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Preprint series, Vol. 46 (2008), 1056

HIGHER-ORDER VORONOI DIAGRAMS ON TRIANGULATED SURFACES

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ISSN 1318-4865

October 2, 2008

Ljubljana, October 2, 2008

Higher-Order Voronoi Diagrams on Triangulated Surfaces

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September 26, 2008

Abstract

We study the complexity of higher-order Voronoi diagrams on triangulated surfaces under the geodesic distance, when the sites may be polygonal regions of constant complexity. More precisely, we show that in a surface defined by n triangles the sum of the combinatorial complexity of the order-j Voronoi diagrams, for $j = 1, \ldots, k$, is $O(k^2n^2 + k^2m + knm)$, which is asymptotically tight in the worst case.

1 Introduction

The diverse generalizations of Voronoi diagrams have important applications in several fields and application areas, such as computer graphics, geometric modelling, geographic information systems, visualization of medical data-sets, pattern recognition, robotics and shape analysis. Given a set of m primitives, called *sites*, in an ambient space equipped with a metric, the order-k Voronoi diagram, where k is an integer between 1 and $m-1$, partitions the ambient space into regions such that each point within a fixed region has the same k closest sites. Many variants of these diagrams can be found in the literature by considering diverse ambient spaces, taking sites of different shape or nature, associating weights to the sites, or changing the underlying metrics [3, 4, 11]. In this paper we concentrate on Voronoi diagrams defined on triangulated surfaces for polygonal sites of constant-description complexity.

Surfaces, sites, and bisectors. Let $\mathcal{P}$ be a triangulated polyhedral surface consisting of n triangles. The surface $\mathcal{P}$ may have nonzero genus. The distance $d(p, q)$ between two points $p, q \in \mathcal{P}$ of the surface is given by their geodesic distance, that is, the length of the shortest path between the two points that is contained in the surface.

A *(generalized) site* s in $\mathcal{P}$ is a region contained in $\mathcal{P}$ with constant-description complexity; it may be a point, a segment, a polygonal path with a bounded number of segments, or a region enclosed by a closed polygonal path with a bounded number of segments. In general, we will drop the term 'generalized' when referring to generalized sites. The distance $d(p, s)$ from a point $p \in \mathcal{P}$ to a site $s \in S$ is defined by the length of the shortest path from p to any point of s, that is, $d(p, s) = \min_{q \in s} d(p, q)$.

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Let $S = \{s_1, \dots, s_m\}$ be a set of m generalized sites in $\mathcal{P}$. We assume that the sites are pairwise interior-disjoint. The *bisector* $\beta(s, s')$ defined by sites s and s' is the locus of points at the same distance from s, s'. When a vertex of the surface $\mathcal{P}$ is equidistant from two sites, it may happen that a bisector contains two-dimensional pieces. We exclude this case as degenerate, and assume henceforth that no vertex of the surface is equidistant from two sites. With this assumption, a bisector consists of straight-line, parabolic and hyperbolic arcs [6, 8]. A *breakpoint* of a bisector is an intersection point between two adjacent arcs of the bisector. A breakpoint corresponds either to a point where the bisector crosses an edge of $\mathcal{P}$ or to a point where there are two different shortest paths from one of the sites defining the bisector[1]. According to previous results [6, 8], a bisector has $O(n^2)$ breakpoints.

Order-k Voronoi diagram. We are interested in the order-k Voronoi diagrams of the sites S. We next recall its definition, together with other related concepts. For a subset S' of the sites, the Voronoi region or cell $V(S')$ of S' is defined as

$$\{p \in \mathcal{P} \mid d(p, s') \leq d(p, s), \forall s' \in S', \forall s \in S \setminus S'\}.$$

For each integer $1 \leq k \leq m - 1$, the order-k Voronoi diagram of S, $Vor_k(S)$, is the family of nonempty Voronoi regions $V(S')$, where S' ranges over all subsets of S with k sites. When $k = 1$ and $k = m - 1$ the order-k Voronoi diagrams are called the *(closest-site) Voronoi* and the *furthest-site Voronoi* diagram, respectively. A *k-region* is a Voronoi region in the order-k Voronoi diagram. A *k-edge* is a connected component in the intersection of two k-regions. A k-edge always lies on the bisector of two sites. A *k-breakpoint* is a breakpoint on a k-edge. A *k-vertex* is the intersection point of three or more k-regions.

