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General lower bounds for the minor crossing number of graphs

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Abstract

There are three general lower bound techniques for the crossing numbers of graphs: the Crossing Lemma, the bisection method and the embedding method. In this contribution, we present their adaptations to the minor crossing number. Using the adapted bounds, we improve on the known bounds on the minor crossing number of hypercubes. We also point out relations of the minor crossing number to string graphs.

Keywords: minor crossing number, crossing number, graph minor, hypercube, string graphs.

1 Preliminaries

The *minor crossing number* of a graph G on a surface Σ, introduced in [6], is defined as the minimum crossing number of all graphs that contain G as a minor:

$$\mathrm{mcr}(G,\Sigma) := \min\{\mathrm{cr}(H,\Sigma) \mid G \leq_m H\}\ .$$

(As usual, the notation $G \leq_m H$ means that G is a minor of H.) By $\mathrm{mcr}(G)$, we denote $\mathrm{mcr}(G,\mathbb{S}_0)$, the crossing number in the sphere $\mathbb{S}_0$.

For each graph G and each surface Σ there exists a *realizing graph* $\bar{G}$, such that $G \leq_m \bar{G}$ and $\mathrm{mcr}(G,\Sigma) = \mathrm{cr}(\bar{G},\Sigma)$. An optimal drawing of $\bar{G}$ in Σ is called a *realizing drawing of* G. We shall assume that G and $\bar{G}$ have the same number of connected components.

G can be obtained as a contraction of a subgraph of $\bar{G}$. In other words, $G = (\bar{G} - R)/C$ for suitable edge sets $R, C \subseteq E_{\bar{G}}$. The edges of R are called the *removed edges* and those in C are the *contracted edges*. Note that the edge-set C is acyclic and that $E_G = E_{\bar{G}} \setminus (R \cup C)$ are the *original edges* of G. It is clear that every graph G has a realizing graph $\bar{G}$ such that $R = \emptyset$. A stronger claim can be established using the following theorem. (Recall that the Euler genus of an (orientable or nonorientable) surface Σ is defined as $g(\Sigma) = 2 - \chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristics of the surface. So the Euler genus of an orientable surface with k handles is $2k$, and the Euler genus of a nonorientable surface with k crosscaps is k.)

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Theorem 1 ([6], Theorem 5.4) *Let Σ be a surface of Euler genus g and let $k \geq 1$ be an integer. Then $\mathrm{mcr}(G,\Sigma) \leq k$ holds for a graph G, if and only if G can be embedded in the nonorientable surface $\mathbb{N}_{g+k}$ of Euler genus $g+k$, so that there exist pairwise noncrossing onesided curves $\gamma_1,\dots,\gamma_k$ in $\mathbb{N}_{g+k}$, each of which intersects G in at most two points, and such that we obtain Σ by pasting disks into $\mathbb{N}_{g+k}$ along those curves.*

Lemma 2 *Let G be a graph and Σ a surface. There exists a realizing drawing $\bar{D}$ of some realizing graph $\bar{G}$ of G in Σ, such that (i) $|V(\bar{G})| - |V(G)| = |E(\bar{G})| - |E(G)| \leq 2\mathrm{mcr}(G,\Sigma)$, (ii) the crossings in $\bar{D}$ involve only contracted edges of $\bar{G}$, and (iii) no two edges of $\bar{G}$ that are contracted to the same vertex of G cross in $\bar{D}$.*

Proof. Let Σ have Euler genus g, let $k = \mathrm{mcr}(G,\Sigma)$, and let $\gamma_1,\dots,\gamma_k$ be the curves guaranteed by Theorem 1. By moving the curves along the edges, we may assume that the curves intersect the embedding D of G in $\mathbb{N}_{g+k}$ in the vertices of G only.

Let $\Delta_1,\dots,\Delta_k$ be the disks we paste along $\gamma_1,\dots,\gamma_k$ into $\mathbb{N}_{g+k}$ to obtain Σ. Each disk has at most two points of G on its boundary, which correspond to four intertwined points in the cut surface. By adding (at most) two edges and precisely one new vertex for every new edge, and by drawing them as paths on the corresponding disk, we obtain a drawing $\bar{D}$ of a graph $\bar{G}$ in Σ, which has G as a minor. In each disk Δ_i, we introduced at most two new vertices, two new edges and one crossing. As G was embedded in $\mathbb{N}_{g+k}$ with no crossings, the drawing satisfies (i) and (ii).

Suppose one of the crossings would be a crossing of two edges that are contracted to the same vertex of G. We can replace this crossing by a vertex: the new graph still has G as a minor, but its drawing in Σ has fewer than $\mathrm{mcr}(G,\Sigma)$ crossings. The contradiction implies there are no such crossings, thus $\bar{D}$ satisfies (iii). □

2 The Crossing Lemma

The following result was conjectured by Erdős and Guy [9] for a general constant c in place of $\frac{1}{64}$, and then proven by Leighton [16] and independently by Ajtai, Chvátal, Newborn, and Szemerédi [2] for $c = \frac{1}{100}$. A folklore beautiful application of the probabilistic method derives it from the Euler Formula [1]. The sketch of the proof below refers to this proof.

Theorem 3 (The Crossing Lemma, [2, 16]) *Let G be a graph of order n with $m \geq 4n$ edges. Then, $\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}$.*

The Crossing Lemma was improved several times and the current strongest version is by Pach, Radoičić, Tardos, and Tóth [18]. Székely found many applications in discrete geometry [29].

Recall the definition of the generalized Randič index of a connected graph G,

$$R^\alpha(G) = \sum_{\{u,v\}\in E(G)} d_u^\alpha d_v^\alpha,$$

where α is a real number and d_v denotes the degree of the vertex v. Further, we use $\Delta(G)$ and $\delta(G)$ for the largest and the smallest degree of a vertex in G, respectively. Somewhat intriguingly, this quantity, coming from mathematical chemistry, provides a lower bound for the minor crossing number of graphs:

Theorem 4 *Let G be a simple graph without isolated vertices that has girth r and let Σ be a surface of Euler genus g and $\alpha \in \mathbb{R}$. For $\alpha \geq 0$, define $d_\alpha = \Delta(G)$, and let $d_\alpha = \delta(G)$ otherwise. Then, the following holds:*

$$\mathrm{mcr}(G,\Sigma) \geq \frac{r-2}{r+2}\frac{R^\alpha(G)}{d_\alpha^{2\alpha}} - \frac{r}{r+2}\left(\sum_{v\in V(G)}\left(\frac{d_v}{d_\alpha}\right)^\alpha + g\right).$$

Proof. Assume that $\bar{G}$ is a realizing graph of G in Σ, i.e. $\text{mcr}(G,\Sigma) = \text{cr}(\bar{G},\Sigma)$. Assume that a vertex v of G has been substituted by $i_v + 1$ vertices (of which i_v are new vertices) and i_v edges. Apply the Crossing Lemma proof to $\bar{G}$, such that we make independent experiments for the $i_v + 1$-tuples of vertices (i.e. include all or none) with probability $(d_v(G)/d_\alpha)^\alpha \le 1$. Note that unlike in the proof of the Crossing Lemma, the probability distribution is no longer uniform.

By Lemma 2, we can simultaneously assume three properties on $\bar{G}$ and its drawing $\bar{D}$:

(i) $\sum_{v\in V(G)} i_v \le 2\text{mcr}(G,\Sigma)$,

(ii) the crossings in $\bar{G}$ involve only contracted edges of $\bar{G}$, and

(iii) edges that are contracted to the same vertex $v \in V(G)$ do not cross each other in $\bar{D}$.

