1).*

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