University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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Thomas Böhme
Ken-ichi Kawarabayashi John Maharry
Bojan Mohar

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$K_{3,k}$-minors in large 7-connected graphs

Thomas Böhme*
Institut für Mathematik
Technische Universität Ilmenau
Ilmenau, Germany

Ken-ichi Kawarabayashi†
National Institute of Informatics
2-1-2 Hitotsubashi, Chiyoda-ku
Tokyo 101-8430, Japan

John Maharry‡
Department of Mathematics
The Ohio State University
Columbus, OH USA

Bojan Mohar§
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6, Canada

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Abstract

It is shown that for any positive integer k there exists a constant $N = N(k)$ such that every 7-connected graph of order at least N contains $K_{3,k}$ as a minor.

1 Introduction

In this paper, all graphs are finite and may have loops and multiple edges. A graph is a *minor* of another graph if the first can be obtained from a subgraph of the second by contracting connected subgraphs. There are many results

*E-mail address: tboehme@theoinf.tu-ilmenau.de

†Research partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research, by C & C Foundation and by Kayamori Foundation. Visiting Department of Mathematics, Simon Fraser University, Burnaby, B.C. V5A 1S6. Email address: k_keniti@nii.ac.jp

‡Supported in part by the International Research Project SLO-US-007 on Graph Minors and by the National Security Agency under Grant Number MDA 904-02-1-0052. E-mail address: maharry@math.ohio-state.edu

§Supported in part by the International Research Project SLO-US-007 on Graph Minors, by by ARRS Research Program P1-0297, by an NSERC Discovery Grant and by the Canada Research Chair program. On leave from Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia. Email address: mohar@sfu.ca

concerning the structure of graphs that do not contain a certain graph as a minor. These excluded graphs include K_5 and $K_{3,3}$ [19], V_8 [14], the 3-cube [9], the octahedron [10], graphs with single crossing [16] and K_6^--minor [7]. There are well-known structures which guarantee a certain minor exists for large graphs. For instance, any 5-connected graph on at least 11 vertices contains the 3-cube as a minor [9]. Any 5-connected non-planar graph on at least 8 vertices contains a V_8-minor [14]. In addition, there are Ramsey-type results similar to the fact that any sufficiently large connected graph contains either a k-path or a k-star. Oporowski, Oxley and Thomas [13] proved that any large 4-connected graph must have a large minor from a set of four families of graphs. Moreover, they found a similar result for large 3-connected graphs. Ding [4] has characterized large graphs that do not contain a $K_{2,k}$-minor. A corollary of his result is that any large 5-connected graph contains a $K_{2,k}$-minor.

Robertson and Seymour [14] have an unpublished result that roughly states that for any infinite family of graphs that do not all embed in a given fixed surface, then for every integer k, there is a graph in the family that contains either a $K_{3,k}$-minor or a minor isomorphic to k Kuratowski graphs identified on $0, 1$, or 2 vertices.

Our result is a cross section of all of these types of results:

Theorem 1.1 *For any positive integer k, there exists a constant $N(k)$ such that every 7-connected graph G on at least $N(k)$ vertices contains $K_{3,k}$ as a minor.*

In a previous paper [2], the authors proved the theorem for the bounded tree-width case:

Theorem 1.2 ([2]) *For any positive integers k and w, there exists a constant $N = N_1(k, w)$ such that every 7-connected graph of tree-width less than w and with at least N vertices contains $K_{3,k}$ as a minor.*

This paper contains the proof of the large tree-width case which completes the proof of Theorem 1.1. This case needs much more work than the proof of Theorem 1.2.

Theorem 1.3 *For any positive integer k, there exist integers $N = N_3(k)$ and $w = w(k)$ such that every 7-connected graph of tree-width at least w and with at least N vertices contains $K_{3,k}$ as a minor.*

These results are best possible in the sense that the connectivity condition cannot be reduced. In [2], we gave a family of arbitrarily large 6-connected graphs none of which contain a $K_{3,7}$-minor. Graphs in that family have tree-width less than 9. On the other hand, no graphs embedded in the torus contain $K_{3,7}$ as a minor, and there are infinitely many 6-connected triangulations of the torus with arbitrarily large tree-width. In fact the family can be generalized to give arbitrarily large $2a$-connected graphs with no $K_{a,2a+1}$-minor. Hence we have made the following conjecture in [2]:

Conjecture 1.4 *For any positive integers a, k, there exists a constant $N(a, k)$ such that every $(2a + 1)$-connected graph G on at least $N(a, k)$ vertices contains $K_{a,k}$ as a minor.*

Toward this conjecture, we proved in [1] the following theorem.

Theorem 1.5 ([1]) *For any integers a, s and k, there exists a constant $N(a, s, k)$ such that every $(3a + 2)$-connected graph of minimum degree at least $\frac{31}{2}(a + 1) - 3$ and with at least $N(a, s, k)$ vertices either contains $K_{a,sk}$ as a topological minor or a minor isomorphic to s disjoint copies of $K_{a,k}$.*

Although our main result, Theorem 1.3, seems to be a special case of a more general Theorem 1.5, the weaker 7-connectivity assumption requires several essentially different proof methods to be used in order to get $K_{3,k}$-minors in large 7-connected graphs. In particular, these methods involve delicate elaboration about graphs embedded in surfaces and particular use of nonplanarity properties within vortices.

The paper is organized as follows. In Section 2 we define the ingredients needed to present the Excluded Minor Theorem of Robertson and Seymour. In Section 3 we start with the proof of Theorem 1.4 and argue about all cases when there is no large vortex. In Section 4 we show that large vortices that are "well-linked" with a "flat grid minor" have special structure when $K_{3,k}$-minor is excluded. In the last section, we continue with the proof and clear up the large vortex case.

2 Structure of near embeddings

In this section, we define some of the structures found in Robertson-Seymour's Excluded Minor Theorem [17] which describes the structure of graphs that do no contain a given graph as a minor. Robertson and Seymour proved a strengthened version of that theorem that gives a more elaborate description of the structure in [18]. This strengthened version enables us to apply a method when finding minors in a vortex structure that is similar to the method used in the bounded tree-width case. In fact, we give a slightly simplified version of the theorem. For a proof of how our version follows from the main results of [18] see the appendix of our paper on $K_{a,k}$-minors [1]. We assume that the reader is familiar with the notion of the tree-width of graphs.

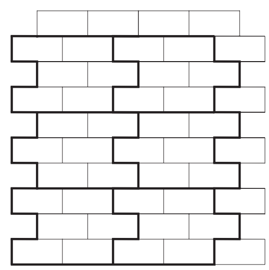

Figure 1: An 11-wall and its subwall

Let us define an *r-wall* as a graph which is isomorphic to a subdivision of the graph W_r defined as follows. We start with vertex set $V = \{(i,j) \mid 1 \le i \le r,\ 1 \le j \le r\}$, and make two vertices (i,j) and (i',j') are adjacent if and only if one of the following possibilities holds:

(1) $i' = i$ and $j' \in \{j-1, j+1\}$.

(2) $j' = j$ and $i' = i + (-1)^{i+j}$.

Some of the vertices of this graph can have degree 1. After deleting them, we get the graph W_r which is 2-connected; see Figure 1 showing the 11-wall W_{11}.

A *surface* is a compact connected 2-manifold (with or without boundary). The components of the boundary are called the *cuffs*.