Let e_v be an edge that is contracted to the vertex $v \in V(G)$. Then,

$$\text{mcr}(G,\Sigma) = \text{cr}(\bar{G},\Sigma) \ge \sum_{e_u,e_v\in E(\bar{G})\ \text{cross}} \left(\frac{d_u}{d_\alpha}\right)^\alpha \left(\frac{d_v}{d_\alpha}\right)^\alpha. \tag{2.1}$$

On the other hand, we take the random subgraph G' of $\bar{G}$ and the corresponding subdrawing of $\bar{D}$ by picking disjoint sets of vertices above with the prescribed probabilities. For any simple graph H, $\text{cr}(H,\Sigma) \ge e(H) - \frac{r}{r-2}(n(H)+g)$ by the Euler Formula. We spell out this inequality for G' and take the expectation. The expected number of vertices in G' is $\sum_{v\in V(G)}(d_v/d_\alpha)^\alpha(i_v+1)$; the expected number of edges in G' is $R^\alpha(G)/d_\alpha^{2\alpha} + \sum_{v\in V(G)}(d_v/d_\alpha)^\alpha i_v$ (corresponding to original and contracted edges), according to the linearity of expectation. Finally, the expected number of crossings of G' in $\bar{D}[G']$ is $\sum_{e_u,e_v\in E(\bar{G})\ cross} \left(\frac{d_u}{d_\alpha}\right)^\alpha \left(\frac{d_v}{d_\alpha}\right)^\alpha$, which is greater than or equal to the expected crossing number of G'. Combining these results with assumption (i), we obtain:

$$\sum_{e_u,e_v\in E(\bar{G})\ \text{cross}} \left(\frac{d_u}{d_\alpha}\right)^\alpha \left(\frac{d_v}{d_\alpha}\right)^\alpha \ge \frac{R^\alpha(G)}{d_\alpha^{2\alpha}} + \sum_{v\in V(G)} \left(\frac{d_v}{d_\alpha}\right)^\alpha i_v - \frac{r}{r-2}\left(\sum_{v\in V(G)}\left(\frac{d_v}{d_\alpha}\right)^\alpha (i_v+1) + g\right)$$

$$\ge \frac{R^\alpha(G)}{d_\alpha^{2\alpha}} - \frac{r}{r-2}\left(\sum_{v\in V(G)}\left(\frac{d_v}{d_\alpha}\right)^\alpha + g\right) - \frac{4}{r-2}\text{mcr}(G,\Sigma),$$

which together with (2.1) implies the claim. □

Combining the same non-uniform probability distribution with the proof of the ordinary Crossing Lemma, we obtain the following:

Theorem 5 *Let G be a simple graph of girth r that has no isolated vertices. For $\alpha \ge 0$, define $d_\alpha = \Delta(G)$, and let $d_\alpha = \delta(G)$ otherwise. If $R^\alpha(G) \ge \frac{3}{2}\frac{r}{r-2} d_\alpha^\alpha \sum_{v\in V(G)} d_v^\alpha$, then the following holds:*

$$\text{cr}(G) \ge \frac{4}{27}\frac{1}{d_\alpha^{4\alpha}}\left(\frac{r-2}{r}\right)^2 \frac{(R^\alpha(G))^3}{\left(\sum_{v\in V(G)} d_v^\alpha\right)^2}.$$

This theorem has little to say about girth in view of results of Pach, Spencer, and Toth [20]. However, we present it as a possible direction of research into combining the Crossing Lemma proofs with non-uniform probability distributions on graph vertices. (In this regard, we acknowledge interesting discussions with Lincoln Lu.)

Note that Theorem 4 simplifies to the bound implied by the Euler Formula in the case $\alpha = 0$; in this case, all probabilities are equal to 1. Similarly, Theorem 5 reduces to the bound of the Crossing Lemma. For $\alpha > 0$, the bound favors vertices with large degree, and, for $\alpha < 0$, it favors vertices with small degree. Note that $\alpha < 0$ may give better results than $\alpha > 0$: an example is a graph $L(n,k)$ obtained from a disjoint union of n copies of a complete graph K_k and an independent set $\bar{K}_k$ by connecting the vertices of complete graphs with corresponding vertices of the independent set, so that in the final graph, all vertices of the independent set have degree n and all vertices of the complete graphs have degree k. Such behavior of the bound from Theorem 4 indicates that the edges incident with vertices of small degree contribute most of the crossings in the graph.

Like the Crossing Lemma, Theorems 4 and 5 can be applied without any knowledge of graph's global structure: the only information we need are the degrees of adjacent vertices. Contrary to the Crossing Lemma, however, Theorem 4 does not require any restrictions on the density of the graph. In fact, it performs best on sparse graphs that have a dense part. For Theorem 5, however, the edge-density condition is replaced by a bound on the generalized Randič index of the graph.

As an example, consider a graph $R(n,m)$, which is obtained by m-times subdividing each edge incident to a fixed vertex v of K_{n+1} and consistently connecting all vertices at a fixed distance $d = 1, 2, \ldots, m$ from v in a cycle (more precisely, number the edges adjacent to v by $1, 2, \ldots, n$, and let $y_{j,i}$ be a subdivision point on edge i at distance j from v. Add the edges $y_{1,i}y_{2,i}$, $y_{2,i}y_{3,i}$, $\ldots$, $y_{m-1,i}y_{m,i}$ and $y_{m,i}y_{1,i}$ for each i) . Then $\mathrm{mcr}(R(n,m)) \le \mathrm{cr}(R(n,m)) = \mathrm{cr}(K_{n+1})$, but the average degree of $R(n,m)$ is close to four for large m, thus, the nonstructural bounds considering just the number of edges and vertices become trivial (Euler bound, Crossing Lemma). Theorem 4, however, produces the following bound:

$$\mathrm{mcr}(R(n,m)) \ge \frac{1}{10}\left(2n(2-3m)\left(\frac{4}{n}\right)^{\alpha} + 2n(2m-1)\left(\frac{4}{n}\right)^{2\alpha} + n^2 - 7n - 6\right).$$

For $n > 4$, $m \ge 0$, and $\alpha \to \infty$, this expression simplifies to $\frac{1}{10}\left(n^2 - 7n - 6\right)$, which is approximately $\frac{2}{5}$ of the best known lower bound for K_n. A similar computation for the graphs $K_{m,n}$ produces $\frac{2}{3}$ of the best known lower bound for $\mathrm{mcr}(K_{m,n})$, applying $\alpha = 0$. Even non-integer values of α can give best results. For instance, the graph $10\,R(199,100) + 10\,(K_{100} * K_{100}^c)$ has largest bound at $\alpha \approx 1.032$ (G^c is the complement of G, $G + H$ is the disjoint union, and $G * H$ is the complete join of G and H).

Similarly for the ordinary crossing number, the edge density condition of the Crossing Lemma is violated for graphs $R(n,m)$ with large m, but with $\alpha \to \infty$, the value of generalized Randič index satisfies the condition of Theorem 5, which thus yields a lower bound of the same order of magnitude and with the same constant factor as the Crossing Lemma. Again, the graph $10\,R(199,100) + 10\,(K_{100} * K_{100}^c)$ is an example with a non-integer optimal $\alpha \approx 2.247$.

3 The bisection method

Leighton was interested in bounded degree graphs for VLSI design. He invented the bisection method and showed that $\mathrm{cr}(G) + n$ is bounded from below by the squared bisection width of the graph mulitplied by a small constant (the bisection width parallels the concepts defined below) [16]. This result was later extended to general graphs by Pach, Shahrokhi and Szegedy [19], who also produced specific constants and replaced n with the sum of degree squares. Independently, Sýkora and Vrťo [28] proved an essentially equivalent result for crossing numbers of general graphs in surfaces of higher genus.

As usual, for $X, Y \subseteq V(G)$, by $\langle X \rangle$ we denote the subgraph of G spanned by X, by $E(X,Y)$ we denote the set of edges $xy \in E(G)$ such that $x \in X$ and $y \in Y$, and $E(X) = E(X,X)$.

Let G be a graph, $\alpha \in (0, 1/2]$, and $W \subseteq V(G)$. A set of edges $F \subseteq E(G)$ is an *α-edge bisection of the vertices of G with respect to W* (in short, *α-edge bisection of G with respect to W*), if $V(G)$ can be partitioned into V_1 and V_2 such that $\min(|V_1 \cap W|, |V_2 \cap W|) \ge \min(\lceil \alpha|W| \rceil, \lfloor \frac{1}{2}|W| \rfloor)$, and every edge between V_1 and V_2 in G belongs to F. We denote by $\mathrm{bw}_e(G, W; \alpha)$ the size of the smallest α-edge bisection of G with respect to W, and use $\mathrm{bw}_e(G, W)$ if $\alpha = 1/2$.

