University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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ON QUADRATIC AND SINGULAR TWO-PARAMETER EIGENVALUE PROBLEMS

Andrej Muhič Bor Plestenjak

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On quadratic and singular two-parameter eigenvalue problems

Andrej Muhič [a,*], Bor Plestenjak [b]

[a] *Institute of mathematics, physics and mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia.*

[b] *Department of Mathematics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia.*

Abstract

We introduce quadratic two-parameter eigenvalue problem and show that we can linearize it as a singular two-parameter eigenvalue problem. This problem, together with another example that comes from model updating, shows the need for numerical methods for singular two-parameter eigenvalue problems and for a better understanding of such problems.

There are various numerical methods for two-parameter eigenvalue problems, but all of them can only be applied to nonsingular problems. We develop a numerical method that can be applied to certain singular two-parameter eigenvalue problems including the linearization of the quadratic two-parameter eigenvalue problem. It is based on the staircase algorithm for the extraction of the common regular part of two singular matrix pencils.

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1 Introduction

We consider the *quadratic two-parameter eigenvalue problem*

* Corresponding author.

Email addresses: andrej.muhic@fmf.uni-lj.si (Andrej Muhič), bor.plestenjak@fmf.uni-lj.si (Bor Plestenjak).

$$
\begin{aligned}
&(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x = 0\\
&(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y = 0,
\end{aligned} \tag{1}
$$

where $A_i, B_i, \ldots, F_i$ are given $n_i \times n_i$ complex matrices, $x \in \mathbb{C}^{n_1}$, $y \in \mathbb{C}^{n_2}$ nonzero vectors and $\lambda, \mu \in \mathbb{C}$. We say that (λ, μ) is an eigenvalue of (1) and the tensor product $x \otimes y$ is the corresponding eigenvector. In the generic case problem (1) has $4n_1n_2$ eigenvalues that are solutions of the following system of two bivariate polynomials

$$
\begin{aligned}
q_1(\lambda,\mu) &:= \det(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1) = 0,\\
q_2(\lambda,\mu) &:= \det(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2) = 0.
\end{aligned} \tag{2}
$$

Recently, a quadratic two-parameter eigenvalue problem of a simpler form, where some of the quadratic terms $\lambda^2, \lambda\mu, \mu^2$ are missing, appeared in the study of linear time-delay systems for the single delay case [9]. Due to the missing terms the problem in [9] has $2n_1n_2$ eigenvalues which makes it easier to solve. Here we study a general case (1) where all quadratic terms are present in both equations.

Similar to the quadratic eigenvalue problem (see, e.g., [11]), where we can linearize the problem to a generalized eigenvalue problem with matrices of double dimension, we can write (1) as a two-parameter eigenvalue problem with matrices of larger dimension. One such two-parameter eigenvalue problem is

$$
\begin{aligned}
&\left(\overbrace{\begin{bmatrix} A_1 & B_1 & C_1\\ 0 & -I & 0\\ 0 & 0 & -I\end{bmatrix}}^{A^{(1)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_1 & E_1\\ I & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}^{B^{(1)}} + \mu \overbrace{\begin{bmatrix} 0 & 0 & F_1\\ 0 & 0 & 0\\ I & 0 & 0\end{bmatrix}}^{C^{(1)}}\right)\overbrace{\begin{bmatrix} x\\ \lambda x\\ \mu x\end{bmatrix}}^{w_1} = 0\\
&\left(\overbrace{\begin{bmatrix} A_2 & B_2 & C_2\\ 0 & -I & 0\\ 0 & 0 & -I\end{bmatrix}}^{A^{(2)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_2 & E_2\\ I & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}^{B^{(2)}} + \mu \overbrace{\begin{bmatrix} 0 & 0 & F_2\\ 0 & 0 & 0\\ I & 0 & 0\end{bmatrix}}^{C^{(2)}}\right)\overbrace{\begin{bmatrix} y\\ \lambda y\\ \mu y\end{bmatrix}}^{w_2} = 0,
\end{aligned} \tag{3}
$$

where matrices $A^{(i)}, B^{(i)}$, and $C^{(i)}$ are of size $3n_i \times 3n_i$ for $i = 1, 2$. One can check that indeed

$$
\begin{aligned}
\det(A^{(1)} + \lambda B^{(1)} + \mu C^{(1)}) &= q_1(\lambda, \mu) \\
\det(A^{(2)} + \lambda B^{(2)} + \mu C^{(2)}) &= q_2(\lambda, \mu).
\end{aligned}
\tag{4}
$$

There are several numerical methods for two-parameter eigenvalue problems, see for instance [7] and references therein, but, unfortunately, as we show later, (3) belongs to a class of singular two-parameter eigenvalue problems whereas all the available methods require that the problem is nonsingular. We present a numerical algorithm that works for singular two-parameter eigenvalue problems of the form (3) and computes all eigenvalues of (1). Up to our knowledge, next to a very special case in [3], this is one of the first numerical methods for singular multiparameter eigenvalue problems.

Let us mention that the linearization (3) is not optimal. Namely, it follows from the theory on determinantal representations [13] that there do exist matrices $A^{(i)}, B^{(i)}$, and $C^{(i)}$ of dimension $2n_i \times 2n_i$ for $i = 1, 2$ such that (4) holds. An appropriate pair of determinantal representations would result in a smaller and, more important, nonsingular two-parameter eigenvalue problem, but as there are no algorithms for the construction of such matrices, this is just a pure theoretical result.

The usual approach for a two-parameter eigenvalue problem of type

$$
\begin{aligned}
\left(A^{(1)} + \lambda B^{(1)} + \mu C^{(1)}\right) w_1 &= 0 \\
\left(A^{(2)} + \lambda B^{(2)} + \mu C^{(2)}\right) w_2 &= 0,
\end{aligned}
\tag{5}
$$

is to define *operator determinants*

$$
\begin{aligned}
\Delta_0 &= B^{(1)} \otimes C^{(2)} - C^{(1)} \otimes B^{(2)}, \\
\Delta_1 &= C^{(1)} \otimes A^{(2)} - A^{(1)} \otimes C^{(2)}, \\
\Delta_2 &= A^{(1)} \otimes B^{(2)} - B^{(1)} \otimes A^{(2)}
\end{aligned}
\tag{6}
$$

on the tensor product space $\mathbb{C}^{3n_1} \otimes \mathbb{C}^{3n_2}$ (see, e.q., [2]) and consider the *coupled generalized eigenvalue problem*

$$
\begin{aligned}
\Delta_1 z &= \lambda \Delta_0 z \\
\Delta_2 z &= \mu \Delta_0 z,
\end{aligned}
\tag{7}
$$

where $z = w_1 \otimes w_2$.

If $A^{(i)}, B^{(i)}$, and $C^{(i)}$ are generic matrices of size $3n_i \times 3n_i$ for $i = 1, 2$ then Δ_0 is nonsingular and we say that (5) is a nonsingular two-parameter eigenvalue problem. In this case it follows (see, e.g., [2]) that matrices $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$

commute, and the problem (5) has $9n_1n_2$ eigenvalues (λ, μ) which can be computed from eigenvalues of $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$.