Let G be a graph and let $W = \{w_0, \ldots, w_n\}$, $n = |W| - 1$, be a linearly ordered subset of its vertices such that w_i precedes w_j in the linear order if and only if $i < j$. The pair (G, W) is called a *vortex* of *length* n, W is the *society* of the vortex and all vertices in W are called *society vertices*. Suppose that for $i = 0, \ldots, n$, there exist vertex sets, called *parts*, $X_i \subseteq V(G)$, with the following properties:

(V1) $X_i \cap W = \{w_i, w_{i+1}\}$ for $i = 0, \ldots, n$, where $w_{n+1} = w_n$,

(V2) $\bigcup_{0 \le i \le n} X_i = V(G)$, (V3) every edge of G has both endvertices in some X_i, and

(V4) if $i \leq j \leq k$, then $X_i \cap X_k \subseteq X_j$.

Then the family $(X_i\,;\, i = 0, \ldots, n)$ is called a *vortex decomposition* of the vortex (G, W). The *width* of the vortex decomposition is the maximum of $\{|X_i|\,;\, i = 0, \ldots, n\}$.

For $i = 1, \ldots, n$, denote by $Z_i = (X_{i-1} \cap X_i) \setminus W$. The *adhesion* of the vortex decomposition is the maximum of $|Z_i|$, for $i = 1, \ldots, n$. The vortex decomposition is *linked* if for $i = 1, \ldots, n-1$, the subgraph of G induced on the vertex set $X_i \setminus W$ contains a collection of disjoint paths linking Z_i with Z_{i+1}. Clearly, in that case $|Z_i| = |Z_{i+1}|$, and the paths corresponding to $Z_i \cap Z_{i+1}$ are trivial. Note that (V1) and (V3) imply that there are no edges between nonconsecutive society vertices of the vortex. Let us remark that every vortex (G, W), in which w_i, w_j are non-adjacent for $|i-j| \geq 2$, admits a linked vortex decomposition; just take $X_i = (V(G) \setminus W) \cup \{w_i, w_{i+1}\}$.

The *width of the vortex* is the minimum width taken over all decompositions of the vortex, and the (*linked*) *adhesion of the vortex* is the minimum adhesion taken over all (linked) decompositions of the vortex. Let us observe that in a linked decomposition of adhesion q, there are q disjoint paths linking Z_1 with Z_n in $G - W$.

Let G_0 be a graph. Suppose that (G_1, G_2) is a separation of G of order $t \leq 3$, i.e., $G_0 = G_1 \cup G_2$, where $G_1 \cap G_2 = \{v_1, \ldots, v_t\} \subset V(G_0)$, $1 \leq t \leq 3$, $V(G_2) \setminus V(G_1) \neq \emptyset$. Let us replace G_0 by the graph G', which is obtained from G_1 by adding all edges v_iv_j $(1 \leq i < j \leq t)$ if they are not already contained in G_1. We say that G' has been obtained from G_0 by an *elementary reduction*. If $t = 3$, then the 3-cycle $T = v_1v_2v_3$ in G' is called the *reduction triangle*. Every graph G'' that can be obtained from G_0 by a sequence of elementary reductions is a *reduction* of G_0.

Let H be an r-wall in the graph G_0 and let G'' be a reduction of G_0. We say that G'' *captures* H if for every elementary reduction used in obtaining G'' from G_0, at most one vertex of degree 3 in H is deleted. (With the above notation, $G_2 \setminus G_1$ contains at most one vertex of degree 3 in H.)

If H is a wall in a graph G_0, we say that the pair (G_0, H) can be *embedded* in a surface Σ *up to* 3-*separations* if there is a reduction G'' of G_0 such that G'' has an embedding in Σ in which every reduction triangle bounds a face of length 3 in Σ and G'' captures H.

Lemma 2.1 *Suppose that G'' is a reduction of the the graph G_0 and that G'' captures an r-wall H in G_0. Then G'' contains an $\lfloor (r+1)/3 \rfloor$-wall, all* *of whose edges are contained in the union of H and all edges added to G'' when performing elementary reductions.*

Proof. Let H' be the subgraph of the r-wall H obtained by taking every third row and every third "column". See Figure 1 in which H' is drawn with thick edges. It is easy to see that for every elementary reduction we can keep a subgraph homeomorphic to H' by replacing the edges of H' which may have been deleted by adding some of the edges v_iv_j involved in the reduction. The only problem would occur when we lose a vertex of degree 3 and when all vertices v_1, v_2, v_3 involved in the elementary reduction would be of degree 3 in H'. However, this is not possible since G'' captures H. □

Let G be a graph, H an r-wall in G, Σ a surface, and $\alpha \geq 0$ an integer. We say that the pair (G, H) can be *α-nearly embedded* in Σ if there is a set of at most α cuffs $C_1, \ldots, C_b$ $(b \leq \alpha)$ in Σ, and there is a set A of at most α vertices of G such that $G - A$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_b$ where $G_0, G_1, \ldots, G_b$ are edge-disjoint subgraphs of G and the following conditions hold:

(N1) H is an r-wall in G_0, and (G_0, H) can be embedded in Σ up to 3-separations, with G'' being the corresponding reduction of G_0.

(N2) If $1 \leq i < j \leq b$, then $V(G_i) \cap V(G_j) = \emptyset$.

(N3) $W_i = V(G_0) \cap V(G_i) = V(G'') \cap C_i$ for every $i = 1, \ldots, b$.

(N4) For every $i = 1, \ldots, b$, the pair (G_i, W_i) is a vortex of adhesion less than α, where the ordering of W_i is consistent with the (cyclic) order of these vertices on C_i.

The vertices in A are called the *apex vertices* of the *α-near embedding*. The subgraph G_0 of G is said to be the *embedded subgraph* with respect to the α-near embedding and the decomposition $G_0, G_1, \ldots, G_b$. The pairs (G_i, W_i), $i = 1, \ldots, b$, are the *vortices* of the α-near embedding. The vortex (G_i, W_i) is said to *be attached to the cuff* C_i of Σ containing W_i.

We shall use the following theorem which is a simplified version of one of the cornerstones of Robertson and Seymour's theory of graph minors, the Excluded Minor Theorem, as stated in [18]. For a detailed explanation of how the version in this paper can be derived from the version in [18], see the appendix of [1].

Theorem 2.2 (Excluded Minor Theorem) *For every graph R, there is a constant α such that for every positive integer w, there exists a positive integer $r = r(R, \alpha, w)$, which tends to infinity with w for any fixed R and α, such that every graph G that does not contain an R-minor either has tree-width at most w or contains an r-wall H such that (G, H) has an α-near embedding in some surface Σ in which R cannot be embedded.*

For our purpose, as proved in [1] (see also [8]), we can add the following assumptions about the r-wall:

Lemma 2.3 *It may be assumed that the r-wall H in Theorem 2.2 has the following properties:*

(a) *H is contained in the reduction G'' of the embedded subgraph G_0.*

(b) *H is planarly embedded in Σ, i.e., every cycle in H is contractible in Σ and the outer cycle of H bounds a disk in Σ that contains H.*

(c) *Every non-contractible curve in the surface Σ intersects G'' in at least two vertices.*

3 Finding a wide vortex

In order to prove Theorem 1.3, we fix k and let G be a 7-connected graph with at least $N_3(k)$ vertices. It will become clear during the proof, which value we may take for $N_3(k)$. Now we apply Theorem 2.2 to G, $R = K_{3,k}$ and a large value of $w = w(k)$ that will be specified later. We let $\alpha = \alpha(k)$, $r_0 = r(K_{3,k}, \alpha(k), w(k))$, H, and Σ be the quantities from Theorem 2.2. By taking large enough w, we can assume that r_0 and hence also the wall H are as large as we need (if the tree-width is bigger than w).