Note that $|W| \ge 2$ implies that $V_i \cap W \ne \emptyset$ for $i \in \{1, 2\}$. The condition $\min(|V_1 \cap W|, |V_2 \cap W|) \ge \min(\lceil \alpha|W| \rceil, \lfloor \frac{1}{2}|W| \rfloor)$ is equivalent with $\max(|V_1 \cap W|, |V_2 \cap W|) \le \max(\lfloor (1-\alpha)|W| \rfloor, \lceil \frac{1}{2}|W| \rceil)$. If $\alpha \le \frac{1}{2} - \frac{1}{2|W|}$ or $|W|$ is even, then $\min(\lceil \alpha|W| \rceil, \lfloor \frac{1}{2}|W| \rfloor) = \lceil \alpha|W| \rceil$, so we can replace the condition on $|V_i \cap W|$ with $\min(|V_1 \cap W|, |V_2 \cap W|) \ge \alpha|W|$, as one would expect. However, if $|W|$ is odd and $\alpha \in (\frac{1}{2} - \frac{1}{2|W|}, \frac{1}{2}]$, then the condition $\min(|V_1 \cap W|, |V_2 \cap W|) \ge \min(\lceil \alpha|W| \rceil, \lfloor \frac{1}{2}|W| \rfloor)$ is equivalent with $\min(|V_1 \cap W|, |V_2 \cap W|) = \lfloor \frac{1}{2}|W| \rfloor < \alpha|W|$.

A set $S \subseteq V$ is an *α-vertex bisection of the edges* of G (in short, an *α-vertex bisection* of G), if the vertices of $G - S$ are partitioned into sets V_1 and V_2, such that the graph induced by V_i, $i = 1, 2$, has at most $(1-\alpha)|E|$ edges; if $E(G) \neq \emptyset$, then we require that, for $i = 1, 2$, $E(V_i \cup S) \neq \emptyset$ and that every path connecting V_1 and V_2 in G contains a vertex from S. Let $\mathrm{bw}_v(G; \alpha)$ denote the cardinality of the smallest α-vertex bisection of G; again we use $\mathrm{bw}_v(G)$ if $\alpha = 1/2$.

A set $S \subseteq V$ is a *strong α-vertex bisection of the edges* of G (in short, a *strong α-vertex bisection* of G), if the vertices of $G - S$ can be partitioned into two sets V_1 and V_2 such that $\min(E(V_1 \cup S), E(V_2 \cup S)) \geq \alpha|E(G)|$, and every path connecting V_1 and V_2 in G contains a vertex from S. Let $\mathrm{bw}^*_v(G; \alpha)$ denote the cardinality of the smallest strong α-vertex bisection of G, again $\alpha = 1/2$ is simply omitted from the notation.

Lemma 6 *Let $\alpha \in (0, 1/2]$. For any $G = (V, E)$, $\mathrm{bw}^*_v(G; \alpha) \geq \mathrm{bw}_v(G; \alpha)$.*

Proof. The statement is trivial if $E(G) = \emptyset$, so assume $E(G) \neq \emptyset$. Let S be a strong α-vertex bisection of G. Then for $i \in \{1, 2\}$ we have that $E(V_i \cup S) \neq \emptyset$ and $|E(V_i)| \leq |E(G)| - |E(V_{3-i} \cup S)| \leq (1-\alpha)|E(G)|$, so S is also an α-vertex bisection of G and the claim follows. □

For a graph $G = (V, E)$, we define $\mathcal{H}_G$ as the set of all graphs $\bar{G}$ that can be obtained from G as follows. For every vertex $v \in V$, subdivide every edge incident to v. Then each edge is subdivided twice. The new vertices are called *leaves*. If v has degree at least three, remove v and attach a tree to the leaves, s.t. the internal vertices of the tree are of degree three. Note that the number of internal tree vertices in every such tree equals the number of leaves less two. The edges of these trees, as well as the edges incident to degree one or two vertices of the original G, are called *tree edges* of $\bar{G} \in \mathcal{H}$. The non-leaf tree vertices are internal vertices. Note that the vertices that have degree one in G are also internal vertices. Let $\bar{G} \in \mathcal{H}_G$, then the number of internal tree vertices at every tree T_v, $v \in G$, equals at least the number of leaves less two (equality holds if the corresponding vertex v in G has degree at least three). For $\bar{G} \in \mathcal{H}_G$ we denote the set of leaves by L. Then $|L| = 2|E|$. Let I be the set of all internal vertices of trees, then $|I| \geq |L| - 2|V| = 2|E| - 2|V|$ (with equality if the minimum degree of G is 3). The newly introduced concepts are related in Lemma 7.

Lemma 7 *Let $\alpha \in (0, 1/2]$. For any $G = (V, E)$ without isolated vertices,*

$$\mathrm{bw}^*_v(G; \alpha) \geq \min_{\bar{G} \in \mathcal{H}_G} \mathrm{bw}_e(\bar{G}, L; \alpha) \geq \mathrm{bw}^*_v(G; \alpha) - 1 \geq \mathrm{bw}_v(G; \alpha) - 1.$$

Proof. Again, if $E = \emptyset$, the statement is trivial, so let G be a graph such that $E \neq \emptyset$, so $|L| = 2|E| \geq 2$. By Lemma 6 it is enough to prove that $\mathrm{bw}^*_v(G; \alpha) \geq \min_{\bar{G} \in \mathcal{H}_G} \mathrm{bw}_e(\bar{G}, L; \alpha) \geq \mathrm{bw}^*_v(G; \alpha) - 1$.

Let $\bar{G} \in \mathcal{H}_G$ be a graph that has minimal α-edge bisection width with respect to L, and let F be the smallest α-edge bisection of $\bar{G}$ with respect to L, i.e., $|F| = \mathrm{bw}_e(\bar{G}, L; \alpha)$. Since $|L| = 2|E|$ is even, the graph $\bar{G} - F$ consists of subgraphs $\bar{G}_1$, $\bar{G}_2$ both having at least $\lceil \alpha|L| \rceil > 0$ leaves.

Any edge $e \in F$ not belonging to any of the trees that replaced the vertices of G in $\bar{G}$ is adjacent with precisely one tree edge e_1 in $\bar{G}_1$ and another tree edge e_2 in $\bar{G}_2$. We obtain a set F' by replacing every such $e \in F$ with one of e_i, such that the sides are chosen evenly; then at most one non-tree edge e is left in F', and $|F| = |F'|$. The set F' separates $\bar{G}$ into $\bar{G}'_1$ and $\bar{G}'_2$. Since every leaf in $\bar{G}'_i$ is incident to a single non-tree edge in $E(\bar{G}'_i) \cup \{e\}$, and every non-tree edge is incident upon precisely two leaves, the number of non-tree edges in $E(\bar{G}'_i) \cup \{e\}$ is at least $\frac{\alpha|L|}{2} = \alpha|E|$. By contracting every tree of $\bar{G}$ to a vertex we get the original graph G. For $i = 1, 2$, let G_i be a subgraph of G spanned by the vertices that were contracted from trees containing leafs from $\bar{G}'_i$. Note that G_1 and G_2 share vertices that correspond to trees containing edges from F'. Then $E(G_i)$ contains the non-tree edges of $E(\bar{G}'_i) \cup \{e\}$, and so $|E(G^*_i)| \geq \alpha|E|$. Let S be the set of vertices in G that correspond to trees containing edges from F' and the endvertices of e. Clearly, the vertices of $G - S$ can be partitioned into V_1 and V_2 such that $V_i \subseteq V(G_i) \subseteq V_i \cup S$ and every path from V_1 to V_2 goes through a vertex of S. Also, $|E(S \cup V_i)| \geq |E(G_i)| \geq \alpha|E|$. Then S is a strong α-vertex bisection of G of size at most $|F| + 1$ (and the edge e is in $\langle V_i \cup S \rangle$ for both $i = 1, 2$). Hence we proved the second inequality:

$$\mathrm{bw}_e(\bar{G}, L; \alpha) = |F| \geq |S| - 1 \geq \mathrm{bw}^*_v(G; \alpha) - 1.$$

For the first inequality, let $S \subseteq V$ be a strong α-vertex bisection in G such that $|S| = \mathrm{bw}_v^*(G;\alpha)$, and let $V_1, V_2 \subseteq V$ be the partition of G corresponding to S. Now either $\lceil \alpha|E| \rceil \leq \lfloor \frac{1}{2}|E| \rfloor$ or $\lceil \alpha|E| \rceil = \lceil \frac{1}{2}|E| \rceil > \lfloor \frac{1}{2}|E| \rfloor$. If $\lceil \alpha|E| \rceil \leq \lfloor \frac{1}{2}|E| \rfloor$, then let $C = \emptyset$. If $\lceil \alpha|E| \rceil = \lceil \frac{1}{2}|E| \rceil > \lfloor \frac{1}{2}|E| \rfloor$, then we must have $E(S) \neq \emptyset$; in this case let $C = \{x_1x_2\}$ for some fixed $x_1x_2 \in E(S)$. Let $c = |C|$ and $\alpha^* = \min(\lceil \alpha|E| \rceil, \lfloor \frac{1}{2}|E| \rfloor)$. In particular, $2\alpha^* \leq |E|$, and if $\lceil \alpha|E| \rceil > \alpha^*$, then $\lceil \alpha|E| \rceil = \alpha^* + 1$ and $|E|$ is odd. Moreover, $|E - C| = 2 \lfloor \frac{1}{2}|E| \rfloor$.