In our case, where matrices $A^{(i)}, B^{(i)}$, and $C^{(i)}$ arise from linearization (3), Δ_0 is singular and (5) is a *singular two-parameter eigenvalue problem*. The theory for singular two-parameter eigenvalue problems is scarce and there are no general results linking the eigenvalues of (5) to the eigenvalues of (7). Some properties of singular two-parameter eigenvalue problems are presented in Section 2. For the particular case (3) we show in Section 3 that, under very mild conditions, the eigenvalues of (1) are exactly the regular eigenvalues of (7).

In order to solve the quadratic two-parameter eigenvalue problem (1) using the linearization (3) we derive an algorithm for the extraction of the common regular part of two matrix pencils in Section 4. The algorithm is based on the staircase algorithm for one matrix pencil from [14]. In our case, the algorithm returns matrices Q and U with orthonormal columns that define matrices $\widetilde{\Delta}_i = Q^*\Delta_i U$ of size $4n_1n_2 \times 4n_1n_2$ for $i = 0, 1, 2$ such that $\widetilde{\Delta}_0$ is nonsingular, matrices $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ commute, and the eigenvalues of the quadratic two-parameter eigenvalue problem (1) are exactly the eigenvalues of the projected regular matrix pencils $\widetilde{\Delta}_1 - \lambda\widetilde{\Delta}_0$ and $\widetilde{\Delta}_2 - \mu\widetilde{\Delta}_0$.

In Section 5 we give some numerical examples. We show that the algorithm can be successfully applied to some other singular two-parameter eigenvalue problems, for example to the polynomial two-parameter eigenvalue problem and to problems that appear in model updating [3].

2 Singular two-parameter eigenvalue problem

Let us consider a general two-parameter eigenvalue problem of form (5) where $A^{(i)}, B^{(i)}$, and $C^{(i)}$ are $m_i \times m_i$ matrices over $\mathbb{C}$, $w_i \in \mathbb{C}^{m_i}$ for $i = 1, 2$ and $\lambda, \mu \in \mathbb{C}$. A pair (λ, μ) is an *eigenvalue* if it satisfies (5) for nonzero vectors w_1, w_2, and the tensor product $w_1 \otimes w_2$ is the corresponding (right) *eigenvector*. Similarly, $v_1 \otimes v_2$ is the corresponding *left eigenvector* if $v_1, v_2 \neq 0$, $v_1^*(A^{(1)} + \lambda B^{(1)} + \mu C^{(1)}) = 0$, and $v_2^*(A^{(2)} + \lambda B^{(2)} + \mu C^{(2)}) = 0$.

Multiparameter eigenvalue problems of this kind arise in a variety of applications [1], particularly in mathematical physics when the method of separation of variables is used to solve boundary value problems [15].

The eigenvalues of (5) are solutions of the following system of two bivariate polynomials

$$
\begin{aligned}
p_1(\lambda,\mu) &:= \det(A^{(1)} + \lambda B^{(1)} + \mu C^{(1)}) = 0,\\
p_2(\lambda,\mu) &:= \det(A^{(2)} + \lambda B^{(2)} + \mu C^{(2)}) = 0.
\end{aligned} \tag{8}
$$

Instead of (5) we study the coupled generalized eigenvalue problem (7) with operator determinants Δ_0, Δ_1, and Δ_2.

Usually, the two-parameter eigenvalue problem (5) is *nonsingular*, i.e., the corresponding operator determinant Δ_0 is nonsingular. In this case matrices $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute and the nonsingular two-parameter eigenvalue problem can be solved using standards tools for the generalized eigenvalue problem, for some algorithms see, e.g., [7,8].

However, several applications lead to singular two-parameter eigenvalue problems where Δ_0 is singular. One such example is the quadratic two-parameter eigenvalue problem, while another one that appears in model updating is presented in the following example.

Example 1 *In model updating [3] one wants to adjust the matrices obtained from the finite element model so that some of the eigenfrequencies of the model match the measured eigenfrequencies. In a matrix formulation we can write the problem for the two frequencies as follows.*

Given $n \times n$ matrices K, L, M and two prescribed eigenvalues $\xi_1 \neq \xi_2$, find values of λ and μ such that two of the eigenvalues of the matrix $K + \lambda L + \mu M$ are equal to ξ_1 and ξ_2. The problem can be expressed as a two-parameter eigenvalue problem

$$
\begin{aligned}
(K - \xi_1 I)x + \lambda Lx + \mu Mx &= 0,\\
(K - \xi_2 I)y + \lambda Ly + \mu My &= 0,
\end{aligned} \tag{9}
$$

which is singular because its operator determinant $\Delta_0 = L \otimes M - M \otimes L$ is singular.

If Δ_0 is singular then there might still exist a linear combination $\Delta = \alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2$ such that Δ is nonsingular. In such case (see [2]) matrices $\Delta^{-1}\Delta_0$, $\Delta^{-1}\Delta_1$, and $\Delta^{-1}\Delta_2$ commute. If we consider the homogeneous problem

$$
\begin{aligned}
(\eta_0 A^{(1)} + \eta_1 B^{(1)} + \eta_2 C^{(1)})w_1 &= 0,\\
(\eta_0 A^{(2)} + \eta_1 B^{(2)} + \eta_2 C^{(2)})w_2 &= 0
\end{aligned} \tag{10}
$$

instead of (5), then we get η_0, η_1, and η_2 from the following three joined generalized eigenvalue problems

$$\begin{aligned}\Delta_0 z &= \eta_0 \Delta z,\\ \Delta_1 z &= \eta_1 \Delta z,\\ \Delta_2 z &= \eta_2 \Delta z.\end{aligned}$$

The solutions of the original problem (5) are then solutions of (10) having $\eta_0 \neq 0$ (we are only interested in finite eigenvalues). For such solution $\lambda = \eta_1/\eta_0$ and $\mu = \eta_2/\eta_0$ give an eigenvalue of (5). As we are interested in singular problems, we assume from now on that $\alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2$ is singular for all values of α_0, α_1, and α_2.

Theorem 2 ([2, Theorem 8.7.1]) *The following two statements for the homogeneous problem (10) are equivalent:*

(1) The matrix $\Delta = \sum_{s=0}^{2} \mu_s \Delta_s$ is singular.
(2) There exist an eigenvalue η of (10) such that $\sum_{s=0}^{2} \eta_s \mu_s = 0$.

Using this theorem we can easily see that Δ_0 in Example 1 is singular. If we look at the homogenized version of problem (9) and put $\eta_0 = 0$, we get two identical equations.