Following the notation of Section 2, the embedded subgraph of G is G_0 and G'' is the reduction of G_0. By Lemmas 2.1 and 2.3 we may assume henceforth that H is contained in G'' and that it is planarly embedded in Σ. Note that the use of Lemma 2.1 may reduce an r_0-wall to an $\frac{1}{3}r_0$-wall, so we assume henceforth that H is an r-wall, where $r = \frac{1}{3}r_0$. In particular, G'' has at least r^2 vertices. Since $K_{3,k}$ cannot be embedded in Σ, the Euler genus of Σ is at most $\frac{k-2}{2}$ (cf. [12]).

Our goal in this section is to prove that G'' has a vortex (G_i, W_i) whose society W_i has many vertices and is linked to the wall H by many disjoint paths. Precise conditions on this linkage will be made more precise later in this section.

The basic idea of the proof is as follows. We know that all vertices in G have degree at least 7, since G is 7-connected. Now we look at G''. Since $|G''| \geq r^2$, where r is large, and G'' is embedded in a surface whose Euler genus is less than $k/2$, we conclude by Euler's formula that G'' has many vertices whose degree is at most 6. Now we look at different possibilities why many vertices of G'' would have degree smaller than in G, and we show that in each case we obtain a $K_{3,k}$-minor.

For every vertex $v \in V(G'')$, whose degree in G'' is less than its degree in G, one of the following holds:

(R) v has been involved in elementary reductions when reducing G_0 to G'',

(S) v is a society vertex in one of the vortices (G_i, W_i), $i = 1, \ldots, b$, or

(A) v is adjacent to a vertex in the apex set A.

Let V^R be the set of vertices of G'' for which (R) holds, let $V^S = \cup_{i=1}^{b} W_i$ be the set of all society vertices, and let V^A be the vertices in $V(G'') \setminus (V^R \cup V^S)$ which have a neighbor in A and whose degree in G'' is at most 6. For $i = 1, 2$, we let V_i^A be the set of those vertices in V^A which have precisely i neighbors in A, and let $V_{\geq 3}^A = V^A \setminus (V_1^A \cup V_2^A)$ be those with at least three neighbors.

For technical reasons we need in the proof of Claim 3.3 that G'' be 2-connected. If it is not, then we contract all blocks of G'' to single vertices except for the block containing the wall H. Let us observe that every former cutvertex in $V(G'') \setminus V^S$ now becomes a vertex with at least three neighbors in A, hence it is contained in (the updated) set $V_{\geq 3}^A$. Of course, we update other sets V_1^A, V_2^A, V^F, V^S accordingly. Note also that this operation may change the surface, but does not increase its Euler genus. From now on we assume that G'' is this reduced 2-connected minor of the original graph.

Claim 3.1 *If $|V^R| \geq 3k\binom{\alpha}{3}$, then G contains a $K_{3,k}$-minor.*

Proof. For each $u \in V^R$, consider the last reduction involving u. Let $G_1^{(u)}$ and $G_2^{(u)}$ be the graphs used in this reduction. Then all vertices of $G_1^{(u)} \cap G_2^{(u)}$ are contained in G'' since any further reduction deleting either of them would also involve all its neighbors, and hence also u. If $|V^R| \geq 3k\binom{\alpha}{3}$, then V^R contains a subset U of cardinality $k\binom{\alpha}{3}$ such that the vertex sets $V(G_2^{(u)}) \setminus V(G_1^{(u)})$ removed in these reductions are pairwise disjoint for different vertices $u \in U$.

For $u \in U$, let u' be a vertex in $V(G_2^{(i)}) \setminus V(G_1^{(i)})$. Since G is 7-connected, there are 7 internally disjoint paths connecting u' with the apex set A in the original graph G. At most three of these paths pass through the vertices in $V(G_1^{(u)}) \cap V(G_2^{(u)})$, so the other paths give rise to a collection of 3 paths joining u' with distinct vertices in A, and all vertices of these paths are contained in $A \cup (V(G_2^{(u)}) \setminus V(G_1^{(u)}))$.

Since $|U| \geq k\binom{\alpha}{3}$, there is a set of k of such vertices u' such that the corresponding three paths end at the same triple of vertices in A. These paths then form a subdivision of $K_{3,k}$ in G. □

The following claim is clear by the pigeonhole principle.

Claim 3.2 *If $|V_{\geq 3}^A| \geq k\binom{\alpha}{3}$, then G contains a subgraph isomorphic to $K_{3,k}$.*

Our goal is to show that V^S is large. We will be able to prove it after we will show that neither V_1^A nor V_2^A are large. In fact, we will only need to show that there are not too many vertices in these sets which are incident with short faces only. For this purpose we define the set V^F of all vertices of G'' that are incident with a face of size at least 12.

Claim 3.3 *If $|V_2^A \setminus V^F| \geq 55k\binom{\alpha}{2}$, then G contains a $K_{3,k}$-minor.*

Proof. By the Pigeonhole Principle, $V_2^A \setminus V^F$ contains a subset U of $55k$ vertices which are adjacent to the same pair a_1, a_2 of apex vertices. Since vertices in U are of degree at most 6 and all their incident faces have at most 11 vertices, each vertex in U is cofacial (i.e., is on the same face) with at most 54 other vertices. Therefore, U contains a subset U' with $|U'| = k$ such that no two vertices in U' are cofacial. Our earlier assumption that G'' is 2-connected now implies that $G'' - u'$ is connected for every $u' \in U'$. By condition (c) of Lemma 2.3 we know that the set of all vertices that are cofacial with u' induces a connected subgraph of G''. This implies that $G'' - U'$ is connected. Therefore, the subgraph $G'' - U'$ contracted to a single vertex, together with a_1, a_2 and with U' yields a minor of G which is isomorphic to $K_{3,k}$. □

In order to exclude many vertices in $V_1^A \setminus V^F$, we can apply a result due to Böhme and Mohar [3] which is stated below. Given a graph Z embedded in some surface and a subset U of its vertices, a set $\mathcal{F}$ of facial walks of Z is a *face cover* of U if each vertex of U belongs to a member of $\mathcal{F}$. Let $\tau(U)$ be the size of the smallest face cover of U. A subgraph of Z that can be contracted to the complete bipartite graph $K_{2,t}$ is called a *U-labeled $K_{2,t}$-minor* if each of the t trees, that are contracted to make the degree-two vertices of $K_{2,t}$, contains a vertex of U.

Theorem 3.4 ([3]) *There is a nondecreasing integer function $f_0 : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f_0(n) = \infty$ and such that the following holds. Let G be a 3-connected plane graph and let $U \subseteq V(G)$. Then G contains a U-labeled $K_{2,t}$-minor where $t \geq f_0(\tau(U))$.*

If U is a subset of vertices in V_1^A, whose neighboring apex vertex is the same, then a U-labeled $K_{2,k}$-minor in G'' together with the apex vertex gives rise to a $K_{3,k}$-minor in G. Theorem 3.4 can be generalized to 3-connected graphs embedded in a fixed surface when the face-width is large. However, this requires many technical details and we are not going to prove it here. Instead, we prove a weaker statement with more elegant proof.

Let $V^H \subseteq V(G'')$ be the set of those vertices in the r-wall H which are of degree 3 in H and are not contained on the outer face of H.