If $c = 0$, then $2\alpha^* + c = 2\alpha^* = 2\lceil \alpha|E| \rceil \geq \lceil 2\alpha|E| \rceil$. If $c = 1$, then E is odd, $2\alpha \leq 1$, and $2\alpha^* + c = |E| \geq \lceil 2\alpha|E| \rceil$. Thus $2\alpha^* + c \geq \lceil 2\alpha|E| \rceil$.

Let a_i be the number of edges in $A_i = E(V_i) \cup E(V_i, S)$, and let b be the number of edges in $E(S) - C$. Set $b_1 = \max(\alpha^* - a_1, 0)$, $b_2 = b - b_1$, and let E_i contain the edges of A_i and b_i edges of $E(S) - C$, such that E_1 and E_2 are disjoint. Then $|E_1| = a_1 + b_1 \geq \alpha^*$ and either $|E_2| = a_2 + b_2 \geq a_2 + b \geq \alpha^*$ or $|E_2| = a_2 + b_2 \geq a_2 + b + a_1 - \alpha^* \geq \alpha^*$. Therefore $|E_i \cup C| \geq \lceil \alpha|E| \rceil$ in all cases.

We design a graph $\bar{G} \in \mathcal{H}_G$ as follows: insert trees on the leaves corresponding to vertices of V_1 and V_2 arbitrarily. If $C \neq \emptyset$, let l_1l_2 be the non-tree edge in $\bar{G}$ corresponding to the edge x_1x_2 in G. For $s \in S, s \notin \{x_1, x_2\}$, let $L_{s,i}$ be the set of leaves corresponding to s that are endpoints of edges in E_i. Insert a tree on the leaves corresponding to s such that a removal of an edge e_s separates the leaves $L_{s,1}$ from $L_{s,2}$. For $F = \{e_s \mid s \in S\} \cup \{l_1l_2\}$, the graph $\bar{G} - F$ consists of two graphs $\bar{G}_i$ containing the set of leaves L_i of size at least $|L_i| \geq 2\alpha^* + c \geq \lceil \alpha|L| \rceil$. Therefore F is an α-edge bisection of $\bar{G} \in \mathcal{H}$ implying $\mathrm{bw}_v^*(G;\alpha) \geq \mathrm{bw}_e(\bar{G}, L; \alpha)$ and proving the first inequality. □

We establish a lower bound on the minor crossing numbers of graphs using Lemma 7 together with the following theorem:

Theorem 8 ([3], Theorem 3.1) *Let G be an n-vertex graph with non-negative vertex-weights w such that G is embeddable in an orientable surface of Euler genus g. For any $\varepsilon \in (0,1)$, there exists a set S, such that $|S| \leq 4\sqrt{\left(\frac{g}{2} + \frac{1}{\varepsilon}\right) n}$ and no component of $V(G - S)$ has weight more than $\varepsilon\, w(G)$.*

Theorem 8 or some similar result on separators together with the standard iterative technique for producing an α-edge bisection of size at most $\frac{c\Delta(G)}{1-\varepsilon^p} n^p$ using separators of size at most cn^p, $0 < p < 1$, [11, 17, 27, 28] implies the following:

Corollary 9 *Let G be an n-vertex graph of maximum degree Δ embeddable in an orientable surface of Euler genus g. Let $L \subset V$ and $\varepsilon \in (0,1)$, $\alpha \in (0, \frac{1}{2}]$. Then there exists an α-edge bisection of G with respect to L of size at most $\frac{4\Delta}{1-\sqrt{\varepsilon}} \sqrt{(\frac{g}{2} + \frac{1}{\varepsilon})n}$.*

Proof. Assign weight 1 to all vertices from the set L and 0 to all remaining vertices. According to Theorem 8, there is a set $S_1 \subseteq V(G)$ of size $4\sqrt{\left(\frac{g}{2} + \frac{1}{\varepsilon}\right) n}$ whose removal leaves no component of weight larger than $\varepsilon|L|$. Let C_1 be the largest component; then we can group the other components into two sets $A_1, B_1 \subseteq V(G)$, such that there are no edges between A_1 and B_1 in $G - S_1$, and $w(A_1), w(B_1) \leq (1-\alpha)|L|$. If $w(B_1 \cup C_1) \leq (1-\alpha)|L|$, we are done, otherwise we proceed by separating C_1 using a set S_1' of size at most $4\sqrt{\left(\frac{g}{2} + \frac{1}{\varepsilon}\right) \varepsilon n}$ into components of weight at most $\varepsilon^2|L|$. Then we can add all but at most one (call it C_2) of those components to sets A_1 and B_1 obtaining $A_2 \supseteq A_1$, $B_2 \supseteq B_1$, such that $w(A_1), w(B_1) \leq (1-\alpha)|L|$ and there are no edges between A_2 and B_2 in $G - S_2$, $S_2 = S_1 \cup S_1'$: the same argument as before applies. Iterating this procedure, we obtain sequences $A_1 \subseteq \ldots \subseteq A_k$, $B_1 \subseteq \ldots \subseteq B_k$, $C_1 \supseteq \ldots \supseteq C_k$, and $S_1 \subseteq \ldots \subseteq S_k$, such that $w(A_i) \leq w(B_i) \leq (1-\alpha)|L|$, $w(C_i) \leq \varepsilon^i|L|$ for $i = 1, \ldots, k-1$, $w(C_k) = 0$, and

$$|S_k| \leq \sum_{i=0}^{k} 4\sqrt{\left(\frac{g}{2} + \frac{1}{\varepsilon}\right) \varepsilon^i n} \leq \frac{4}{1-\sqrt{\varepsilon}} \sqrt{\left(\frac{g}{2} + \frac{1}{\varepsilon}\right) n}.$$

Let F be the set of at most $\Delta|S_k|$ edges, incident with vertices of S_k. The vertices of S_k are isolated vertices in $G - F$ and can be properly distributed among the sets A_k and B_k, so that we obtain A_{k+1} and B_{k+1} that are a partition of $V(G)$ with $w(A_{k+1}), w(B_{k+1}) \geq \lfloor \alpha|L| \rfloor$. Thus F is an α-edge bisection of G with respect to L. □

Theorem 10 *Let $G = (V, E)$ be a graph with minimum degree at least three, Σ an orientable surface of Euler genus g. For every $\alpha \in \left(0, \frac{1}{2}\right]$, there exists $c_g > 0$ such that*

$$\mathrm{mcr}(G, \Sigma) \geq c_g \, \mathrm{bw}_v^*(G; \alpha)^2 - 4|E|.$$

Proof. We prove the theorem with $c_{g,\varepsilon} = \frac{\varepsilon(1-\sqrt{\varepsilon})^2}{128(g\varepsilon+2)}$ for any $\varepsilon \in (0, 1)$. Let $\bar{G}'$ be a realizing graph of G, i.e., $\mathrm{mcr}(G, \Sigma) = \mathrm{cr}(\bar{G}', \Sigma)$. As G has minimum degree three, we may assume that $\bar{G}'$ is cubic (cf. [6]) and we can obtain a graph $\bar{G} \in \mathcal{H}_G$ by subdividing each original edge of $\bar{G}'$ twice. Using a standard technique of Leighton [16], we will prove that

$$\mathrm{cr}(\bar{G}, \Sigma) \geq c_{g,\varepsilon} \, \mathrm{bw}_e(\bar{G}, L; \alpha)^2 - 4|E| + 2|V|. \tag{3.2}$$

Then Lemma 7 will imply $\mathrm{mcr}(G, \Sigma) \geq c_{g,\varepsilon} \, (\mathrm{bw}_v^*(G; \alpha) - 1)^2 - 4|E| + 2|V| \geq c_{g,\varepsilon} \, \mathrm{bw}_v^*(G; \alpha)^2 - 4|E|$.