Some results about specific hermitian singular problems can be found in [3]. In the situation where all Δ_i matrices are hermitian and $\mathrm{Im}(\Delta_1), \mathrm{Im}(\Delta_2) \subseteq \mathrm{Im}(\Delta_0)$ one can use a generalized inverse of Δ_0 to obtain matrices $\Delta_0^+\Delta_0$, $\Delta_0^+\Delta_1$, and $\Delta_0^+\Delta_2$. All new matrices are of the form

$$\begin{array}{c c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \end{array},$$

where k is the dimension of $\mathrm{Ker}\,\Delta_0$. Let $\widehat{\Delta}_0 = I$, $\widehat{\Delta}_1$, and $\widehat{\Delta}_2$ be $m \times m$ leading submatrices of $\Delta_0^+\Delta_0$, $\Delta_0^+\Delta_1$, and $\Delta_0^+\Delta_2$, respectively. When all eigenvalues are semisimple, matrices $\widehat{\Delta}_1$ and $\widehat{\Delta}_2$ commute. It turns out that this is a special case of the algorithm for the extraction of the common regular part that is presented in Section 4.

3 Quadratic two-parameter eigenvalue problem

Let us take a closer look at the general quadratic two-parameter eigenvalue problem (1). To simplify things, we will assume from now on that $n_1 = n_2 = n$. By inspecting the Kronecker canonical structure of the two matrix pencils (7) obtained by the linearization, we will show that we get exactly $4n^2$ regular eigenvalues in the generic case. This is equal to the number of common zeros of polynomials q_1 and q_2 defined in (2).

Let us denote

$$W_i(\lambda,\mu) = A_i + \lambda B_i + \mu C_i + \lambda^2 D_i + \lambda\mu E_i + \mu^2 F_i$$

for $i = 1, 2$. We are looking for λ, μ and nonzero vectors x, y such that

$$\begin{aligned} W_1(\lambda,\mu)x &= 0,\\ W_2(\lambda,\mu)y &= 0. \end{aligned}$$

We form the two-parameter eigenvalue problem

$$\left(\overbrace{\begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}}^{A^{(1)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_1 & E_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{B^{(1)}} + \mu \overbrace{\begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}}^{C^{(1)}}\right) w_1 = 0$$

$$\left(\overbrace{\begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}}^{A^{(2)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{B^{(2)}} + \mu \overbrace{\begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}}^{C^{(2)}}\right) w_2 = 0.$$

The matrix of the first equation

$$\begin{bmatrix} A_1 & B_1 + \lambda D_1 & C_1 + \lambda E_1 + \mu F_1 \\ \lambda I & -I & 0 \\ \mu I & 0 & -I \end{bmatrix}$$

can be transformed multiplying it from left by

$$E(\lambda,\mu) = \begin{bmatrix} I & B_1 + \lambda D_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & C_1 + \lambda E_1 + \mu F_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and from right by

$$F(\lambda,\mu)=\begin{bmatrix} I & 0 & 0\\ 0 & I & 0\\ \mu I & 0 & I\end{bmatrix}\begin{bmatrix} I & 0 & 0\\ \lambda I & I & 0\\ 0 & 0 & I\end{bmatrix}$$

to

$$\begin{bmatrix} W_1(\lambda,\mu) & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}.$$

This shows that (3) is a weak linearization of (1) in a sense of [10]. For a definition of the weak linearization see Appendix A, where we show that we can apply a similar approach to linearize every polynomial two-parameter eigenvalue problem to obtain a two-parameter eigenvalue problem with matrices of higher dimension.

Matrices of the corresponding pair of generalized eigenvalue problems (7) are

$$\begin{aligned}\Delta_0 &= B^{(1)}\otimes C^{(2)} - C^{(1)}\otimes B^{(2)},\\ \Delta_1 &= C^{(1)}\otimes A^{(2)} - A^{(1)}\otimes C^{(2)},\\ \Delta_2 &= A^{(1)}\otimes B^{(2)} - B^{(1)}\otimes A^{(2)}.\end{aligned}$$

In order to simplify the proofs of the next two lemmas, we will apply the Tracy–Singh product of partitioned matrices [12].

Definition 3 *Let an $m\times n$ matrix A be partitioned into the $m_i\times n_j$ blocks A_{ij} and a $p\times q$ matrix B into the $p_k\times q_l$ blocks B_{kl} such that $m=\sum_{i=1}^r m_i$, $n=\sum_{j=1}^s n_j$, $p=\sum_{k=1}^t p_k$, $q=\sum_{l=1}^u q_l$. The Tracy–Singh product $A\circ B$ is a $mp\times nq$ matrix, defined as*

$$A\circ B=(A_{ij}\circ B)_{ij}=((A_{ij}\otimes B_{kl})_{kl})_{ij},$$

where the (ij)th block of the product is the $m_ip\times n_jq$ matrix $A_{ij}\circ B$, of which the (kl)th subblock equals the $m_ip_k\times n_jq_l$ matrix $A_{ij}\otimes B_{kl}$.

Basically, $A\circ B$ is a block matrix, where each block is a pairwise Kronecker product for each pair of partitions in the two matrices. For instance, if A and

B are 2×2 *block matrices, then*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \circ \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} \end{bmatrix}.$$

Theorem 4 ([12, Theorem 5]) *In the case of balanced partitioning, where all blocks in matrix A and B are of the same size, respectively, the Tracy–Singh product $A \circ B$ is permutation equivalent to the Kronecker product $A \otimes B$.*

All our block matrices have balanced partition and it turns out that some properties are easier to see if we work with the Tracy-Singh product instead of the Kronecker product. Since this is just a reordering of columns and rows, we will denote by TS the map that reorders the elements of $A \otimes B$ so that $TS(A \otimes B) = A \circ B$.

Lemma 5 *In the generic case, matrices Δ_1 and Δ_2 are of rank $8n^2$.*

Proof. Let us observe the problem obtained by putting $\lambda = 0$ in $W_1(\lambda, \mu)$

$$\overbrace{(A_1 + \mu C_1 + \mu^2 F_1)}^{W_1(0,\mu)} x = 0,$$
$$\overbrace{(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)}^{W_2(\lambda,\mu)} y = 0,$$

which is a modified version of the original problem (1). Its linearization is

$$\begin{gathered} \left(\overbrace{\begin{bmatrix} A_1 & C_1 \\ 0 & -I \end{bmatrix}}^{A'^{(1)}} + \mu \overbrace{\begin{bmatrix} 0 & F_1 \\ I & 0 \end{bmatrix}}^{C'^{(1)}} \right) w_1 = 0 \\ \left(\overbrace{\begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}}^{A^{(2)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{B^{(2)}} + \mu \overbrace{\begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}}^{C^{(2)}} \right) w_2 = 0. \end{gathered} \qquad (11)$$