Claim 3.5 *If $|V_1^A \cap V^H| \geq 2\alpha k^2$, then G contains a $K_{3,k}$-minor.*

Proof. There is an apex vertex a_1 which is adjacent to at least $2k^2$ vertices in $V_1^A \cap V^H$. Let U be a set of neighbors of a_1 in $V_1^A \cap V^H$ such that $|U| = 2k^2$. Then either $2k$ of these vertices are in the same row of the r-wall H, or k of them are in distinct rows. In each case, a k-subset of them can be linked by using the columns (or by using the rows) of the wall to the upper and to the lower (or to the left and the right) side of the outer cycle of H. This is easily seen to give rise to a U-labaled $K_{2,k}$-minor in H. This is also such a minor in G''. Together with the vertex a_1, this gives a $K_{3,k}$-minor in G. □

Next we prove that the claims established above imply that there is a large vortex. In fact, we need a large vortex and many disjoint paths joining the society of this vortex with the wall H. Moreover, we want that every society vertex participating in this linkage has at least three neighbors in the vortex. These requirements are specified in conditions (a) and (b) below. Having obtained such a vortex and the corresponding linkage, we shall develop, in the next section, a refinement of the method used in [2] to find a $K_{3,k}$-minor in G.

We say that a society vertex $v \in W_i$ is *essential* if v has at least three neighbors in $G_i \setminus W_i$. We let V_0^S denote the subset of V^S containing all essential society vertices. We say that the vortex (G_i, W_i) attached to the cuff C_i is *n-wide* if the following two properties are satisfied:

(a) The vortex contains n essential society vertices $w_1, \ldots, w_n \in W_i$ that appear in this order in W_i, and there are a path P in G'' and n disjoint paths $Q_1, \ldots, Q_n$, where $Q_j \cap P$ is a single vertex w'_j and Q_j joins w_j with w'_j $(1 \le j \le n)$, and where $w'_1, \ldots, w'_n$ appear in this order on P.

(b) There exists a connected subgraph S in G'' disjoint from $P \cup (\bigcup_{j=1}^{n} Q_j) \cup W_i$ such that there is an edge joining S with the segment of P from w'_j to w'_{j+1} (but excluding w'_{j+1}) for every $1 \le j \le n$.

Claim 3.6 *Let n be a positive integer. If G has no $K_{3,k}$-minor and $r \ge 22k\alpha(\alpha+g)n$, then there exists an n-wide vortex.*

Proof. For each cuff C_i $(1 \le i \le b)$, let L_i be the set of all essential vertices in C_i. Let R_0 be the outer cycle of the planarly embedded r-wall H in G'', and let R be the outer cycle of the graph $H - V(R_0)$ (viewed as being embedded in the plane). Note that R bounds a planarly embedded $(r-4)$-subwall of H.

Suppose that for some i, there are $12(\alpha+g)n^2$ disjoint paths from L_i to R. In this case we consider homotopy of these paths in the surface of Euler genus at most $\alpha+g$ which is obtained from Σ by adding a crosscap into each cuff. By a well-known result (cf. [12, Proposition 4.2.6]), a subset of $4n^2$ of these paths will be homotopic to each other. By another well-known result, the famous theorem of Erdős and Szekeres [5], there are at least $2n$ of these paths whose endvertices $w_1, \ldots, w_{2n} \in W_i$ appear in the society W_i in the same order as their endvertices in R appear on R (traversed in one or the other way). Since these paths are homotopic, their shortenings Q_j $(1 \le j \le 2n)$ to R_0 also end up in the same order on R_0. By a *shortening* we mean the segment of the path from W_i until the path hits R_0 for the first time. This proves part (a) from the definition of an n-wide vortex (with twice as many paths than needed), where we take for the path P the corresponding segment of R_0. For Q_j, consider how it continues from R_0 towards R. Let r_j be the last vertex of the path on R_0. Since Q_j are homotopic, the next edge after r_j joins the segment of R_0 between the Q_{j-1} and Q_{j+1} with the component of $G'' - (P \cup (\bigcup_{l=1}^{2n} Q_l))$ containing R. This almost proves (b) except that this edge may precede the endvertex of Q_j. But if we take every second of these paths, we achieve property (b) as well.

Otherwise, by Menger's theorem, for each i, there is a separation (I_i, J_i) of order at most $3(\alpha+g)n^2-1$ such that J_i contains all the vertices in L_i and $R \subseteq I_i$. (Here we allow that $J_i \setminus I_i$ is empty.) Let G''_1 be the graph obtained from G'' by deleting $J_i - I_i$ for all i. We consider the induced embedding of G''_1 in the same surface as G''. Observe that this may no longer be a 2-cell embedding. Let $J = \bigcup_{i=1}^{b}(J_i \cap I_i)$. Then $G_1'' - J$ has no essential society vertices. Since $R \subseteq G_1''$, the subgraph G_1'' also contains the disk bounded by R (assuming all separations (I_i, J_i) are of minimal order). Therefore, G_1'' contains the whole $(r-4)$-subwall of H. In particular, $|G_1''| \geq (r-4)^2$.

We may assume that the separations (I_i, J_i) obey the following requirement: For every edge $e \in E(J_i)$, there is a path $P_e \subseteq J_i$, which starts in the cuff C_i, passes through e, ends in $J_i \cap I_i$, and its only vertex in $J_i \cap I_i$ is its endvertex. This condition assures that all vertices in $I_i \cap J_i$ lie on the same (possibly non-simply connected) face of G_1'' and that every face of G_1'' that is not a face of G'' contains one or more of the cuffs. In particular, the number of new faces is at most α.

If a vertex $v \in J$ has at most one neighbor u in $G_1'' \setminus J$, then we change every separation (I_i, J_i) with $v \in I_i \cap J_i$ into the separation $(I_i \setminus \{v\}, J_i \cup \{v\})$. This change does not affect the property discussed in the previous paragraph. By repeating such changes as long as possible, each vertex $v \in J$ will have at least two neighbors in $G_1'' \setminus J$, which we assume henceforth.

Let $V^{F'}$ denote the set of all vertices of G_1'' that lie on faces of G_1'' of length at least 12. Note that a vertex $v \in V(G_1'') \cap V^F$ may not be in $V^{F'}$. However, this happens only when a large face in G'' is replaced by a short face in G_1''. But there are at most α faces in G_1'' that are not faces in G''. These faces contain not only all vertices in $V(G_1'') \cap V^F \setminus V^{F'}$ but also all vertices in $V(G_1'') \cap V^S \setminus V^{F'}$. Therefore,

$$|V(G_1'') \cap V^F \setminus V^{F'}| + |V(G_1'') \cap V^S \setminus V^{F'}| \leq 11\alpha. \tag{1}$$

In the rest of the proof we will use the discharging method on G_1'' to arrive to a contradiction. Let us assign the value $c(v) = \deg_{G_1''}(v) - 6$ to each vertex $v \in V(G_1'')$. This value is called the *charge* at v. Similarly, we assign charge $c(f) = 2\deg(f) - 6$ to each face f of G_1'', where $\deg(f)$ is the length of f. Euler's formula implies that

$$\sum_{v \in V(G_1'')} c(v) + \sum_{f \in \mathcal{F}(G_1'')} c(f) \leq -12 + 6g. \tag{2}$$

If f is a face of length $d \geq 12$, let us change its charge to $c'(f) = c(f) - \frac{3}{2}d = \frac{1}{2}d - 6 \geq 0$, and then redistribute the difference to vertices incident with f by adding charge $\frac{3}{2}$ to each of them. After doing this for all long faces f, let us denote by $c'(v)$ the new charge at a vertex v. This way the total sum in (2) remains the same, and we conclude that

$$\sum_{v \in V(G_1'')} c'(v) \leq -12 + 6g. \tag{3}$$

Observe that each vertex v of degree at least 7 has $c'(v) \geq 1$. If $v \in V(G_1'') \setminus J$ is a vertex in the set $(V_1^A \cup V_2^A) \cap V^F$, then $\deg_{G_1''}(v) \geq 5$ and v gets charge $\frac{3}{2}$ from at least one of the incident faces. Therefore, $c'(v) \geq \frac{1}{2}$.