The terms edges and vertices of the order-k Voronoi diagram are chosen because of their resemblance with a graph embedded in the surface $\mathcal{P}$. The removal of edges from the surface leaves a set of *faces*. The closure of a face corresponds to a order-k Voronoi cell, and vice versa. Finally, let us remark the difference between vertices and breakpoints: while vertices correspond to endpoint of edges, breakpoints can only appear along the relative interior of edges.

The *(combinatorial) complexity* of the order-k Voronoi diagram is the total number of k-regions, k-edges, k-vertices and k-breakpoints.

Our results. Let $Vor_{\leq k}(S)$ denote the family of all order-j Voronoi diagrams of S for $j = 1, \dots, k$. Let us define the complexity of $Vor_{\leq k}(S)$ as the sum of the complexities of the order-j Voronoi diagrams, $j = 1, \dots, k$. We show that the complexity of $Vor_{\leq k}(S)$ is $O(k^2n^2 + k^2m + knm)$, and that this bound is asymptotically tight in the worst case. We also prove that the complexity of $Vor_{\leq k}(S)$ in so-called realistic terrains (the concept is explained in Section 4) is $O(k^2n + k\sqrt{n}m + k^2m)$.

Related work. In the Euclidean plane $\mathbb{R}^2$, which can be regarded as a surface of constant complexity, the order-j Voronoi diagram of a set of m points has complexity $O(j(m-j))$, and hence the complexity of $Vor_{\leq k}(S)$ is $O(mk^2)$ [7, 12]. In the case of a triangulated surface consisting of n triangles and a set of m sites, the complexity of the closest Voronoi diagram is $O(n(n + m))$, while the number of vertices is $O(m)$ [8, 10]. Aronov et al. [1] show that the furthest-site Voronoi diagram has complexity $\Theta(mn^2)$ in the worst-case. These previous results for surfaces were written for the case where the sites are points; in this case the edges consist of straight-line and hyperbolic

[1]This definition of breakpoint is consistent with [1] but different from [2, 9].

arcs. However, the results directly extend to generalized sites, where the bisectors may also include parabolic arcs.

2 Pathologies of order-k Voronoi diagrams.

2.1 Edges without vertices

Although the definition of edges and vertices resemble those of graphs embedded in $\mathcal{P}$, the situation may be quite different.

Lemma 1 *There exist order-k Voronoi diagrams with k-edges that form closed curves in P and are not adjacent to any k-vertex.*

Proof. Let us consider a prism with triangular base. Consider a line ℓ parallel to an edge of the prism from one base to the other, as in Fig. 1. We place m sites on ℓ, no pair of bisectors intersect and some of them define the boundary a k-cells. Consequently k-cells may have closed curve edges without vertices. □

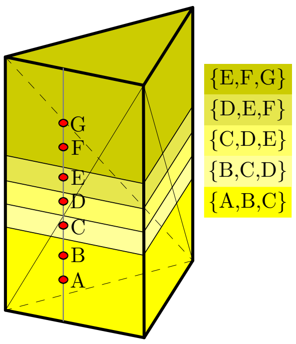

Figure 1: Example of a order-3 Voronoi diagram with all the regions without vertices.

A key property that we use below is that a k-edge that is not adjacent to any k-vertex must contain a breakpoint, as otherwise it would consist of a single arc contained in a triangle of $\mathcal{P}$, which is not possible.

2.2 Path-connectivity of the cells

The cells of the closest-site or furthest-site Voronoi diagram are path-connected [1]. In the Euclidean plane, any k-cell is also path-connected, since in fact any k-cell is convex. The situation is different for the general case.