Let us look at the homogenized version of (11). The first equation in the linearization of the modified problem has no infinite eigenvalues, because matrix $C'^{(1)}$ is invertible in the generic case. Therefore, if the original problem (1) does not have an eigenvalue with $\lambda = 0$, then that is also the case for the modified version. A nonexistence of an eigenvalue with $\lambda = 0$ means that polynomials $q_1(0, \mu)$ and $q_2(0, \mu)$ do not have a common zero, which is the situation in the generic case. It follows that the $6n^2 \times 6n^2$ matrix

$$\Delta_1' = C'^{(1)} \otimes A^{(2)} - A'^{(1)} \otimes C^{(2)}$$

from the coupled generalized eigenvalue problem of the modified problem (11) is nonsingular. Block structure of $TS(\Delta_1)$ is

$$\begin{array}{c} \\ 3n^2 \\ 3n^2 \\ 3n^2 \end{array} \begin{array}{c} \begin{array}{ccc} 3n^2 & 3n^2 & 3n^2 \end{array} \\ \left[\begin{array}{ccc} \times & \times & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{array}\right] \end{array},$$

where the four corner blocks represent nonsingular matrix $TS(\Delta_1')$ of the modified problem. The central $3n^2 \times 3n^2$ block of matrix $TS(\Delta_1)$ is

$$\begin{array}{c} \\ n^2 \\ n^2 \\ n^2 \end{array} \begin{array}{c} \begin{array}{ccc} n^2 & n^2 & n^2 \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & I \otimes F_2 \\ 0 & 0 & 0 \\ I \otimes I & 0 & 0 \end{array}\right] \end{array}.$$

This matrix is of maximal rank $2n^2$ in the generic case, where we assume that matrix F_2 is nonsingular. It follows that matrix Δ_1 is of rank $8n^2$.

Similarly we can show that if the problem (1) does not have an eigenvalue with $\mu = 0$ and if the matrix $[D_2\ E_2]$ is of full rank, then matrix Δ_2 has rank $8n^2$. □

Lemma 6 *Matrix Δ_0 has rank $6n^2$ in the generic case.*

Proof. If we rewrite the matrix Δ_0 in the Tracy–Singh reordering, we obtain the following block structure

$$TS(\Delta_0) = \begin{array}{c} \\ {\scriptstyle 3n^2} \\ {\scriptstyle 6n^2} \end{array} \!\! \begin{array}{c} \begin{array}{cc} {\scriptstyle 3n^2} & {\scriptstyle 6n^2} \end{array} \\ \left[\begin{array}{cc} 0 & S \\ T & 0 \end{array} \right] \end{array},$$

where

$$S = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & D_1 \otimes F_2 & 0 & -F_1 \otimes D_2 & E_1 \otimes F_2 - F_1 \otimes E_2 \\ \hline 0 & 0 & 0 & -F_1 \otimes I & 0 & 0 \\ \hline D_1 \otimes I & 0 & 0 & E_1 \otimes I & 0 & 0 \end{array} \right]$$

and

$$T = \left[\begin{array}{c|c|c} 0 & 0 & I \otimes F_2 \\ \hline 0 & 0 & 0 \\ \hline I \otimes I & 0 & 0 \\ \hline 0 & -I \otimes D_2 & -I \otimes E_2 \\ \hline -I \otimes I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right].$$

From the above block representations of S and T it is easy to see, under the general assumption, that matrices D_1, F_1, D_2, and F_2 are all nonsingular, that each of the matrices S and T is of rank $3n^2$. It follows that in the generic case the rank of Δ_0 is indeed $6n^2$. $\square$

Lemma 7 *In the generic case, where we assume that matrices D_1, D_2, F_1, F_2 are nonsingular, we can construct basis for kernels of Δ_0, Δ_1, and Δ_2 as follows:*

(1) A basis for $\mathrm{Ker}(\Delta_1)$ *consists of vectors*

$$\begin{bmatrix} 0 \\ e_i \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ e_j \\ 0 \end{bmatrix}, \quad i, j = 1, \ldots, n.$$

(2) A basis for $\mathrm{Ker}(\Delta_2)$ *consists of vectors*

$$\begin{bmatrix} 0 \\ D_1^{-1}E_1e_i \\ -e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ D_2^{-1}E_2e_j \\ -e_j \end{bmatrix}, \quad i,j=1,\ldots,n.$$

(3) Kernels of Δ_1 *and* Δ_2 *are included in the kernel of* Δ_0*. A basis for* $\mathrm{Ker}(\Delta_0)$ *consists of vectors in (1) and (2), and vectors*

$$\begin{bmatrix} 0 \\ D_1^{-1}(E_1-F_1)e_i \\ -e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ D_2^{-1}(E_2-F_2)e_j \\ -e_j \end{bmatrix}, \quad i,j=1,\ldots,n.$$

Proof. One can confirm the lemma by a direct computation. □

In a similar way we can find basis for $\mathrm{Ker}(\Delta_0^*)$, $\mathrm{Ker}(\Delta_1^*)$, and $\mathrm{Ker}(\Delta_2^*)$.

Lemma 8 *A basis for* $\mathrm{Ker}(\Delta_0^*)$ *is*

$$\overbrace{\begin{bmatrix} 0 \\ e_i \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ e_j \\ 0 \end{bmatrix}}^{\mathrm{Ker}(\Delta_1^*)}, \overbrace{\begin{bmatrix} 0 \\ 0 \\ e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ e_j \end{bmatrix}}^{\mathrm{Ker}(\Delta_2^*)}, \begin{bmatrix} 0 \\ e_i \\ e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ e_j \\ e_j \end{bmatrix}, \quad i,j=1,\ldots,n.$$

Proof. It is easy to see that the above vectors are indeed in the spaces $\mathrm{Ker}(\Delta_1^*)$, $\mathrm{Ker}(\Delta_2^*)$, and $\mathrm{Ker}(\Delta_0^*)$, respectively. From Lemmas 5 and 6 it follows that these vectors form a basis for the mentioned kernels. □

Let us show that for all $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$, not all equal to zero, the linear combination $\alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2$ is singular. This means that the two-parameter eigenvalue problem is singular even if we study it in the homogeneous form (10).

Lemma 9 *Matrices* Δ_1^* *and* Δ_2^* *act on* $\mathrm{Ker}(\Delta_0)$ *as*

$$\Delta_1^* \begin{bmatrix} 0 \\ x \\ x \end{bmatrix} \otimes \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} = -\Delta_2^* \begin{bmatrix} 0 \\ x \\ x \end{bmatrix} \otimes \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ x \end{bmatrix} \otimes \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ y \\ y \end{bmatrix}$$

$$\Delta_1^* \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \otimes \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix}$$

$$\Delta_2^* \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \otimes \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}.$$

In fact images of Δ_1^* and Δ_2^* restricted to $\operatorname{Ker}\Delta_0^*$ *coincide.*

Using the above straightforward lemma one can easily check that there exist a, b, and c, not all equal zero, such that

$$(\alpha_0\Delta_0^* + \alpha_1\Delta_1^* + \alpha_2\Delta_2^*)\left(a \begin{bmatrix} 0 \\ x \\ x \end{bmatrix} \otimes \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} + b \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} \right) = 0,$$

one solution is $a = \alpha_1\alpha_2$, $b = \alpha_1^2 - \alpha_1\alpha_2$, and $c = \alpha_2^2 - \alpha_1\alpha_2$. The problem is therefore singular.