Let D be an optimal drawing of $\bar{G}$ and replace every crossing of D with a new vertex. We get graph G_D on $\mathrm{cr}(\bar{G}, \Sigma) + |L| + |I| = \mathrm{cr}(\bar{G}, \Sigma) + 4|E| - 2|V|$ vertices, whose maximum degree is four, embedded in Σ. (Recall that the internal vertices I and the leaf vertices L were defined before Lemma 7.) Assign the weight 1 to every leaf vertex and weight zero to other vertices in G_D. Corollary 9 implies that there exists an α-edge bisection F_D of G_D with respect to L of size

$$|F_D| \leq \tfrac{16}{1-\sqrt{\varepsilon}} \sqrt{(\tfrac{g}{2} + \tfrac{1}{\varepsilon})|V_D|}.$$

Each edge of F_D corresponds to a unique edge of $\bar{G}$, but two edges in F_D may correspond to the same edge of $\bar{G}$. Thus the set F of edges of $\bar{G}$ corresponding to the edges of F_D has size at most $|F_D|$. Since F is an α-edge bisection of $\bar{G}$ with respect to L, we established

$$\tfrac{16}{1-\sqrt{\varepsilon}} \sqrt{(\tfrac{g}{2} + \tfrac{1}{\varepsilon})} \sqrt{\mathrm{cr}(\bar{G}, \Sigma) + 4|E| - 2|V|} \geq |F| \geq \mathrm{bw}_e(\bar{G}, L; \alpha),$$

which implies (3.2). □

4 The embedding method

Let H, G be two graphs. An *embedding* of H into G is a pair of injections $\omega = \langle \lambda, \Lambda \rangle$, $\lambda : V(H) \to V(G)$, $\Lambda : E(H) \to \{P \mid P \text{ is a path in } G\}$, such that $\Lambda(e)$ is a path in G from $\lambda(u)$ to $\lambda(v)$ for any edge $e = uv \in E(H)$. The paths $\Omega_\omega = \{\Lambda(e) \mid e \in E(H)\}$ are called *ω-active paths*. The *edge congestion* $\mu_\omega(e)$ of an edge $e \in E(G)$ is the number of active paths using e, and the *vertex congestion* $m_\omega(v)$ of a vertex $v \in V(G)$ is the number of active paths using the vertex $v \in V(G)$. Edge congestion μ_ω and vertex congestion m_ω of the embedding ω are the maximum corresponding values over all the edges or vertices. Given an embedding of H into G, the following theorem bounds the crossing number of G in terms of the crossing number of H:

Theorem 11 ([25]) *Let G be a graph of order n, ω an embedding of a graph H into G with edge-congestion μ_ω and vertex congestion m_ω, and Σ any surface. Then,*

$$\mathrm{cr}(G, \Sigma) \geq \frac{\mathrm{cr}(H, \Sigma)}{\mu_\omega^2} - \frac{n}{2} \left(\frac{m_\omega}{\mu_\omega} \right)^2.$$

For our purposes, we need to refine the above statement. Let ω be an embedding of a graph H into a graph G. For a pair of edges $e, f \in E(H)$ let P be a component of $\Lambda(e) \cap \Lambda(f)$. Clearly, P is a path in G. If e and f are adjacent and P contains a λ-image of their common endvertex, then P is a *starting component*, otherwise P is a *non-starting component* of $\Lambda(e) \cap \Lambda(f)$. We denote with $o_\omega(e, f)$ the number of non-starting components of $\Lambda(e) \cap \Lambda(f)$. Let $o_\omega = \sum_{\{e,f\} \in \Pi_\omega} o_\omega(e, f)$. With $\Pi_\omega \subseteq \binom{E(H)}{2}$ we denote the set of entangled edge pairs of H: a pair $\{e, f\} \subseteq E(H)$ is *entangled*, if $o_\omega(e, f) > 0$.

Theorem 12 *Let G be a graph and $\omega = (\lambda, \Lambda)$ an embedding of a graph H into G with edge-congestion μ_ω. Then* $\mathrm{cr}(G) \geq \frac{1}{\mu_\omega^2} \left\{ \mathrm{cr}(H, \Sigma) - o_\omega \right\}$.

Proof. Let D be a drawing of G and let D' be the subdrawing, induced by the edges of $\Lambda(E(H))$. Using the embedding ω as in [25], we construct a drawing D_H of H as follows. First, we draw each vertex $v \in V(H)$ into the D-image of $\lambda(v)$. Second, we draw each edge $e \in E(H)$ in a small neighborhood of the drawing $D[\Lambda(e)]$ of the embedding path $\Lambda(e)$ parallel with that path. We say that such D_H *respects* D and ω.

In D_H, there are precisely two types of crossings. Crossings of type (i) arise in small neighborhood of some crossing x of D: if $\Lambda(e)$ and $\Lambda(f)$ each uses a different edge of G that crosses at x, then e and f cross in a crossing of type (i) in the neigbhorhood of x.

Crossings of type (ii) arise in small neighborhoods of some vertex v of G: if $v \in \Lambda(e) \cap \Lambda(f)$, then e and f may cross in a crossing of type (ii) in the neigbhorhood of v.

The construction of such a drawing D_H alone implies Theorem 11, as there are at most μ_ω^2 crossings of type (i) at every crossing of D, and at most $m_\omega^2/2$ crossings of type (ii) at every vertex of G. The improvement follows from elimination and a more sharp counting of crossings of type (ii).

The sharper counting relies on the obvious fact that crossings of type (ii) appear only at vertices of G. Thus, they can appear only at vertices of $\Lambda(e) \cap \Lambda(f)$ for some $e, f \in E(H)$.

We claim that there exists D_H with (a) at most one e, f-crossing of type (ii) per non-starting component of $\Lambda(e) \cap \Lambda(f)$ and with no type (ii) e, f-crossings at starting components of $\Lambda(e) \cap \Lambda(f)$.

Assume no such D_H exists and let $\bar{D}$ be the drawing respecting D and ω with the smallest number of violations of (a). Further, assume that $P \subseteq G$ is a component of $\Lambda(e) \cap \Lambda(f)$.

If $v \in P$ is an endvertex of both $\Lambda(e)$ and $\Lambda(f)$ and e, f cross in $\bar{D}$ in a small neighborhood of $w \in P$, we can flip e and f in a small neighborhood of P such that the crossing is eliminated. The new drawing still respects D and ω as P is a component of $\Lambda(e) \cap \Lambda(f)$, and both drawings $\bar{D}[e]$ and $\bar{D}[f]$ are routed in small neighborhoood of P. But the new drawing has a smaller number of violations of (a) than $\bar{D}$, a contradiction to the choice of $\bar{D}$.

So we may assume that P is a non-starting component with at least two type (ii) crossings of e and f. Let x, y be the vertices of P in whose small neighborhoods in $\bar{D}$ the two crossings appear. If we flip the edges e and f in small neighborhoods of the crossings, the new drawing still respects $\bar{D}$ and ω as in the previous paragraph and has a smaller number of violations of (a) than $\bar{D}$, another contradiction to the choice of $\bar{D}$. We conclude that (a) holds.

In D_H, there are at most μ_ω^2 type (i) crossings at any crossing of G, and altogether at most o_ω type (ii) crossings. Thus,

$$\mu_\omega^2 \operatorname{cr}(G, \Sigma) + o_\omega \geq \operatorname{cr}(H, \Sigma),$$

and the claim follows. □

In special circumstances that are of interest in [5], Theorem 12 could be improved:

Theorem 13 *Let G be a graph and $\omega = (\lambda, \Lambda)$ an embedding of a graph H into G with edge-congestion $\mu_\omega = 1$, such that all ω-entangled pairs of edges of H have the same common endvertex v. Then any drawing D of G has at least $\operatorname{cr}(H, \Sigma)$ crossings in $D[\Omega_\omega]$ that do not involve two edges of Λ-images of ω-entangled pairs of edges.*

Proof. Let D_H be a drawing of H respecting D and ω with smallest number of type (ii) crossings, as in the proof of Theorem 12. We augment D_H to D'_H and produce an embedding $\omega' = (\lambda', \Lambda')$, such that D'_H respects D and ω', $\mu_{\bar{\omega}} = 1$, $\Lambda'(E(H))$ contain the same edges as $\Lambda(E(H))$, and D'_H has no type (ii) crossings.