Lemma 2 *For any given $k \geq 2$, there exists a polyhedral surface and a set of sites such that the order-k Voronoi diagram has some disconnected cells.*

Proof. We describe a polyhedral surface $\mathcal{P}$ embedded in 3-dimensional Euclidean space. Consider three squares, Q_i, $i = 1 \ldots 3$, with sides of length 1.1, 1 and 0.9, respectively. They are placed in planes parallel to the xy-plane, with their centers on the line $x = y = 0$, and with their sides parallel to the x, y coordinate axis. Squares Q_1 and Q_3 are placed at height zero and Q_2 at height

1000. Let $\mathcal{P}$ be the polyhedral surface obtained by gluing the xy-plane minus the interior of Q_1, the pyramid with bases Q_1 and Q_2 (without the bases), the pyramid with bases Q_2 and Q_3 (without the bases), and the square Q_3. Let ℓ be the curve obtained by intersecting $\mathcal{P}$ with the vertical plane $y = 0$. See Fig. 2 for a projection onto the xy-plane.

We place $3k$ point-sites along ℓ, as follows. First, we place k sites in Q_3, symmetrically respect to its center. We then place k sites along ℓ in each side outside Q_1, distributed in four groups: S_1^l, S_2^l, S_1^r and S_2^r. Groups S_1^l, S_1^r contain $\lceil k/2 \rceil$ points, while groups S_2^l, S_2^r contain $\lfloor k/2 \rfloor$ points. Within each group, the points are regularly spaced at distance ϵ, where $0 < \epsilon \ll 1/k$. The points S_1^l are placed to the left of Q_1, along ℓ, at distance ϵ from Q_1, and the points S_2^l at distance $1/2$ from S_1^l. We do the same on the right of Q_1 but placing S_2^r closer to Q_1 than S_1^r (see Fig. 2). The construction is symmetric with respect to the plane $x = 0$. (It is also symmetric with respect to the plane $y = 0$ when k is even.)

Consider the Voronoi region R having $S_1^l \cup S_2^m$ as closest sites. First, note that R is nonempty because it contains the point $(0, 1, 0)$ in its interior, and hence R is a k-region. However, no point on ℓ is in R, and since the construction is symmetric, this means that R has at least two path-connected components. The construction for $k = 7$ is shown in Fig. 2 (xy-projection). □

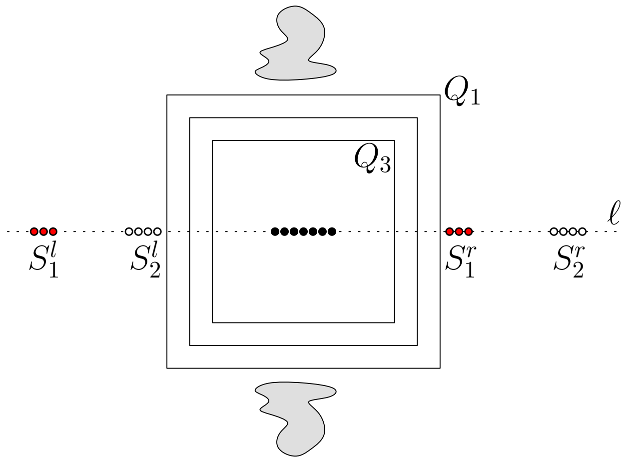

Figure 2: Example of an order-7 Voronoi diagram with disconnected cells.

3 Complexity of order-($\leq k$) Voronoi diagrams

In this section we will give bounds for the complexity of $V_{\leq k}(S)$. Our approach is similar to the approach used by Clarkson and Shor [5] or Sharir [13]. Let B_j be the number of j-breakpoints, and let $B_{\leq k}$ be the number of j-breakpoints for $j = 1, \dots, k$. Similarly, let V_j be the number of j-vertices and let $V_{\leq k}$ be the number of j-vertices for $j = 1, \dots, k$.