The remaining vertices of G_1'' may have negative charge (but still $c'(v) \geq -5$). However, their number cannot be too large. The vertices in $V_1^A \setminus J$ have degree at least 6 and thus have $c'(v) \geq 0$. Also, G_1'' may contain some non-essential society vertices. They have degree at least 5. They get charge $\frac{3}{2}$ if they lie on a big face, hence their charge $c'(v)$ is positive. Since all society vertices corresponding to the same cuff are on the same face, we conclude that at most 11α of them can be on faces of size less than 12, and each of them has charge at least -1. The same counting includes vertices in $V(G_1'') \cap V^F \setminus V^{F'}$, see (1). By using Claims 3.1, 3.2, 3.3, 3.5, and the inequality (3), we now conclude:

$$\begin{aligned}
6g &> \sum_{v \in V(G_1'')} c'(v) \\
&\geq \frac{1}{2}(|G_1''| - |V_1^A \cap G_1''|) - (5 + \tfrac{1}{2})(|J| + |V^R| + |V_{\geq 3}^A| + |V_2^A \setminus V^F|) - 11\alpha \\
&\geq \frac{1}{2}(r-4)^2 - \alpha k^2 - \tfrac{11}{2}\left(12\alpha(\alpha+g)n^2 + 4k\binom{\alpha}{3} + 55k\binom{\alpha}{2}\right) - 11\alpha \\
&\geq \frac{1}{2}(r-4)^2 - 225k^2\alpha^2(\alpha+g)n^2.
\end{aligned}$$

In passing from the second to the third line in the above chain of inequalities, we have only involved the $(r-4)^2$ vertices in V^H and have used Claim 3.5 to conclude that at most $2\alpha k^2$ of them can be in V_1^A. In the last inequality, which is very crude, we have assumed, without the loss of generality, that $\alpha \geq 1$. Another crude estimate now implies that $r < 22k\alpha(\alpha+g)n$, contrary to our assumption. This completes the proof. □

4 The structure in a wide vortex

In this and the next section we complete the proof of Theorem 1.3. First, for any positive integers a, k, and w, we define the constants that will be used in the proofs:

$$\begin{aligned}
n_1 &= (2n_2)^p, \quad \text{where } p = 2^{w+1} \\
n_2 &= 2n_3^q, \quad \text{where } q = 2^{w(w+1)/2} \\
n_3 &= 30k\alpha.
\end{aligned}$$

At this point we can assume that for a fixed k, we have a 7-connected graph G, with at least $N_3(k)$ vertices. Now we apply Theorem 2.2 to G for the excluded minor $R = K_{3,k}$ and for the width parameter $w = w(k)$, which is chosen large enough to ensure that $r_0 = r(K_{3,k}, \alpha(k), w(k)) \geq 66k\alpha(\alpha + g)n_1$. Recall that in the previous section, we used an r-wall H, where $r = \frac{1}{3}r_0 \geq 22k\alpha(\alpha + g)n_1$. Hence by Claim 3.6, we can also assume that the α-near embedding of G in Σ has an n_1-wide vortex (G_1, W_1). We shall be using the notation from Section 3. Recall, in particular, that there are paths $Q_1, \ldots, Q_{n_1}$ joining essential society vertices w_i $(1 \leq i \leq n_1)$ in W_1 with a path P in G'', and there is a connected subgraph $S \subseteq G''$ as specified in (a)–(b) in the definition of an n_1-wide vortex.

At this point, we find it easier to include all apex vertices in G_1 and include all of them in every part of the vortex decomposition. In this way, the decomposition remains linked (with the trivial paths consisting of the added apex vertices) and the adhesion of the vortex decomposition increases by $|A| \leq \alpha$, so it is still bounded by 2α. In order that the extended vortex preserves the properties of the vortex, we add all edges of G connecting a vertex in A with a vertex in G_1.

Let (G_1, W_1) be an n_1-wide vortex of adhesion $q \leq 2\alpha$, modified as in the previous paragraph, with $w_1, \ldots, w_{n_1}$ being the corresponding essential society vertices. Let $P_1, \ldots, P_q$ be the paths of the linked decomposition of the vortex. Let $Q_1, \ldots, Q_{n_1}$, P, and S be as in the definition of an n_1-wide vortex. We will denote the union of the path P and all paths Q_i $(1 \leq i \leq n_1)$ by P_0 and will call it the *society path*. (Note that P_0 is not a path but its role will be similar to the paths $P_1, \ldots, P_q$, and after contracting $Q_1, \ldots Q_{n_1}$ to single vertices, it will actually become a path joining the essential society vertices $w_1, \ldots, w_{n_1}$.) Moreover, the connected subgraph S can be contracted to a vertex, denoted henceforth by u_0, and called the *surface node*.

We shall use other notation introduced in previous sections. For instance, X_i are parts of the linked decomposition of the vortex (G_1, W_1) (extended with the apex vertices), and $Z_i = (X_{i-1} \cap X_i) \setminus W_1$.

Let $Z = W_1 \cup P_1 \cup \cdots \cup P_q$. Since a Z-bridge in G_1 can be attached to at most two society vertices, the 7-connectivity of the graph G is enough to assure that the paths $P_1, \ldots, P_q$ can be chosen such that every Z-bridge in G_1 is attached to at least two of the paths (cf., e.g., [6]), where we consider the society vertices $W_1 \subseteq Z$ as one of the "paths". We shall assume this property henceforth. Note that every Z-bridge in G_1 is confined to a single part X_i.

Notice that for any parts X_i and X_l and for every $j \in \{1, \ldots, q\}$, there is a unique subpath of P_j with one end in Z_i and the other end in Z_l. Denote this subpath by $P_j(i,l)$.

The path P_j is said to be *trivial* if it consists of a single vertex, and it is said to be *everywhere non-trivial* w.r.t. the sequence $r_1, \ldots, r_n$ if $P_j(r_i, r_{i+1})$ contains at least three vertices for each $i = 1, \ldots, n-1$. The proofs of the following two claims can be found in [2] and [1].

Claim 4.1 *For $i = 1, \ldots, n_1$, let r_i be the index such that $w_i \in X_{r_i-1} \cap X_{r_i}$. There is a subsequence $q_1, q_2, \ldots, q_{n_2}$ of $r_1, r_2, \ldots, r_{n_1}$ of length n_2 such that for each $j = 0, 1, \ldots, q$, the path segment $P_j(q_1, q_{n_2})$ is either trivial or everywhere non-trivial (w.r.t. the subsequence).*

The paths P_j and P_l are said to be *everywhere bridge connected* (resp. *everywhere bridge disconnected*) with respect to a subsequence $p_1, \ldots, p_n$ of $r_1, r_2, \ldots, r_{n_1}$ if for every $i = 1, \ldots, n-1$, there exists (resp. does not exist) a Z-bridge which has a vertex of attachment in $P_j(p_i, p_{i+1})$ and a vertex of attachment in $P_l(p_i, p_{i+1})$.