If D_H has no type (ii) crossings, then $\omega' = \omega$ and $D'_H = D_H$, otherwise let x be a type (ii) crossing in a small D_H-neighborhood of a vertex $w \in V(G)$ and let e, f be the two edges of H crossing at x.

As $\mu_\omega = 1$, x is a non-starting component of $\Lambda(e) \cap \Lambda(f)$. So let P_e and P_f be the maximum common $w - v$ segments of $\Lambda(e)$ and $\Lambda(f)$, respectively. We alter $\omega = (\lambda, \Lambda)$ to $\omega' = (\lambda, \Lambda')$, so that $\Lambda'(e)$ uses P_f and $\Lambda'(f)$ uses P_e, but otherwise they are equal. By flipping e and f in a small D_H-neighborhood of x, we obtain D'_H that respects ω'. After performing such a change at every type (ii) crossing of D_H, the final D'_H respects the final ω', and D_H has no type (ii) crossings.

We further produce a drawing D''_H, which has no type (i) crossings on the Λ-images of the ω-entangled pairs of H-edges. As all such edges are incident with v, we can uncross them at any type (i) crossing involving such edges, even if the two edges are not from the same entangled pair. The new drawing is still a drawing of H and has at least $\operatorname{cr}(H, \Sigma)$ crossings, and we deduce that all these crossings are type (i) crossings appearing in small neighborhoods of crossings of D. As $\mu_\omega = 1$, each such type (i) crossing corresponds to a unique crossing of D, so D has at least $\operatorname{cr}(H, \Sigma)$ crossings of which none involves two Λ-images of ω-entangled pairs of edges. □

The ideas behind Theorem 13 could be applied in more general settings, too. By eliminating the crossings in a specific setting of a given embedding and a given drawing respecting that embedding, the bounds could be further improved, either by decreasing the multiplicative factor (in our case μ_ω) or by decreasing the subtracted constant (in our case o_ω).

Let G be a graph and $\bar{G}$ its realizing graph in some surface Σ. For a vertex v in $V(G)$, let T_v be the tree in $\bar{G}$ that is contracted to v. For any path $P = u_0 \ldots u_t$ in G of positive length, we define its *lift* $\bar{P}$ to be the path in $\bar{G}$ that uses every edge e_i of $\bar{G}$ corresponding to $u_{i-1}u_i \in E(G)$, $i = 1, \ldots, t$, and connects the edges e_i and e_{i+1} with the unique path in T_{u_i} connecting their endvertices. Formally, $\bar{P} = e_1 T_{u_1} e_2 T_{u_2} \ldots T_{u_{t-1}} e_t$.

If $\omega = \langle \lambda, \Lambda \rangle$ is an embedding of a graph H into G and $\bar{G}$ a realizing graph of G in some surface Σ, then a *lift* of ω is any embedding $\bar{\omega} = \langle \bar{\lambda}, \bar{\Lambda} \rangle$ of H into $\bar{G}$, for which $\bar{\lambda}(v) \in V(T_{\lambda(v)})$ for every $v \in V(H)$; and, for $e = uv$, $\bar{\Lambda}(e)$ is the path containing the lift $\bar{P}$ of $\Lambda(e)$ extended by the path connecting $\bar{\lambda}(u)$ with the initial vertex of $\bar{P}$ and the path connecting the endvertex of $\bar{P}$ with $\bar{\lambda}(v)$. Note that we have the freedom of choosing $\bar{\lambda}(v) \in T_{\lambda(v)}$. After that, the lifts of paths are uniquely defined.

Lemma 14 *Let G, H be two graphs and $\bar{G}$ a realizing graph of G in some surface Σ. Further, let $\omega = \langle \lambda, \Lambda \rangle$ be an embedding of H into G and let $\bar{\omega} = \langle \bar{\lambda}, \bar{\Lambda} \rangle$ be a lift of ω. Then $o_{\bar{\omega}}(e, f) = o_\omega(e, f)$ for any pair of edges $e, f \in E(H)$ and consequently $o_{\bar{\omega}} = o_\omega$.*

Proof. Let $e, f \in E(H)$ and let $P = v_1 e_1 v_2 e_2 \ldots e_{k-1} v_k$ be a component of $\Lambda(e) \cap \Lambda(f)$. If P is a starting component, then (by symmetry) we may assume v_1 is the λ-image of a common endvertex w of e and f. By definition of the lift $\bar{\omega}$, $\bar{\lambda}(w)$ is a vertex in T_{v_1}. Furthermore, $\bar{\Lambda}(e)$ and $\bar{\Lambda}(f)$ contain the edges $e_1, \ldots, e_k$, and as there are unique endvertices of corresponding edges in T_{v_i}, $i = 1, \ldots, k-1$ and unique paths connecting these endvertices in T_{v_i}, the respective lifts of P into $\bar{G}$ are in the same component $\bar{P}$ of $\bar{\Lambda}(e) \cap \bar{\Lambda}(f)$ in $\bar{G}$. In T_{v_k}, however, only the endvertex w' of e_{k-1} is a common T_{v-k}-leaf of the lift, the other leaf is distinct for each of $\bar{\Lambda}(e)$, $\bar{\Lambda}(f)$. But, as T_{v_k} contains no cycles, $\bar{\Lambda}(e) \cap \bar{\Lambda}(f) \cap T_{v_k}$ has only one component that is a path $\bar{P}'$ and contains w'. As w' is in $\bar{P}$ and $\bar{P}'$, $\bar{P}'$ is a part of $\bar{P}$. Thus there is a unique component of $\bar{\Lambda}(e) \cap \bar{\Lambda}(f) \cap T_{v_k}$ that corresponds to P. A similar reasoning applies if P is not a starting component: in that case, the T_{v_1}-endvertex of e_1 defines the only component $\bar{P}$ of $\bar{\Lambda}(e) \cap \bar{\Lambda}(f) \cap T_{v_1}$ which contains $e_1 T_{u_1} e_2 T_{u_2} \ldots T_{u_{t-1}} e_t$. Thus, for every component P of $\Lambda(e) \cap \Lambda(f)$ there exists a component $\bar{P}$ of $\bar{\Lambda}(e) \cap \bar{\Lambda}(f)$. As $\bar{\Lambda}(e)$ is a path for every $e \in E(H)$, there is only one component $\bar{P}$ for each P, and the claim follows. □

We define $s_\omega(v)$ to be the number of ω-active paths starting at $v \in V(G)$ and $t_\omega(v)$ to be the number of active paths passing through v. Then $m_\omega(v) = s_\omega(v) + t_\omega(v)$, but we define $\nu_\omega(v) = \left\lfloor \frac{1}{2} s_\omega(v) \right\rfloor + t_\omega(v)$ and $\nu_\omega = \max(\max_{v \in V(G)} \nu_\omega(v), \max_{e \in E(G)} \mu_\omega(e))$. As follows, this refinement strengthens the translation of the embedding method to the minor crossing number in such a way, that the result generalizes the lower bound on $\mathrm{mcr}(G, \Sigma)$ in terms of $\mathrm{cr}(G, \Sigma)$ and $\Delta(G)$ from [6, 10].

Theorem 15 *Let G be a graph and $\omega = \langle \lambda, \Lambda \rangle$ an embedding of a connected nonempty graph H into G. Then,*

$$\mathrm{mcr}(G, \Sigma) \geq \frac{1}{\nu_\omega^2} \left\{ \mathrm{cr}(H, \Sigma) - o_\omega \right\}.$$

Proof. Let $\bar{G}$ be a realizing graph of G.

Claim 0: There exists a lift $\bar{\omega} : H \to \bar{G}$, $\bar{\omega} = \langle \bar{\lambda}, \bar{\Lambda} \rangle$, such that for every $v \in V(G)$ and $e \in E(T_v)$, $\mu_{\bar{\omega}}(e) \leq \left\lfloor \frac{1}{2} s_\omega(v) \right\rfloor + t_\omega(v)$.

Claim 0 implies $\mu_{\bar{\omega}} \leq \nu_\omega$. As $o_{\bar{\omega}} = o_\omega$ by Lemma 14, Theorem 12 applied to $\bar{G}$ implies the theorem.