We would like to show that the eigenvalues of the initial quadratic two-parameter eigenvalue problem (1) are exactly the finite regular eigenvalues of the coupled generalized eigenvalue problem (7).

Definition 10 *Normal rank of a square matrix pencil $A - \lambda B$ is*

$$n_r = \max_{s\in\mathbb{C}} \operatorname{rank}(A - sB).$$

We say that $z \in \mathbb{C}$ is a finite regular eigenvalue of matrix pencil if $\operatorname{rank}(A - zB) < n_r$.

Definition 11 *A pair $(\lambda, \mu) \in \mathbb{C}^2$ is a finite regular eigenvalue of two-parameter eigenvalue problem (5) if*

$$\operatorname{rank}(A^{(i)} + \lambda B^{(i)} + \mu C^{(i)}) < \max_{(s,t)\in\mathbb{C}^2} \operatorname{rank}(A^{(i)} + sB^{(i)} + tC^{(i)})$$

for $i = 1, 2$.

Definition 12 *Let $A-\lambda B \in \mathbb{C}^{m\times n}$ be a matrix pencil. There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}(A-\lambda B)Q = \widetilde{A} - \lambda\widetilde{B} = \mathrm{diag}(A_1 - \lambda B_1, \ldots, A_k - \lambda B_k)$$

is the Kronecker canonical form. Each block $A_i - \lambda B_i$, $i = 1, \ldots, k$, must be of one of the following forms: $J_j(\alpha), N_j, L_j$, or L_j^T, where blocks

$$J_j(\alpha) = \begin{bmatrix} \alpha-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha-\lambda \end{bmatrix}, \quad N_j = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix},$$

$$L_j = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \quad L_j^T = \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix},$$

represent finite regular blocks, infinite regular blocks, right singular blocks, and left singular blocks respectively. More about Kronecker canonical form can be found in, e.g., [4], [5], [6], and [14].

It follows from the weak linearization that all eigenvalues of the initial quadratic two-parameter eigenvalue problem (1) are finite regular eigenvalues of the linearized two-parameter eigenvalue problem (3).

Definition 13 *A pair $(\lambda, \mu) \in \mathbb{C}^2$ is a finite regular eigenvalue of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if all of the following statements are true:*

(1) λ is a finite regular eigenvalue of $\Delta_1 - \lambda\Delta_0$,
(2) μ is a finite regular eigenvalue of $\Delta_2 - \mu\Delta_0$,
(3) there exists a common eigenvector z in the intersection of finite regular subspaces of pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ such that

$$(\Delta_1 - \lambda\Delta_0)z = 0,$$
$$(\Delta_2 - \mu\Delta_0)z = 0.$$

Now we can show that all eigenvalues of the initial quadratic two-parameter eigenvalue problem (1) are finite regular eigenvalues of the coupled generalized eigenvalue problem (7). The equivalence of both sets of eigenvalues is established later in Theorem 17.

Lemma 14 *The common zeros of $q_1(\lambda, \mu)$ and $q_2(\lambda, \mu)$ from (2) are finite regular eigenvalues of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ from (7).*

Proof. It follows from Lemmas 5, 6, and 7 that the normal rank of pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ is exactly $8n^2$.

Let a vector of the form

$$\begin{bmatrix} x \\ \lambda x \\ \mu x \end{bmatrix} \otimes \begin{bmatrix} y \\ \lambda y \\ \mu y \end{bmatrix}$$

be an eigenvector for the eigenvalue (λ, μ) that we get from the linearization. Such vector has nonzero first block components x and y. Vectors in the kernels of Δ_1 and Δ_2 have the first block component zero, so we have $\operatorname{rank}(\Delta_1 - \lambda\Delta_0) < 8n^2$ and $\operatorname{rank}(\Delta_2 - \mu\Delta_0) < 8n^2$. □

Now we have enough information to determine the Kronecker canonical structure of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$.

Lemma 15 *The pencil $\Delta_1^* - \lambda\Delta_0^*$ has at least $2n^2$ first root vectors for infinite eigenvalues. The same is true for the pencil $\Delta_2^* - \mu\Delta_0^*$.*

Proof. The first root vector for an infinite eigenvalue is vector x_1 in the chain $\Delta_0^* x_0 = 0$, $\Delta_1^* x_0 = \Delta_0^* x_1$ such that $\Delta_1^* x_1 \neq 0$. We have to show that we can find $2n^2$ such linearly independent vectors.

Froma Lemma 9 it follows that all vectors in $\operatorname{Ker}(\Delta_0)$, which are of the form

$$\begin{bmatrix} 0 \\ \times \\ \times \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \times \\ \times \end{bmatrix}$$

by Lemma 7, are obviously orthogonal to $\Delta_1^* \operatorname{Ker}(\Delta_0^*)$. As the whole space is an orthogonal sum of $\operatorname{Im}(\Delta_0^*)$ and $\operatorname{Ker}(\Delta_0)$, it follows that $\Delta_1^* \operatorname{Ker}(\Delta_0^*)$ is a subspace of $\operatorname{Im}(\Delta_0^*)$. So, there exist $2n^2$ linearly independent vectors x_1 such that $\Delta_0^* x_1$ is in $\Delta_1^* \operatorname{Ker}(\Delta_0^*)$. □

Lemma 16 *Kronecker canonical form of pencil $\Delta_1 - \lambda\Delta_0$ has n^2 L_0, n^2 L_0^T, $2n^2$ N_2 blocks, and the finite regular part of size $4n^2$.*

Proof. Regular Kronecker canonical structure of the transposed pencil $\Delta_1^* - \lambda\Delta_0^*$ is the same as of $\Delta_1 - \lambda\Delta_0$. Right (left) singular structure of $\Delta_1^* - \lambda\Delta_0^*$ is left (right) singular structure of $\Delta_1 - \lambda\Delta_0$. The pencil $\Delta_1 - \lambda\Delta_0$ has a regular part of size at least $4n^2$ by Lemma 14. Number of L_0 and L_0^T blocks is n^2 by

Lemmas 5, 7, and 8. It follows from Lemma 15 that the pencil in addition has $2n^2$ N_2 blocks. Thus we have completely determined the Kronecker canonical structure. $\square$

Theorem 17 *The eigenvalues of the initial quadratic two-parameter eigenvalue problem (1) are exactly the finite regular eigenvalues of the coupled generalized eigenvalue problem (7).*

Proof. We know that (1) has $4n^2$ eigenvalues which are also finite regular eigenvalues of the linearized two-parameter eigenvalue problem (5) and we proved in Lemma 15 that all eigenvalues of (5) are finite regular eigenvalues of (7). As it follows from Lemma 16 that (7) can not have more than $4n^2$ finite regular eigenvalues, the sets of eigenvalues must be equal. $\square$

In the next section we describe the algorithm that computes the common regular part of two matrix pencils. Using this algorithm we can solve the quadratic two-parameter eigenvalue problem applying the proposed linearization.