Claim 4.2 *There is a subsequence $p_1, p_2, \ldots, p_{n_3}$ of $q_1, \ldots, q_{n_2}$ of length n_3 such that for every distinct pair of indices $j, l \in \{0, 1, \ldots, q\}$, $P_j(p_1, p_{n_3})$ and $P_l(p_1, p_{n_3})$ are either everywhere bridge connected or everywhere bridge disconnected (w.r.t. the new subsequence).*

We introduce the following notation. Let $Z(i) = \cup_{j=0}^{q} P_j(p_i, p_{i+1})$, and let $\hat{H}_i$ be the subgraph of G_1 consisting of $Z(i)$ together with all Z-bridges in G_1 that have all their vertices of attachment in $Z(i)$. For reader's convenience we assemble the medium scale picture of the vortex structure between p_{i-1} and p_{i+1} in Figure 2. It is also worthwhile to point out the distinction between $p_{i\pm1}$ and $p_i \pm 1$ and between $w_{p_{i\pm1}}$ and $w_{p_i \pm 1}$, which are all sketched in Figure 2.

Next, we shall study how the paths P_j are (everywhere) connected to each other. For this purpose, we define an auxiliary graph Γ with vertex set $V(\Gamma) = \{P_0, \ldots, P_q\}$, and the paths P_j and P_l are adjacent vertices in Γ if they are everywhere bridge connected w.r.t. $p_1, \ldots, p_{n_3}$ (cf. Claim 4.2).

Note that the surface node u_0, although it is not a vertex of Γ, could be considered as a trivial path that is everywhere bridge connected to the society path P_0, and that P_0 is everywhere non-trivial.

Let Γ_1 be the induced subgraph of Γ on the everywhere nontrivial paths. Let Γ_0 be the connected component of Γ_1 containing P_0. Further, let Γ_0^+ be the the graph Γ_0 together with the trivial paths adjacent to Γ_0 in Γ.

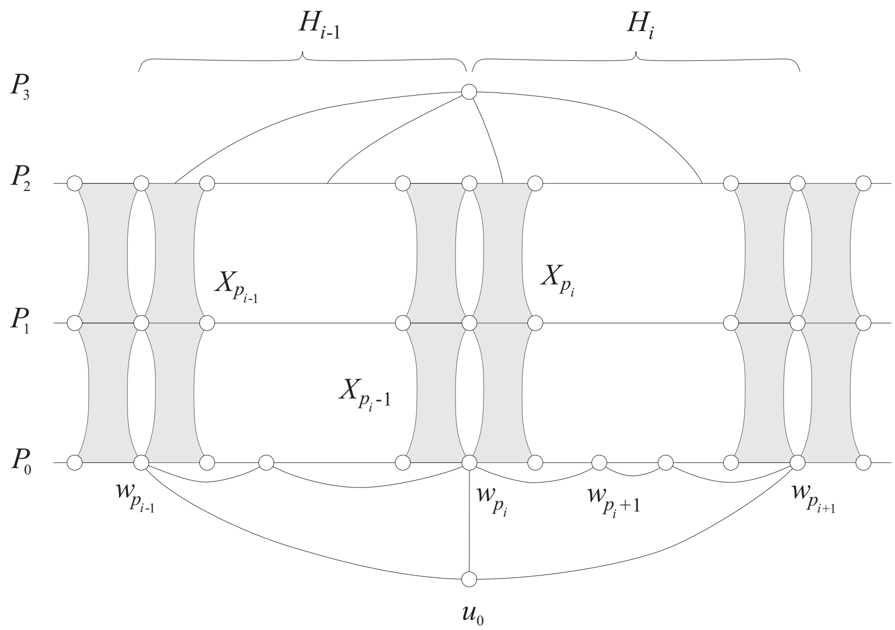

Figure 2: Vortex structure between p_{i-1} and p_{i+1}

Let $q_0 = |\Gamma_0| - 1$. Note that $0 \leq q_0 \leq q$. We may assume that $V(\Gamma_0) = \{P_0, P_1, \ldots, P_{q_0}\}$.

As in [1], we introduce the graph $H_i \subseteq \hat{H}_i$ which consists of all segments $P_j(p_i, p_{i+1})$ for $j = 0, \ldots, q_0$ together with all Z-bridges in $\hat{H}_i$ that are attached to at least one of the paths $P_0, \ldots, P_{q_0}$. Observe that H_i contains all paths in Γ_0^+.

Claim 4.3 *There is at most one trivial path in Γ_0^+.*

Proof. Suppose there are two trivial paths u_1, u_2 in Γ_0^+. Since they are in Γ_0^+, there is a path L_i^1 in H_i from u_1 to P_0. Similarly, there is a path L_i^2 joining u_2 and P_0 in H_i. These paths, taken for $i = 3, 5, 7, \ldots, 2k-1$ together with the surface node u_0 would give a $K_{3,k}$-minor after contracting $\Gamma_0 \cap H_i$ to a vertex for each i and contracting all edges except the ones adjacent to u_1 or u_2 in L_i^1 and L_i^2. Note that we used the fact that $n_3 > 2k$. $\square$

We also introduce vertex sets

$$\begin{aligned} S_i &= V(X_{p_i-1} \cap X_{p_i}) \cap H_i, \\ S_i^- &= V(X_{p_i-2} \cap X_{p_i-1}) \cap H_{i-1}, \\ S_i^+ &= V(X_{p_i} \cap X_{p_i+1}) \cap H_i \end{aligned}$$

and the corresponding society vertices: let z_i^-, z_i and z_i^+ be the vertices from W_1 that are contained in S_i^-, S_i and S_i^+, respectively. Note that $z_i = w_{p_i}$. Let us observe that S_i, S_i^-, and S_i^+ need not be disjoint. However, vertices z_i^-, z_i and z_i^+ are distinct. See Figure 3.

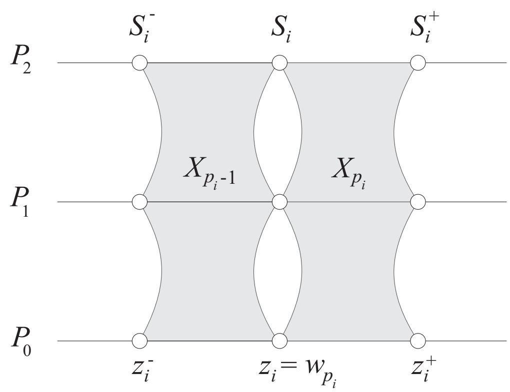

Figure 3: Vortex structure around $z_i = w_{p_i}$

Claim 4.4 *The graph* Γ_0 *has at least three vertices.*

Proof. If Γ_0^+ contains a trivial path, denote this path by u_1. Let us assume (reductio ad absurdum) that $q_0 \leq 1$. Since z_i is an essential society vertex, it has at least three neighbors in $G_1 \setminus W_1$. If one of them is not in the set $S = S_i^- \cup S_i^+ \cup \{z_i, u_1\}$, then S is a separating set. Therefore $7 \leq |S| \leq 2(q_0 + 1) + 2$, and we conclude that $q_0 \geq 2$. This contradiction shows that all neighbors of z_i in $G_1 \setminus W_1$ are in S. However, the only possibility is that $q_0 = 1$, when two of these neighbors are on the path P_1 and the third one is u_1. But this case occurring more than $2k$ times gives rise to a $K_{3,k}$-minor, whose vertices of degree k correspond to u_0, u_1, and to the contracted path P_1, and whose vertices of degree 3 are $z_1, z_3, \ldots, z_{2k-1}$.
□

Claim 4.5 *$deg_{\Gamma_0^+}(P_j) \leq 2$ for $j = 1, \ldots, q_0$, and $deg_{\Gamma_0^+}(P_0) = 1$.*

Proof. If the degree of P_j were at least 3, then the three paths adjacent to P_j could be used to construct a $K_{3,k}$-minor. Note that P_0 is everywhere connected with the surface node u_0. Hence, if it had at least two neighbors in Γ_0^+, contractions of those two paths along with u_0 and the connecting paths to P_0 could be used to form a $K_{3,k}$-minor. □

To conclude, we can assume that Γ_0^+ is a path on consecutive vertices $P_0, P_1, \ldots, P_{q_0}$ ($2 \leq q_0 \leq \alpha$), which may be appended by one additional vertex u_1 (adjacent to P_{q_0}) if there is a trivial path in Γ_0^+.