Lemma 3 *The complexity of the order-k Voronoi diagram is proportional to $B_k + V_k$.*

Proof. Each edge that is not adjacent to a vertex (c.f. Lemma 1) contains some breakpoints. Therefore, the number of k-edges that are not adjacent to any vertex is bounded by B_k. The number of k-edges that are adjacent to some vertex is bounded by $O(V_k)$, as follows by applying Euler's formula to the graph defined by the k-vertices and the k-edges adjacent to some k-vertices. Finally, the number of k-cells is bounded by the number of k-edges because of Euler's formula. □

Lemma 4 *We have $B_{\leq k} = O(kn(kn+m))$ for any $1 \leq k \leq m-1$.*

Proof. Draw a random sample R of S, by independently drawing each element of S with probability p. We will set below p to an appropriate value. Let $B(R)$ be the number of breakpoints of the order-1 Voronoi diagram of R in $\mathcal{P}$. From the bounds on the complexity of order-1 Voronoi diagrams, discussed in Section 1, we know that $B(R) = O(n^2 + |R|n)$. Since $|R|$ follows a binomial distribution, we have

$$\mathbb{E}\left[|R|\right] = mp,$$

and therefore

$$\mathbb{E}\left[B(R)\right] = O(n^2 + nmp). \tag{1}$$

Consider a breakpoint b in some j-edge e of the order-j Voronoi diagram of S. Let s, s' be the two sites defining the bisector that contains b. Note that a disk centered at b and with radius $d(b,s)$ contains s, s' and $j-1$ sites, which we denote S_b. The point b is a breakpoint in the order-1 Voronoi diagram of the random sample R if and only if s, s' are in R and any other of the $j-1$ sites S_b are not in R. Therefore, the probability that b is a breakpoint in the order-1 Voronoi diagram of R is precisely $p^2(1-p)^{j-1}$. By linearity of expectation we then have

$$\begin{aligned}
\mathbb{E}\left[B(R)\right] &= \sum_{j=1}^{m-1} B_j p^2 (1-p)^{j-1} \\
&\geq p^2 \sum_{j=1}^{k} B_j (1-p)^{j-1} \\
&\geq p^2 (1-p)^{k-1} \sum_{j=1}^{k} B_j \\
&= p^2 (1-p)^{k-1} B_{\leq k}.
\end{aligned}$$

Manipulating and substituting equation (1) we see

$$\begin{aligned}
B_{\leq k} &\leq \frac{\mathbb{E}\left[B(R)\right]}{p^2(1-p)^{k-1}} \\
&= \frac{O(n^2 + nmp)}{p^2(1-p)^{k-1}}.
\end{aligned}$$

Finally, setting $p = 1/k$ we obtain

$$\begin{aligned}
B_{\leq k} &= O\left(\frac{n^2 + nm/k}{(1/k)^2(1-1/k)^{k-1}}\right) \\
&= O\left(k^2 n^2 + knm\right),
\end{aligned}$$

where we have used that

$$\frac{1}{(1-1/k)^{k-1}} = \left(\frac{k}{k-1}\right)^{k-1} \leq e.$$

□

Lemma 5 *We have $V_{\leq k} = O(k^2 m)$ for any $1 \leq k \leq m-1$.*

Proof. This proof is very similar to the previous one. Draw a random sample R from S, by independently drawing each element of S with probability p. Let $V(R)$ be the number of vertices of the order-1 Voronoi diagram of R in $\mathcal{P}$. From the known bounds, as discussed in Section 1, we have $V(R) = O(|R|) = O(mp)$.

Consider a j-vertex v, and let s_1, s_2, s_3 be the three sites defining v: v is in the intersection of the bisectors $\beta(s_1, s_2)$, $\beta(s_1, s_3)$, $\beta(s_2, s_3)$. A disk centered at v and with radius $d(v, s_1)$ contains s_1, s_2, s_3 and possibly some other sites, which we denote by S_v. The cardinality of S_v is either $j-2$ or $j-1$. The point v is a vertex in the order-1 Voronoi diagram of the random sample R if and only if s_1, s_2, s_3 are in R and any other of the sites S_v are not in R. Therefore, the probability that b is a breakpoint in the order-1 Voronoi diagram of R is precisely $p^3(1-p)^{|S_v|} \geq p^3(1-p)^{j-1}$. By linearity of expectation we then have

$$\begin{aligned}
\mathbb{E}[V(R)] &= \sum_{j=1}^{m-2} V_j p^3 (1-p)^{j-1} \\
&\geq p^3 \sum_{j=1}^{k} V_j (1-p)^{j-1} \\
&= p^3 (1-p)^{k-1} V_{\leq k}.
\end{aligned}$$