Now we prove Claim 0. According to a previous remark, it is enough to define $\bar{\lambda}(v')$ for any $v' \in V(H)$ so that the bound of Claim 0 holds. Let v be a vertex of G and T_v the tree in $\bar{G}$ contracted to v. If $s_\omega = 0$, then there are t_ω active paths using v, none of them as a starting vertex. In $\bar{\omega}$, at most the corresponding t_ω lifted paths can use any $e \in T_v$, so the claim holds. Therefore we may assume that $s_\omega \geq 1$, so $v = \lambda(v')$ for some $v' \in V(H)$ with $d_{v'} \geq 1$.

As the paths going through v contribute at most $t_\omega(v)$ to $\mu_{\bar{\omega}}(e)$ as in the previous paragraph, we may for simplicity assume that $t_\omega(v) = 0$. Let $e = u_1 u_2$ be an edge of T_v. The forest $T_v - e$ has two components T_1 and T_2

with $u_i \in V(T_i)$, $i=1,2$. Let E_i be the set of all original edges of G, incident with T_i, and let $\mu_i = \sum_{e \in E_i} \mu_\omega(e)$. Note that, $(*)$ if $\bar{\lambda}(v') \in V(T_i)$, then $\mu_{\bar{\omega}}(e) = \mu_{3-i}$. If $\mu_i < \mu_{3-i}$, we direct the edge e from u_i to u_{3-i}, otherwise we leave the edge e undirected. With T_e we denote the component T_{3-i} and with T'_e the component T_i. If $\mu_1 = \mu_2$, then T_{e,u_i} denotes the tree T_i. With $\mu(T)$ we denote the sum of $\mu_\omega(e)$ for all original edges incident with T.

Claim 1: *Each vertex $u \in V(T_v)$ has at most one incident outgoing edge.* Suppose the edges e and f are both directed away from u. Since T'_e contains T_f and T'_f contains T_e, this would imply $\mu(T_f) \le \mu(T'_e) < \mu(T_e) \le \mu(T'_f)$, a contradiction.

Claim 2: *If $u \in V(T_v)$ is incident with an undirected edge, then there is no outgoing edge incident with u.* Let u be incident with an outgoing edge e and an undirected edge $f = uu'$. Since f is undirected, $\mu(T_{f,u}) = \mu(T_{f,u'}) = \frac{1}{2}m_v$. Then, $\mu(T_e) > \mu(T'_e) \ge \mu(T_{f,u'}) = \frac{1}{2}m_v$, since $T_{f,u'}$ is a subtree of T_e. This contradicts $\mu(T_e) + \mu(T'_e) = m_v$.

Claim 3: *$u \in V(T_v)$ can have at most two incident undirected edges.* Let $e_i = uu_i$, $i = 1,2,3$, be three incident undirected edges. Then $\mu_1 = \mu_2 + \mu_3 + \mu'$, $\mu_2 = \mu_1 + \mu_3 + \mu'$, and $\mu_3 = \mu_1 + \mu_2 + \mu'$ for $\mu_i = \mu(T_{e_i,u_i})$. Since $\mu_i, \mu' \ge 0$, this implies $\mu_1 = \mu_2 = \mu_3 = \mu' = 0$, contradicting $d_{v'} \ge 1$.

Claim 4: *The subgraph of T_v induced by undirected edges is connected.* By Claim 1 and Claim 2, the unique path $e_1 \dots e_t$ of T_v connecting two undirected edges e_1 and e_t has only incoming edges, therefore the path can contain undirected edges only.

Claims 3 and 4 establish that the subgraph spanned by the undirected edges of T_v is a connected graph P of maximum degree two, and by Claim 1 and 2, all edges of T incident with P are directed into P. If there is no undirected edge, then T_v is a directed acyclic graph and must have a vertex $P = u$ of out-degree zero. If we embed v' into P, then Claim 0 follows by $(*)$. □

The inequality $\mathrm{mcr}(G,\Sigma) \ge \mathrm{cr}(G,\Sigma) / \left\lfloor \frac{\Delta(G)}{2} \right\rfloor^2$, proved in [10] for $\Delta(G) = 4$ and in [6] for general $\Delta(G)$, is a simple consequence of Theorem 15: If we embed G into G using the canonical injection ι, then $s_\iota = \Delta(G)$ and $t_\iota = 0$ imply $\nu_\iota = \left\lfloor \frac{\Delta(G)}{2} \right\rfloor$, which together with $o_\iota = 0$ implies the inequality.

Note that there are two ways of using an embedding $\omega : H \to G$ to obtain a lower bound for $\mathrm{mcr}(G,\Sigma)$ in terms of $\mathrm{cr}(H,\Sigma)$. We can apply it directly using Theorem 15, in which case the lower bound is roughly $\mathrm{cr}(H,\Sigma)/\nu_\omega^2$, or we can first apply Theorem 12 to obtain a lower bound on $\mathrm{cr}(G,\Sigma)$, and then use the embedding $\iota : G \to G$ from the previous paragraph, in which case we obtain a bound, roughly equal to $4\,\mathrm{cr}(H,\Sigma)/(\Delta\mu_\omega)^2$. The direct approach is preferable whenever $2\nu_\omega \le \Delta\mu_\omega$, otherwise the indirect approach yields a better bound.

5 Applications

5.1 Hypercubes

Theorem 16 *For the n-dimensional hypercube Q_n, $\mathrm{bw}_v(Q_n) \ge \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} = \frac{2^{n+1}}{\sqrt{2\pi n}}(1 - o(1))$.*

Proof. We prove the claim for odd n. The even case is similar. Consider an optimal vertex bisection S of edges of Q_n, which separates the hypercube into $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $|V_1| \ge |V_2|$, such that $|E_1| \le 2^{n-2}n$ and $|E_2| \le 2^{n-2}n$. By contradiction, we prove

$$|S| \ge \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} = \frac{2^{n+1}}{\sqrt{2\pi n}}(1 - o(1)).$$

For $A \subseteq V$, define the vertex boundary of A as $\partial_v(A) = \{u \in V - A : \text{there exists } w \in A, uw \in E\}$ and the edge boundary of A as $\partial_e(A) = \{uv \in E : u \in A, v \in V - A\}$.
Case 1: $|V_1| \le 2^{n-1} + \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Since $|V_1| \ge |V_2|$, this implies

$$|V_2| = 2^n - |V_1| - |S| > 2^{n-1} - \binom{n}{\lfloor \frac{n}{2} \rfloor} - \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{\lfloor \frac{n}{2} \rfloor - 2},$$

$$|V_2| \le 2^{n-1} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Distinguish two subcases. If

$$|V_2| \leq \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1},$$

then

$$|V_2| = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{\lfloor \frac{n}{2} \rfloor - 2} + \alpha \binom{n}{\lfloor \frac{n}{2} \rfloor - 1},$$

for some $0 < \alpha \leq 1$. According to Bollobás and Leader [8, Corollary 2],

$$|\partial_v(V_2)| \geq (1-\alpha)\binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + \alpha \binom{n}{\lfloor \frac{n}{2} \rfloor} > \binom{n}{\lfloor \frac{n}{2} \rfloor - 1}.$$

This, however, contradicts to

$$|\partial_v(V_2)| \leq |S| < \binom{n}{\lfloor \frac{n}{2} \rfloor - 1}.$$

If

$$|V_2| > \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1},$$

then

$$|V_2| = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + \alpha \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Similarly,

$$|\partial_v(V_2)| \geq (1-\alpha)\binom{n}{\lfloor \frac{n}{2} \rfloor} + \alpha \binom{n}{\lfloor \frac{n}{2} \rfloor + 1} = \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

A contradiction again.
Case 2: $|V_1| > 2^{n-1} + \binom{n}{\lfloor \frac{n}{2} \rfloor}$. We have $2|E_1| + |\partial_e(V_1)| = |V_1|n$, implying

$$2^{n-1}n + |\partial_e(V_1)| > 2^{n-1}n + \binom{n}{\lfloor \frac{n}{2} \rfloor} n,$$

$$|\partial_e(V_1)| > \binom{n}{\lfloor \frac{n}{2} \rfloor} n.$$

This is a contradiction to

$$|\partial_e(V_1)| \leq |\partial_e(S)| \leq |S|n < \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} n.$$

□

Applying Theorem 16 in combination with Lemma 7 and Theorem 10, we obtain the following corollary:

Corollary 17 *For every orientable surface Σ of Euler genus g, there exists a constant $c_g > 0$, such that*

$$\mathrm{mcr}(Q_n, \Sigma) \geq c_g \frac{4^n}{n}(1 - o(1)).$$

for the n-dimensional hypercube Q_n.