5 Finding the minor in the vortex

In this section, we will show that the non-planarity of the vortex can be used to "weave" the paths and construct a $K_{3,k}$-minor.

Let $R, R' \in V(\Gamma_0^+)$ be paths that are adjacent in Γ_0^+. For $i = 1, 2, \ldots, n_3$ define the graph $D_i = D_i(R, R')$ as follows. First, take $S = (R \cup R') \cap (H_{i-2} \cup H_{i-1} \cup H_i \cup H_{i+1})$ together with all Z-bridges in $H_{i-2} \cup H_{i-1} \cup H_i \cup H_{i+1}$ that have all vertices of attachment on S. Finally, let us add two edges e_1, e_2, where e_1 joins the "left" endvertices, λ in $R \cap S_{i-2}$ and λ' in $R' \cap S_{i-2}$, and e_2 joins the "right" endvertices, ρ and ρ' in S_{i+2}, of these two paths. Then $S + e_1 + e_2 =: C_0$ is a cycle in D_i. If R (R') is everywhere trivial, then $\lambda = \rho$ ($\lambda' = \rho'$).

Claim 5.1 *For every value of i, there are adjacent vertices R, R' of Γ_0^+ such that $D_i(R, R')$ has no embedding in the plane in which the cycle C_0 would bound a face.*

Proof. Suppose that for every adjacent pair R, R' in Γ_0^+, the graph $D_i(R, R')$ has an embedding in the plane in which the cycle C_0 bounds the outer face. By using Claim 4.5 it is easy to see that such embeddings can be combined together to get an embedding of $H_{i-2} \cup H_{i-1} \cup H_i \cup H_{i+1}$ in the plane. This embedding would look like the representation shown in Figure 2. We shall now get a contradiction by showing that the subgraph L_i exhibited in Figure 3 cannot be planar. To do this, we first give a precise definition of L_i.

Let us recall that paths $P_0, \ldots, P_{q_0}$ form a path in Γ_0^+ and that possibly there is a trivial path u_1 adjacent to P_{q_0} in Γ_0^+. Let m be the smallest index such that $P_m(p_i - 1, p_i + 1)$ is either a single vertex or a single edge. If such index does not exist and u_1 exists, then let $m = q_0 + 1$. Otherwise, let $m = q_0$. Let us remark at this point that the 7-connectivity of G and the fact that $z_i = w_{p_i}$ is an essential society vertex imply that $m \geq 2$.

We will consider the graph $L_i \subseteq H_{i-1} \cup H_i$ which is a subgraph of $X_{p_i-1} \cup X_{p_i}$ (see Figure 3), in which we have only the paths $P_0, \dots, P_m$ and all Z-bridges in $H_{i-1} \cup H_i$ with all their attachments in $P_0 \cup \cdots \cup P_m$. Let us define the following sets:

$$\begin{aligned} T_i &= S_i \cap \big(\cup_{j=0}^{m} P_j \big), \\ T_i^- &= S_i^- \cap \big(\cup_{j=0}^{m} P_j \big), \quad \text{and} \\ T_i^+ &= S_i^+ \cap \big(\cup_{j=0}^{m} P_j \big). \end{aligned}$$

Consider the set $B = T_i^- \cup T_i^+ \cup \{z_i, z_i^-, z_i^+\} \subseteq V(L_i)$. Suppose that $H_{i-2} \cup H_{i-1} \cup H_i \cup H_{i+1}$ is embedded in the plane. This induces an embedding of L_i with the vertices in B appearing on the boundary of the 'infinite face'. Let $I = V(L_i) \setminus B$ be the set of remaining vertices of L_i. We will apply Euler's formula to the planar graph L_i^* obtained from L_i by first adding edges (if they are not already present) to create a cycle on the vertex set B and then adding an extra vertex of degree $|B|$ in the infinite face adjacent to all vertices of B. Note that $\deg_{L_i^*}(v) = \deg_G(v) \geq 7$ for any $v \in I$. The $2(m-1)$ endvertices in B of the first $m-1$ paths all have degree at least 4 in L_i^*, while z_i^-, z_i^+ and the (one or two) vertices in the path P_m all have degree at least 3. Since z_i is an essential society vertex, its degree in L_i^* is at least 6. Finally, the vertex in the infinite face has degree $2m+3$ or $2m+2$. Euler's formula applied to L_i^* shows that

$$12 \leq \sum_{v \in V(L_i^*)} (6 - \deg_{L_i^*}(v)).$$

By using the degree restrictions listed above, this inequality yields the following. If P_m has only one vertex, then

$$12 \leq (6-7)|I| + (6-4)(2m-2) + (6-3)(3) + (6-(2m+2)). \tag{4}$$

If P_m has two vertices and at least one of them is of degree more than 3 in L_i^*, then

$$12 \leq (6-7)|I| + (6-4)(2m-1) + (6-3)(3) + (6-(2m+3)). \tag{5}$$

Finally, if P_m has two vertices of degree 3 in L_i^*, then we contract the edge joining them and apply the above formula to the resulting graph, obtaining (4) again. In each case, (4) or (5), we conclude that

$$|I| \leq 2m - 2. \tag{6}$$

On the other hand, we will show that I must have at least $2m$ vertices, and thus get a contradiction.

For $l = 1, \dots, m$, let $D^l = \cup_{j=1}^{l} D_i(P_{j-1}, P_j) \cap L_i$ and let $I^l = |V(D^l) \cap I|$. A vertex $v \in V(D^l) \cap I$ has degree at least seven, unless it is contained in the path P_l. In the latter case, we say that v is *d-deficient* if it has at least d incident edges that are not contained in D^l.

We claim that $I^l \geq 2l - 1$, and if equality holds, then there exists a 4-deficient vertex in $P_l \cap I$. Moreover, if $I^l = 2l$, then there exists a 3-deficient vertex. This claim will be proved by induction on l. Clearly, we arrive at a contradiction with (6) when $l = m$, so the proof of this claim will complete the proof of Claim 5.1.

For $l = 1$, consider three neighbors of z_i distinct from z_i^- and z_i^+. Note that at least one of them, let us call it z, is in I. We may assume that $I^1 \leq 2$ since otherwise there is nothing else to prove. If $I^1 = 1$, then z is the only vertex in $D^1 \cap I$, and it is easy to see that it must be on P_1 and that it is 4-deficient. Suppose now that $I^1 = 2$. If $z \notin V(P_1)$, then it has degree at least seven, and it is easy to see that this gives a contradiction to $I^1 = 2$. Therefore $z \in V(P_1)$. Since there is only one other vertex in $D^1 \cap I$ and z is of degree at least 7 in L_i^*, the vertex z is 3-deficient. This completes the proof of the case when $l = 1$.

For the induction step, let us assume that $l \geq 2$ and that the claim holds for values less than l. The proof is easy if $l = m$, so we may also assume that $l < m$. Then P_l contains a vertex $z \in I$ by the definition of m. This completes the proof if $I^{l-1} \geq 2(l-1) + 2$.