Like in the previous proof, we manipulate, substitute $V(R) = O(mp)$, and set $p = 1/k$ to get

$$\begin{aligned}
V_{\leq k} &\leq \frac{\mathbb{E}[V(R)]}{p^3(1-p)^{k-1}} \\
&= \frac{m/k}{(1/k)^3(1-1/k)^{k-1}} \\
&= O(k^2 m).
\end{aligned}$$

□

Combining Lemmas 3–5 we can conclude our main result:

Theorem 1 *Let $\mathcal{P}$ be a polyhedral surface with n triangles, let S be a set of m sites on $\mathcal{P}$, and let k be an integer between 1 and $m-1$. The complexity of $Vor_{\leq k}(S)$ is $O(k^2n^2 + k^2m + knm)$.* □

3.1 Lower bounds.

We will show that the bound of Theorem 1 is asymptotically optimal in the worst case. For this, we give three constructions, each showing the need of one summand in the bound $O(k^2n^2 + k^2m + knm)$.

A construction attaining complexity $\Omega(mk^2)$ can be done as follows. Consider the Euclidean plane and observe that the complexity of $Vor_{\leq k}(S)$ for a set S of m points is $\Omega(mk^2)$ [7]. Clipping the plane with a large enough rectangle, so that $n = O(1)$, we obtain the desired bound.

A construction attaining complexity $\Omega(knm)$ can be done as follows. Consider the Euclidean plane, and take the point sites $s_i = (i, 0)$, $i = 1, \ldots, m$. Sites with consecutive indices, like $s_i, s_{i+1}, \ldots, s_{i+j-1}$, define a j-cell, which is a vertical strip. Therefore, Vor_j consists of $m - j + 1$ cells, and $Vor_{\leq k}$ has $\Omega(km)$ cells. Clip the plane with a sufficiently large axis-parallel rectangle,

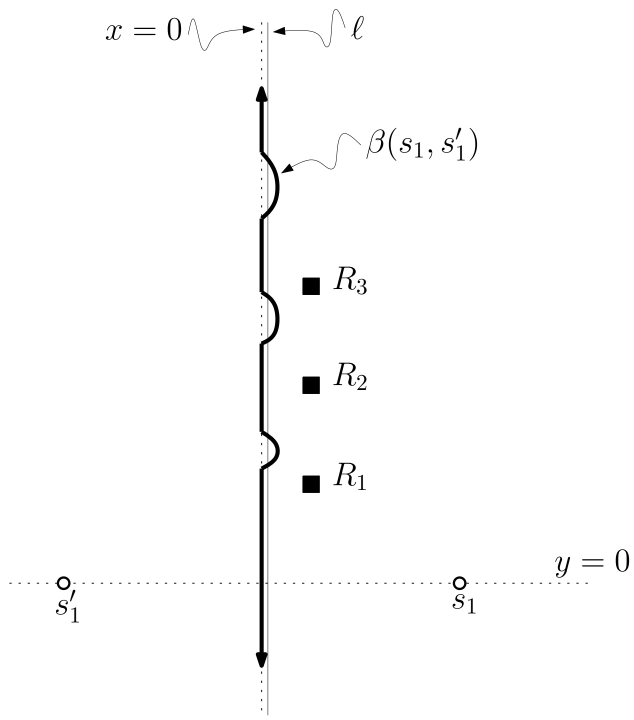

Figure 3: Schema for the lower bound $\Omega(k^2n^2)$ when $n = 3$; the hyperbolic arcs are symbolic and only the sites s_1, s_1' are shown.

and triangulate it with n 'long' edges that connect the left boundary to the right boundary. Each such 'long' edge intersects each cell in $Vor_{\leq k}$, and hence the complexity of $Vor_{\leq k}$ is $\Omega(kmn)$.