For the sake of completness, we provide an upper bound:

Theorem 18 *For the n-dimensional hypercube Q_n,*

$$\mathrm{mcr}(Q_n) \leq \frac{4^n}{\sqrt{\pi n}} - 2n.$$

Proof. We make a staircase drawing of Q_n by first identifying the vertices of Q_n with the subsets of $[n]$. Each vertex corresponds to a line in the drawing; lines whose sets have even number of elements are horizontal, lines whose sets have odd number of elements are vertical. Lines intersect other lines only when the cardinalities of the two sets differ by 1. This gives the bound

$$\operatorname{mcr}(Q_n) \le \sum_{i=1}^{n-2} \binom{n}{i}\binom{n}{i+1} = \frac{n4^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+2)} - 2n \le \frac{4^n}{\sqrt{\pi n}} - 2n.$$

□

A similar approach can be applied to the Hamming graphs $K_p^n = \prod_{i=1}^n K_p$. They are $n(p-1)$-regular, have p^n vertices, $n(p-1)p^n/2$ edges, and are a natural generalization of hypercubes, which are Hamming graphs for $p = 2$. It is known that $\operatorname{cr}(K_p^n) = \Theta(p^{2n+2})$ [30], thus the embedding method lower bound gives $\operatorname{mcr}(K_p^n) = \Omega(\frac{p^{2n}}{n^2})$. Combining Theorem 10 with the vertex boundary estimations in [12] and the approach used in Theorem 16, we get a better estimate $\operatorname{mcr}(K_p^n) = \Omega(\frac{p^{2n}}{n})$.

5.2 String representation of graphs in the plane

A graph is called a *string graph*, if its vertices are represented by simple curves in the plane, and two vertices are connected by an edge if and only if the corresponding simple curves intersect. Benzer [4] was motivated by biology and Sinden [26] by electrical engineering to ask which graphs are string graphs. Ron Graham deserves much credit for recognizing the importance of the problem and making it known. Although Kratochvíl [14, 15] showed that the recognition problem of string graphs is NP-hard, only recently were Pach and Tóth [21] and independently Schaefer, Sedgwick, and Štefankovič [23, 24] were able to show that the recognition problem of string graphs is decidable and is in NP. The basis of this result is an upper bound on how many crossings a drawing proving that an n-vertex graph is a string graph may need (note that a pair of crossing curves may intersect many times, and this may be even needed for the string graph representation).

In view of this history, it is surprising that the following analogue of planar graph drawings has not been considered before. Represent the vertices of the graph G by simple curves in the plane, and make sure that any two curves representing the endpoints of an edge of G intersect, but allow intersection of curves representing non-adjacent vertices. Minimize the total number of intersections over all pairs of curves. We call this quantity minus $|E(G)|$ the *string crossing number* of G and denote it by $i(G)$. Note that this definition is analogous to $i(.)$ in Richter and Thomassen [22] and Juarez and Salazar [13], where it was applied to closed curves. We have an interesting observation, supported by Propositions 19 and 20: $i(G)$ is intimately related to $\operatorname{mcr}(G)$.

Proposition 19 *Let G be a graph. Then, $i(G) \le 4\operatorname{mcr}(G)$.*

Proof. Let $\bar{G}$ be a realizing graph of G as in Lemma 2, T_v be the tree in $\bar{G}$ corresponding to $v \in V(G)$, and let $e_1, e_2, ..., e_m$ be the edges leaving T_v in $\bar{G}$. Recall that all crossings in the optimal drawing of $\bar{G}$ occur between tree edges of different trees. Extend T_v into a bigger tree by adding to it "half" of the edges $e_1, e_2, ..., e_m$ (till their midpoint). Draw now a closed curve C'_v "very near" around this extended tree in such a way that if T_u and T_v share an extended edge e that C'_u and C'_v touch at the midpoint of e, but have no more points in common. To obtain the string C_v cut open the closed curve C'_v. Note that for $uv \in E(G)$, $|C_u \cap C_v| = 1$, and a crossing of $e \in E(T_v)$ and $f \in E(T_u)$ results in at most 4 common points of C_u and C_v. $v \mapsto C_v$ is the required string representation of G. □

Proposition 20 *For any graph G with $t(G)$ tree components,* $\operatorname{mcr}(G) \le i(G) + |E(G)| - |V(G)| + t(G)$.

Proof. First we prove $\operatorname{mcr}(G) \le i(G) + |E(G)| - |V(G)|$ for a graph G with $\delta(G) \ge 2$. Assume that $v \mapsto C_v$ is a string representation of G with a drawing D that realizes $i(G)$. In other words, the strings intersect $|E(G)| + i(G)$ times in D. We can assume without loss of generality that no three curves pass through any point. Observe the $|E(G)|$ intersection points in the string representation that represent edges of G. If C_u and C_v have a point p in common that represents an edge e of G, choose vertices $u_e \in C_u$ and $v_e \in C_v$ very close to p such that they can be connected with a curve C_e not creating any additional crossings with C_u, C_v, or any additional C_f. Furthermore, we can make sure that the first and last point thus added to any curve C_v is within the segment of C_v bounded by the first and last crossings on C_v. We create a new graph $\hat{G}$ with a drawing $\hat{D}$ as follows: the points of this graph are the points v_e on the curves C_v. The edges of $\hat{G}$ are drawn as follows: along the curve C_v connecting neighboring points v_e and v_f, and the curves C_e. We obtained a drawing $\hat{D}$ of a graph $\hat{G}$ containing G as a minor. Since $\delta(G) \geq 2$, this drawing removes a crossing from both ends of each string; so it removes a total of $|V(G)|$ crossings from D. The number of crossings in $\bar{D}$ is therefore at most $i(G) + |E(G)| - |V(G)|$; wherever the curves C_u and C_v touch (not cross), we could eliminate an additional crossing.

To conclude, let G be a general graph and let G_1 be the graph obtained from G by removing all tree components of G. Clearly $\mathrm{mcr}(G_1) = \mathrm{mcr}(G)$, $i(G_1) = i(G)$, $|E(G_1)| - |V(G_1)| = |E(G)| - |V(G)| + t(G)$, and every component in G_1 has at least one cycle. Create G_2 from G_1 by iteratively removing degree one vertices. Since every component of G_1 has a cycle, G_2 is nonempty, has the same number of components as G_1, $\delta(G_2) \geq 2$, $\mathrm{mcr}(G_2) = \mathrm{mcr}(G_1)$, $i(G_2) = i(G_1)$, and $|E(G_2)| - |V(G_2)| = |E(G_1)| - |V(G_1)|$. The lower bound follows by the previous lower bound applied to G_2. □

Propositions 19 and 20 easily extend to general surfaces, but not Corollary 21. Combining these propositions with known results on the minor crossing number from previous sections and from [6, 7], we obtain bounds on the string crossing number for several families of graphs as well as general bounds. Perhaps most interesting is the following observation that follows from Proposition 19 and the results of [7], which claim that $\mathrm{mcr}(G) \leq c_H|V(G)|$ for H-minor-free graphs G:

Corollary 21 *For every graph H, there exists a constant c_H, such that every H-minor-free graph G has string crossing number $i(G) \leq c_H|V(G)|$.*

In other words, Corollary 21 implies that graphs with no prescribed minor have their deficiency to being string graphs linear in their order. Proposition 20 with Corollary 17 implies that

$$i(Q_n) = \Omega\left(\frac{4^n}{n}\right) = \Omega\left(\frac{|V(Q_n)|^2}{n}\right),$$

thus exclusion of a minor is necessary in Corollary 21. It is easy to see that the bound in the proposition is best possible.

We conclude with an observation that string crossing number of G can be defined in different ways analoguously with the crossing number: the *faithful string crossing number* counts just the crossings among strings representing non-adjacent vertices of G, and the *pair string crossing number* counts just the pairs of non-adjacent vertices whose strings cross. The pair string crossing number actually counts the minimum number of edges that need to be added to G to obtain a string graph containing G. The inequalities between these variants of string crossing number are obvious, but it is unclear whether there are any equalities.

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