4 Algorithm for the extraction of common regular subspace of two singular matrix pencils

We would like to recover the finite regular eigenvalues of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$. In this paper we are not interested in the infinite part.

Instead of the Kronecker canonical form we will use the generalized upper-triangular form, where the transformation matrices P and Q are unitary, see, e.g., [14] or [4]. For the matrix pencil $A - \lambda B$ there exist unitary matrices P and Q such that

$$P^*(A-\lambda B)Q = \left[\begin{array}{cc|cc} A_\mu - \lambda B_\mu & & & \\ \times & A_\infty - \lambda B_\infty & & \\ \hline \times & \times & A_f - \lambda B_f & \\ \times & \times & \times & A_\epsilon - \lambda B_\epsilon \end{array}\right]. \qquad (12)$$

Pencils $A_\mu - \lambda B_\mu$, $A_\infty - \lambda B_\infty$, $A_f - \lambda B_f$, and $A_\epsilon - \lambda B_\epsilon$ contain the left singular structure, the infinite regular structure, the finite regular structure, and the right singular structure, respectively. We are particularly interested in the lower right block of (12). There we find the finite regular structure together with the right singular structure. We partition $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ and $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ in such a way that

$$\begin{bmatrix} A_f - \lambda B_f & \\ \times & \lambda A_\epsilon - \lambda B_\epsilon \end{bmatrix} = P_2^*(A - \lambda B)Q_2.$$

The columns of Q_2 now represent a basis for the eigenspace of the regular part with the right singular structure. The most simple case of a right singular structure is when $\mathrm{Ker}(A) \cap \mathrm{Ker}(B)$ is nontrivial. Eigenvectors of the projected pencil are then not well defined.

Below we provide a sketch of the algorithm that computes a pencil representing the regular structure together with the right singular structure of pencil $\Delta_1 - \lambda\Delta_0$. The algorithm, which is based on the staircase algorithm presented in [14], starts with two matrices Δ_0 and Δ_1. It reduces them using consequent row and column compressions, until D_0 has full row rank.

Algorithm 1 *$D_0 = \Delta_0$; $D_1 = \Delta_1$;*
Repeat,

(1) (a) Compute SVD(D_0). Matrix D_0 has size $m \times n$. We get matrices with orthonormal columns U_0, V_0 and a diagonal matrix Σ_0 such that $U_0\Sigma_0 V_0^* = D_0$. Rank r of Δ_0 is the number of nonzero singular eigenvalues.

(b) If matrix D_0 has full row rank, exit and return $D_0 = P^*\Delta_0 Q$, $D_1 = P^*\Delta_1 Q$.

(2) Compute the row compression of matrix D_0.

$$U_0^* D_0 = \begin{matrix} \\ r \\ m-r \end{matrix} \overset{n}{\begin{bmatrix} \times \\ 0 \end{bmatrix}}.$$

Compute block H of

$$U_0^* D_1 = \begin{matrix} \\ r \\ m-r \end{matrix} \overset{n}{\begin{bmatrix} \times \\ H \end{bmatrix}}$$

and compress it to full column rank c. Compute SVD(H). We get matrices

U_1, V_1, Σ_1. We now have

$$U_0^*(D_1 - \lambda D_0)V_1 = \begin{array}{c} \\ r \\ m-r \end{array}\begin{array}{c} \begin{array}{cc} c & n-c \end{array} \\ \begin{bmatrix} \times & \widehat{D_1} \\ 0 & 0 \end{bmatrix} \end{array} - \lambda \begin{array}{c} \\ r \\ m-r \end{array}\begin{array}{c} \begin{array}{cc} c & n-c \end{array} \\ \begin{bmatrix} \times & \widehat{D_0} \\ \times & 0 \end{bmatrix} \end{array}.$$

(3) *Assign $D_0 = \widehat{D_0}$, $D_1 = \widehat{D_1}$ and proceed to* 1.

The algorithm has a dual form that computes a pencil representing the regular structure together with the left singular structure of pencil $\Delta_1 - \lambda\Delta_0$. The algorithm starts with two matrices Δ_0 and Δ_1. It reduces them using consequent column and row compressions, until D_0 has full column rank. For the reduction we use the singular value decomposition or rank revealing QR.

Algorithm 2 *$D_0 = \Delta_0$; $D_1 = \Delta_1$;*
Repeat,

(1) (a) *Compute SVD(D_0). Matrix D_0 has size $m \times n$. We get matrices with orthonormal columns U_0, V_0 and a diagonal matrix Σ_0, such that $U_0\Sigma_0 V_0^* = D_0$. Rank c of Δ_0 is the number of nonzero singular eigenvalues.*
   (b) *If matrix D_0 has full column rank, exit and return $D_0 = P^*\Delta_0 Q$, $D_1 = P^*\Delta_1 Q$.*

(2) *Compute column compression of matrix D_0.*

$$D_0V_0 = {}_m\begin{array}{c} \begin{array}{cc} c & n-c \end{array} \\ \begin{bmatrix} \times & 0 \end{bmatrix} \end{array}.$$

Compute block H of

$$D_1V_0 = {}_m\begin{array}{c} \begin{array}{cc} c & n-c \end{array} \\ \begin{bmatrix} \times & H \end{bmatrix} \end{array}$$

and compress it to the full row rank r. Compute SVD(H). We get matrices U_1, V_1, Σ_1. We now have

$$U_1^*(D_1 - \lambda D_0)V_0 = \begin{array}{c} \\ r \\ m-r \end{array}\begin{array}{c} \begin{array}{cc} c & n-c \end{array} \\ \begin{bmatrix} \times & 0 \\ \widehat{D_1} & 0 \end{bmatrix} \end{array} - \lambda \begin{array}{c} \\ r \\ m-r \end{array}\begin{array}{c} \begin{array}{cc} c & n-c \end{array} \\ \begin{bmatrix} \times & \times \\ \widehat{D_0} & 0 \end{bmatrix} \end{array}.$$

(3) *Assign $D_0 = \widehat{D_0}$, $D_1 = \widehat{D_1}$ and proceed to* 1.

We will apply these two algorithms to compute the common regular structure of two matrix pencils. In the first phase of the algorithm we compute the common regular structure and the common right singular structure of $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ separately using Algorithm 1. We get $P_1^*\Delta_1 Q_1 - \lambda P_1^*\Delta_0 Q_1$ and $P_2^*\Delta_2 Q_2 - \mu P_2^*\Delta_0 Q_2$. Let the columns of matrix Q be an orthogonal basis of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ and the columns of matrix P an orthogonal basis for $\mathcal{P}_1 + \mathcal{P}_2$. We now continue with $\Delta_0 = P^*\Delta_0 Q$, $\Delta_1 = P^*\Delta_1 Q$, and $\Delta_2 = P^*\Delta_2 Q$. We stop if matrix Δ_0 has full row rank. In the second phase of the algorithm we separate the regular part and the right singular structure using the Algorithm 2. At the end we get square matrices, where Δ_0 is invertible.