A construction attaining complexity $\Omega(k^2n^2)$ can be done as follows. Set $m = 2k$. Consider the Euclidean plane and define the $m/2$ point sites $s_i = (2, 0) + (\varepsilon \cdot i, 0)$, for $i = 1, \ldots, k$, where $\varepsilon > 0$ is a sufficiently small value to be set below. Define also the sites s_i' by $s_i' = -s_i$, for $i = 1, \ldots, k$. Note that the bisector between s_i and s_i' is the line $x = 0$. Let us place n small rectangular obstacles along the line $x = 1/2$. For concreteness, let R_t be the rectangle with endpoints $(1/2-1/2n, t-1/2n), (1/2-1/2n, t+1/2n), (1/2+1/2n, t+1/2n), (1/2+1/2n, t-1/2n)$, for $t = 1, \ldots, n$. We erect an almost vertical wall in each of the edges of each R_t. This is like effectively considering each R_t as an obstacle. Note that there is a line through s_i that separates the rectangle R_t from R_{t+1}, for any $i = 1, \ldots, k$ and $t = 1, \ldots, n$. See Figure 3 for a schema.

Let β denote the bisector between sites s_1 and s_1'. The bisector β is not a line anymore, but contains $\Theta(n)$ straight-line arcs—contained in the line $x = 0$–and $\Theta(n)$ hyperbolic arcs—contained in the halfplane $x \geq 0$. Consider a value $\delta > 0$ sufficiently small such that the vertical line $x = \delta$ intersects each hyperbolic arc in β. Let ℓ denote the vertical line $x = \delta/2$. We can take ε to be a sufficiently small value such that the line ℓ intersects all the hyperbolic arcs in any of the bisectors between a site s_i and a site s_j', for any $i, j \in \{1, \ldots, k\}$. This is possible because, looking at a bounded subregion of the plane, the bisector $\beta(s_1, s_1')$ changes continuously with a small perturbation of the sites s_1, s_1'. Therefore, the line ℓ intersects $\Theta(n)$ times the bisector $\beta(s_i, s_j')$, for any $i, j \in \{1, \ldots, k\}$. We clip the surface with a large enough rectangle, and triangulate it adding n edges that are parallel to ℓ and very close together; each of the added edges should intersect $\Theta(n)$ times any bisector $\beta(s_i, s_j')$.

Let $S_{i,j}$ denote the subset of sites

$$\{s_1, s_2, \ldots, s_i\} \cup \{s'_1, s'_2, \ldots, s'_j\}.$$

Note that a point p in the plane whose x coordinate is between -1 and 1 is closer to s_i than to s_{i+1} for any $i \in \{1, \ldots, k-1\}$. Similarly, such point is closer to s'_j than to s'_{j+1}. This implies that the $i + j$ sites $S_{i,j}$ define a nonempty $(i + j)$-cell in $Vor_{i+j}(S)$ whose boundary is determined by the bisectors $\beta(s_i, s'_{j+1})$ and $\beta(s_{i+1}, s'_j)$. Therefore, the boundary of $V(S_{i,j})$ has complexity $\Omega(n^2)$ because bisector $\beta(s_i, s'_{j+1})$ is intersected n times by each of the n edges parallel to ℓ. We conclude that any set $S_{i,j}$ defines a $(i+j)$-cell of complexity $\Omega(n^2)$. Since there are $\Theta(k^2)$ pairs of values i, j adding to at most k, this means that $Vor_{\leq k}$ has at least $\Omega(k^2)$ cells of complexity $\Omega(n^2)$. Therefore in this construction the complexity of $Vor_{\leq k}$ is $\Omega(k^2 n^2)$.

4 Complexity of order-k Voronoi diagrams on Realistic surfaces

We next restrict ourselves to so-called realistic terrains [9]. A *terrain* $\mathcal{T}$ is a polyhedral surface embedded in $\mathbb{R}^3$ that is homeomorphic to a topological disk and is xy-monotone—a vertical line intersects $\mathcal{T}$ at most once. The *domain* of a terrain is the projection of $\mathcal{T}$ onto the xy-plane. Let $\alpha, \beta, \gamma, \delta$ be constants independent of n, m. A *realistic terrain* is a terrain where:

1. the domain is a rectangle whose aspect ratio is upper bounded by α;
2. the slope of any segment contained in $\mathcal{T}$ is upper bounded by $\beta < \frac{\pi}{2}$;
3. the ratio between the longest and the shortest edge in the xy-projection of $\mathcal{T}$ is upper bounded by γ;
4. the triangulation of the domain induced by the xy-projection is a γ-low-density triangulation: any square Q intersects at most γ edges of length greater than the side length of Q.