If $I^{l-1} = 2(l-1) + 1$, we are done if there is another new vertex besides z contributing to I^l. If z has at most four neighbors in D^l, then it is 3-deficient, and we are done. Otherwise, we apply Euler's formula to D^l in the same way as we did in deriving (6), and we arrive at a contradiction.

Suppose now that $I^{l-1} = 2(l-1)$. By the induction hypothesis, P_{l-1} contains a 3-deficient vertex. The argument is now exactly the same as in the case when $l = 1$, so we omit the details. Finally, if $I^{l-1} = 2(l-1) - 1$, then P_{l-1} contains a 4-deficient vertex z'. We may assume that $I^l \leq 2l$, so one of the neighbors of z' is not in I. So it is one of the ends of $P_l \cap D^l$. The proof is easy to complete when $D^l \setminus D^{l-1}$ has a vertex that is not on P_l. Otherwise, (at least) two of the neighbors of z' are in $P_l \cap I$. It is clear that one of them is 4-deficient. This completes the proof. $\square$

After lots of preparation, we will now be able to construct a $K_{3,k}$-minor by exploiting the crossing paths forced by the nonplanarity of the segments of the vortex. The following claim about crossing paths follows from a result by Robertson and Seymour [15] (see also [11]). It uses the fact that G is 7-connected and that the paths R and R' are chosen in such a way that every Z-bridge in G_1 is attached to at least two of the paths.

Claim 5.2 *If $D_i(R,R')$ is nonplanar, then one of the following holds:*

(a) *$D_i(R,R')$ contains disjoint paths Q_1, Q_2 connecting λ with ρ' and λ' with ρ, respectively.*

(b) *$D_i(R,R')$ contains a path Q (resp., Q') disjoint from R' (resp., R) which connects λ and ρ (resp., λ' and ρ') such that after replacing R (resp., R') by Q (resp., Q'), there is a Z-bridge in H_i which is attached to (at least) three among the paths $P_0, \dots, P_q$ and the surface node u_0.*

We are ready to complete the proof of Theorem 1.3. Recall that Γ_0^+ is a path on consecutive vertices $P_0, \dots, P_{q_0}$ (and u_1, if u_1 exists), where $2 \le q_0 \le \alpha$. Let $D_i^j = D_i(P_j, P_{j+1})$, $j = 1, \dots, q_0$, where P_{q_0+1} is the trivial path u_1 (if it exists).

Claim 5.1 shows that in each of the segments there is a pair of adjacent paths (R, R') such that $D_i(R,R')$ is non-planar. As $n_3 \ge 30k\alpha$, either case (a) occurs for at least $4k\alpha$ indices, or case (b) occurs for at least $2k\alpha$ indices, and all these indices are at least five apart from each other. We need the latter condition in order that the subgraphs $D_i \subseteq H_{i-2} \cup H_{i-1} \cup H_i \cup H_{i+1}$ are disjoint for distinct indices i.

Let us first assume that the case (a) of Claim 5.2 occurs at least $4k\alpha$ times. Then, there exits a pair of paths P_j and P_{j+1} such that, for at least $4k$ indices i_l, the two paths Q_1, Q_2 of Claim 5.2(a) exist in $D_{i_l}(P_j, P_{j+1})$ for $1 \le l \le 4k$. We will now show how to construct a $K_{3,k}$-minor in the graph. Let us construct two paths P' and P'' by first taking the paths P_j and P_{j+1} and then exchanging their segments in $D_{i_l}(P_j, P_{j+1})$ by the crossing paths Q_1 and Q_2 in all of the D_{i_l} with l even. Let us suppose first that $j \ge 1$. If we consider the auxiliary graph with respect to the new paths and the subsequence $p_{i_{2l}}$ $(l = 1, \dots, 2k)$, we see that either the path P_{j-1} (if $j \ge 1$) or the path P_{j+2} (if $j \le q_0 - 2$) is everywhere bridge connected to three other paths. This gives rise to a $K_{3,k}$-minor as it was shown in Claim 4.5.

Let us now assume that the case (b) of Claim 5.2 occurs $2k\alpha$ or more times. Then there is an index $j \in \{0, \dots, q_0\}$, and there are indices $1 \le i_1 < i_2 < \cdots < i_k \le n_3$ such that each of $D_{i_1}^j, D_{i_2}^j, \dots, D_{i_k}^j$ contains a path Q (or each of $D_{i_1}^j, D_{i_2}^j, \dots, D_{i_k}^j$ contains a path Q') as stated in Claim 5.2(b).

For any $D_{i_l}^j$ we replace the segment of P_j (resp., P_{j+1}) by the corresponding path Q (resp., Q') such that there is a Z-bridge (where Z is defined as the union of the new paths) attached to P_j, P_{j+1}, and P_{j+2} (or P_{j-1}). We may assume that k of these bridges, $B_1, \ldots, B_k$ are attached to P_j, P_{j+1}, and P_{j+2}. Now, there is a $K_{3,k}$-minor obtained by contracting P_j, P_{j+1}, P_{j+2} into single vertices and adding paths in $B_1, \ldots, B_k$ to these vertices. This completes the proof of Theorem 1.3.

References

[1] T. Böhme, K. Kawarabayashi, J. Maharry, and B. Mohar, Linear connectivity forces large complete bipartite minors, submitted.

[2] T. Böhme, J. Maharry, B. Mohar, $K_{a,k}$ minors in graphs of bounded tree-width, *J. Combin. Theory Ser. B* **86** (2002) 133–147.

[3] T. Böhme, B. Mohar, Labeled $K_{2,t}$ minors in plane graphs, *J.Combin. Theory Ser. B* **84** (2002) 291–300.

[4] G. Ding, *personal communication.*

[5] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935) 463–470.

[6] M. Juvan, J. Marinček, B. Mohar, Elimination of local brigdes, *Math. Slovaca* **47** (1997) 85–92.

[7] K. Kawarabayashi, J. Maharry, Minors in large 5-connected nonplanar graphs, submitted.

[8] K. Kawarabayashi, B. Mohar, Some recent progress and applications in graph minor theory, *Graphs Combin.* **23** (2007) 1–46.

[9] J. Maharry, A characterization of graphs with no cube minor, *J. Combin. Theory Ser. B* **80** (2000) 179–201.

[10] J. Maharry, An excluded minor theorem for the octahedron, *J. Graph Theory* **31** (1999) 95–100.

[11] B. Mohar, Obstructions for the disk and the cylinder embedding extension problems, *Combin. Probab. Comput.* **3** (1994), 375–406.

[12] B. Mohar, C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press, Baltimore, MD, 2001.

[13] B. Oporowski, J. Oxley, R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Combin. Theory Ser. B* **57** (1993) 239–257.

[14] N. Robertson, *personal communication.*

[15] N. Robertson, P. D. Seymour, Graph minors. IX. Disjoint crossed paths, *J. Combin. Theory Ser. B* **49** (1990) 40–77.

[16] N. Robertson, P. D. Seymour, Excluding a graph with one crossing, *Contemporary Mathematics* **147** (1993) 669–675.

[17] N. Robertson, P. D. Seymour, Graph minors. XVI. Excuding a non-planar graph, *J. Combin. Theory Ser. B* **89** (2003), 43–76.

[18] N. Robertson, P. D. Seymour, Graph minors. XVII. Taming a vortex, *J. Combin. Theory Ser. B* **77** (1999) 162–210.

[19] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937) 570–590.