In the following algorithm we denote the vector space spanned by the columns of a matrix A as $\mathcal{A}$.

Algorithm 3 *$P = I_m$, $Q = I_n$, where m is the number of rows of Δ_0 and n is the number of columns of Δ_0.*

(1) Separate infinite and finite part.
  - *(a) Apply Algorithm 1 to $P^*\Delta_1 Q - \lambda P^*\Delta_0 Q$ and $P^*\Delta_2 Q - \mu P^*\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.*
  - *(b) Compute matrices Q and P with orthonormal columns such that $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$.*
  - *(c) If $\mathcal{Q} = \mathcal{Q}_1$ return P, Q and proceed to (2.a). Otherwise, proceed to (1.a).*

(2) Separate the finite regular part from the right singular part.
  - *(a) Apply Algorithm 2 to $P^*\Delta_1 Q - \lambda P^*\Delta_0 Q$ and $P^*\Delta_2 Q - \mu P^*\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.*
  - *(b) Compute matrix Q with orthonormal columns such that $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ and matrix P with orthonormal columns such that $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$.*
  - *(c) If $\mathcal{Q} = \mathcal{Q}_1$ return P, Q and exit. Otherwise, proceed to (2.a).*

Algorithm 3 stops in a finite number of steps. In the first phase the row rank of $P^*\Delta_0 Q$ and the number of columns in Q decrease until $P^*\Delta_0 Q$ has full row rank. In the second phase the column rank of $P^*\Delta_0 Q$ and the number of columns in Q decrease until $P^*\Delta_0 Q$ has full column rank. Moreover, $P^*\Delta_0 Q$ is a full rank square matrix.

Let us mention that the above algorithm has a dual form. We can start with Algorithm 2 in the first phase and use Algorithm 1 in the second phase, but then we have to compute Q as an orthogonal basis for $\mathcal{Q}_1 + \mathcal{Q}_2$ and P as an orthogonal basis for $\mathcal{P}_1 \cap \mathcal{P}_2$ in the first step. In the second step we then compute Q as an orthogonal basis for $\mathcal{Q}_1 \cap \mathcal{Q}_2$ and P as an orthogonal basis for $\mathcal{P}_1 + \mathcal{P}_2$.

In the following lemma we show that Algorithms 1, 2, and 3 work for the special singular two-parameter eigenvalues problems that appear in model updating [3].

Lemma 18 *Let matrices* Δ_0, Δ_1*, and* Δ_2 *be hermitian and* $\mathrm{Im}(\Delta_1), \mathrm{Im}(\Delta_2) \subseteq \mathrm{Im}(\Delta_0)$*. Then matrices* $\Delta_0^+\Delta_0$*,* $\Delta_0^+\Delta_1$*, and* $\Delta_0^+\Delta_2$*, where* Δ_0^+ *is a generalized inverse of* Δ_0*, are of the form*

$$\Delta_0^+\Delta_0 = \begin{array}{c} \\ m \\ k \end{array}\!\!\begin{array}{c} m\ \ k \\ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad \Delta_0^+\Delta_1 = \begin{array}{c} \\ m \\ k \end{array}\!\!\begin{array}{c} m\ \ k \\ \begin{bmatrix} \widehat{\Delta}_1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad \Delta_0^+\Delta_2 = \begin{array}{c} \\ m \\ k \end{array}\!\!\begin{array}{c} m\ \ k \\ \begin{bmatrix} \widehat{\Delta}_2 & 0 \\ 0 & 0 \end{bmatrix} \end{array},$$

where k *is the dimension of* $\mathrm{Ker}(\Delta_0)$*.*

Transformations in Algorithm 2 can be chosen so that $U_1^*\widehat{\Delta}_1 U_1 = D_1^{-1}\widetilde{\Delta}_1$*,* $U_1^*\widehat{\Delta}_2 U_1 = D_1^{-1}\widetilde{\Delta}_2$*, and* $\widetilde{\Delta}_0 = D_1$*. Matrix* D_1 *is diagonal and matrix* U_1 *is unitary.*

Proof. Matrix Δ_0 is hermitian. There exist a unitary matrix $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ and a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_m, 0, \ldots, 0)$, where $m = \dim(\mathrm{Im}(\Delta_0))$ and the columns of U_1 span $\mathrm{Im}(\Delta_0)$, such that $\Delta_0 = UDU^*$. In the first phase in Algorithm 1 we compress Δ_0 to the full row rank. We multiply with U^* and obtain

$$U^*\Delta_0 = {}^{m}\!\begin{bmatrix} \times \\ 0 \end{bmatrix}, \quad U^*\Delta_1 = {}^{m}\!\begin{bmatrix} \times \\ 0 \end{bmatrix}, \quad U^*\Delta_2 = {}^{m}\!\begin{bmatrix} \times \\ 0 \end{bmatrix}.$$

We know that $\mathrm{Ker}(\Delta_0) \subseteq \mathrm{Ker}(\Delta_1), \mathrm{Ker}(\Delta_2)$, and so we compressed Δ_0 to full row rank.

We can do the same to compress Δ_0 to the full column rank. We get the following matrices

$$U^*\Delta_0 U = {}^{m}\!\overset{m}{\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}}, \quad U^*\Delta_1 U = {}^{m}\!\overset{m}{\begin{bmatrix} \widetilde{\Delta}_1 & 0 \\ 0 & 0 \end{bmatrix}}, \quad U^*\Delta_2 U = {}^{m}\!\overset{m}{\begin{bmatrix} \widetilde{\Delta}_2 & 0 \\ 0 & 0 \end{bmatrix}}.$$

It is easy to check that UD^+U^* is a generalized inverse of Δ_0. Let us compute

$$U^*\Delta_0^+\Delta_1 U = U^*UD^+U^*\Delta_1 U = D^+ \begin{bmatrix} \widetilde{\Delta}_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The proof for Δ_2 is analogous. $\square$