Aronov et al. [2] have recently showed that in a realistic terrain the closest-site Voronoi diagram of m point sites has complexity $O(n + m\sqrt{n})$; this improves the previous bound by Moet et al. [9]. Using this bound in Lemma 4 we obtain the following result.

Lemma 6 *In a realistic terrain* $B_{\leq k} = O(k^2 n + k\sqrt{n}m)$ *for any* $1 \leq k \leq m - 1$.

Proof. Consider the proof of Lemma 4 for a realistic terrain. Using the bound

$$B(R) = O(n + |R|\sqrt{n}) = O(n + \sqrt{n}mp),$$

instead of the equation (1), one readily obtains

$$B_{\leq k} = O(k^2 n + k\sqrt{n}m),$$

□

Combining Lemmas 3, 5, and 6 we can conclude the following.

Theorem 2 *Let* $\mathcal{T}$ *be a realistic terrain with* n *triangles, let* S *be a set of* m *point sites, and let* k *be an integer between* 1 *and* $m - 1$. *The complexity of* $Vor_{\leq k}(S)$ *is* $O(k^2 n + k\sqrt{n}m + k^2 m)$. □

5 Conclusions

We have have given tight bounds for the combinatorial complexity of order-($\leq k$) Voronoi diagrams in triangulated surfaces. Our upper bound of course also holds for the combinatorial complexity of the order-k Voronoi diagram. Obtaining better upper bounds for the order-k Voronoi diagram seems a much more challenging problem, even if restricted only to terrains or realistic terrains.

References

[1] B. Aronov, M. van Kreveld, R. W. van Oostrum, and K. Varadarajan. Facility location on terrains. *Discrete and Computational Geometry*, 30:257–372, 2003.

[2] B. Aronov, M. de Berg, and S. Thite. The complexity of bisectors and Voronoi diagrams on realistic terrins. To appear in: *Algorithms - ESA 2008, 16th Annual European Symposium*, 2008.

[3] F. Aurenhammer. Voronoi diagrams: a survey of a fundamental geometric data structure. *ACM Comput. Surv.*, 23:345–405, 1991.

[4] F. Aurenhammer and R. Klein. Voronoi diagrams. In: J. R. Sack, J. Urrutia (Eds.), Handbook of Computational Geometry, 2000, Elsevier, 201–290.

[5] K. L. Clarkson and P. W. Shor. Application of random sampling in computational geometry, II, *Discrete and Computational Geometry*, 4:387–421, 1989.

[6] M. Fort and J. A. Sellarès. Generalized higher-order Voronoi diagrams on polyhedral terrains. *4th International Symposium on Voronoi Diagrams in Science an Engineering*, pages 74–84, 2007.

[7] D. T. Lee. On k-nearest neighbor Voronoi diagrams in the plane. *IEEE Trans. Computers*, 31:478–787, 1982.

[8] J. S. B. Mitchell, D. M. Mount, and C. H. Paparimitriou. The discrete geodesic problem. *SIAM J. Computation*, 16:647–668, 1987.

[9] E. Moet, M. van Kreveld, and F. van der Stappen, On realistic terrains. In *SCG '06: Proceedings of the 22nd Symposium on Computational Geometry*, pages 177–186, 2006.

[10] D. M. Mount. Voronoi diagrams on the terrain of a polyhedron. Technical report, University of Maryland, 1985.

[11] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial Tessellation: Concepts and Application of Voronoi Diagrams*. John Wiley and Sons, 2000.

[12] D. Schmitt and J. C. Spehner. Order-k Voronoi diagrams, k-sections, and k-sets. In *JCDCG '98: Japanese Conference on Discrete and Computational Geometry*, LNCS 1763, pages 290–304, 1998.

[13] M. Sharir. The Clarkson-Shor technique revisited and extended. *Combinatorics, Probability & Computing*, 12:191–201, 2003.

[14] V. Surazhsky, T. Surazhsky, D. Kirsanov, S. J. Gortler, and H. Hoppe. Fast exact and approximate geodesics on meshes. *ACM Trans. Graph.*, 24(3):553–560, 2005.