5 Numerical examples

Example 19 *We have the following quadratic two-parameter eigenvalue problem*

$$\left(\begin{bmatrix}-3 & 4\\ 6 & -1\end{bmatrix} + \lambda\begin{bmatrix}7 & 2\\ -2 & 1\end{bmatrix} + \mu\begin{bmatrix}4 & -1\\ 9 & 4\end{bmatrix} + \lambda^2\begin{bmatrix}6 & 7\\ 5 & 2\end{bmatrix} + \lambda\mu\begin{bmatrix}10 & -3\\ 7 & 1\end{bmatrix} + \mu^2\begin{bmatrix}4 & 8\\ 6 & -3\end{bmatrix}\right)x = 0,$$

$$\left(\begin{bmatrix}-1 & 3\\ 2 & -1\end{bmatrix} + \lambda\begin{bmatrix}-1 & -4\\ 8 & 2\end{bmatrix} + \mu\begin{bmatrix}2 & 3\\ -4 & -1\end{bmatrix} + \lambda^2\begin{bmatrix}2 & 6\\ 1 & 3\end{bmatrix} + \lambda\mu\begin{bmatrix}7 & -2\\ 3 & 7\end{bmatrix} + \mu^2\begin{bmatrix}3 & -5\\ -5 & 2\end{bmatrix}\right)y = 0$$

which has 16 *eigenvalues.*

Matrices Δ_0*,* Δ_1*, and* Δ_2 *obtained from the weak linearization are of size* 36×36*. Algorithm 3 returns matrices* $\widetilde{\Delta}_0$*,* $\widetilde{\Delta}_1$*, and* $\widetilde{\Delta}_2$ *of size* 16×16 *such that* $\widetilde{\Delta}_0$ *is nonsingular and matrices* $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ *and* $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ *commute. From* $\widetilde{\Delta}_0$*,* $\widetilde{\Delta}_1$*, and* $\widetilde{\Delta}_2$ *we can compute all* 16 *eigenvalues of the quadratic two-parameter eigenvalue problem. The largest and the smallest (by absolute value) eigenvalue truncated to* 3 *decimal places are* $(1.799, -2.166)$ *and* $(0.007 + 0.167i, -0.507 \pm 0.1i)$*, respectively.*

Example 20 *A cubic two-parameter eigenvalue problem has the form*

$$\begin{aligned}(S_{00} + \lambda S_{10} + \mu S_{01} + \cdots + \lambda^3 S_{30} + \lambda^2\mu S_{21} + \lambda\mu^2 S_{12} + \mu^3 S_{03})x &= 0\\ (T_{00} + \lambda T_{10} + \mu T_{01} + \cdots + \lambda^3 T_{30} + \lambda^2\mu T_{21} + \lambda\mu^2 T_{12} + \mu^3 T_{03})y &= 0.\end{aligned} \tag{13}$$

If S_{ij} *and* T_{ij} *are general* $n\times n$ *matrices, then the problem has* $9n^2$ *eigenvalues. In a similar way as we linearized the quadratic two-parameter eigenvalue problem we can linearize (13) as a two-parameter eigenvalue problem, a possible linearization is*

$$\left(\begin{bmatrix}S_{00} & S_{10} & S_{01} & S_{20} & S_{11} & S_{02}\\ & -I & & & & \\ & & -I & & & \\ & & & -I & & \\ & & & & -I & \\ & & & & & -I\end{bmatrix} + \lambda\begin{bmatrix}0 & 0 & 0 & S_{30} & S_{21} & S_{12}\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} + \mu\begin{bmatrix}0 & 0 & 0 & 0 & 0 & S_{03}\\ 0 & 0 & 0 & 0 & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\end{bmatrix}\right)\widetilde{x} = 0$$

$$\left(\begin{bmatrix} T_{00} & T_{10} & T_{01} & T_{20} & T_{11} & T_{02} \\ & -I & & & & \\ & & -I & & & \\ & & & -I & & \\ & & & & -I & \\ & & & & & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & T_{30} & T_{21} & T_{12} \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & T_{03} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}\right) \widetilde{y} = 0,$$

where $\widetilde{x} = \begin{bmatrix} 1 & \lambda & \mu & \lambda^2 & \lambda\mu & \mu^2 \end{bmatrix}^T \otimes x$ *and* $\widetilde{y} = \begin{bmatrix} 1 & \lambda & \mu & \lambda^2 & \lambda\mu & \mu^2 \end{bmatrix}^T \otimes y$*. The corresponding operator determinant* Δ_0 *of the above two-parameter eigenvalue problem is of rank* $20n^2$ *and thus singular.*

Using software package GUPTRI [5] for the evaluation of the generalized upper-triangular form we observe the following interesting structure:

- *a) Kronecker structure of* $\Delta_1 - \lambda\Delta_0$ *(and same for* $\Delta_2 - \mu\Delta_0$*) consists of* $4n^2$ L_0, $4n^2$ L_0^T, n^2 L_1, n^2 L_1^T, $6n^2$ N_1, $2n^2$ N_2, $2n^2$ N_3, n^2 N_4, *and the regular part of size* $9n^2$.
- *b)* $\dim(\mathrm{Ker}(\Delta_0)) = 16n^2$, $\dim(\mathrm{Ker}(\Delta_1)) = 5n^2$, *and* $\dim(\mathrm{Ker}(\Delta_2)) = 5n^2$.
- *c)* $\dim(\mathrm{Ker}(\Delta_1) \cap \mathrm{Ker}(\Delta_0)) = 4n^2$, $\dim(\mathrm{Ker}(\Delta_2) \cap \mathrm{Ker}(\Delta_0)) = 4n^2$, *and* $\dim(\mathrm{Ker}(\Delta_0) \cap \mathrm{Ker}(\Delta_1) \cap \mathrm{Ker}(\Delta_2)) = n^2$.

Due to a complex Kronecker canonical structure, we did not attempt to prove the structure in theory as we did for the quadratic case.

Using Algorithm 3 for the extraction of the common regular part, we are able to compute all eigenvalues of the cubic two-parameter eigenvalue problem. For the test case we reuse the matrices from Example 19 and add the matrices

$$S_{30} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}, \; S_{21} = \begin{bmatrix} -1 & 7 \\ 2 & 8 \end{bmatrix}, \; S_{12} = \begin{bmatrix} -4 & -9 \\ 1 & 1 \end{bmatrix}, \; S_{03} = \begin{bmatrix} 5 & 8 \\ -6 & 3 \end{bmatrix},$$

$$T_{30} = \begin{bmatrix} 2 & 3 \\ -2 & -7 \end{bmatrix}, \; T_{21} = \begin{bmatrix} -6 & 5 \\ 9 & 1 \end{bmatrix}, \; T_{12} = \begin{bmatrix} 5 & 7 \\ 8 & 8 \end{bmatrix}, \; T_{03} = \begin{bmatrix} 3 & 1 \\ -3 & 5 \end{bmatrix}.$$

Matrices Δ_0, Δ_1 *and* Δ_2 *obtained from the weak linearization are of size* 144×144*. Algorithm 3 returns matrices* $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, *and* $\widetilde{\Delta}_2$ *of size* 36×36*. Matrices* $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ *and* $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ *commute as assumed. From* $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, *and* $\widetilde{\Delta}_2$ *we can compute all* 36 *eigenvalues of the cubic two-parameter eigenvalue problem. The largest and the smallest (by absolute value) eigenvalue truncated to* 3 *decimal places are* $(-1.227 \pm 0.495i, 1.758 \mp 0.178i)$ *and* $(0.090 \mp 0.245i, -0.439 \mp 0.0142i)$*, respectively.*

Example 21 *For the last example we simulate a problem from model updating. We take the following three matrices*

$$A = \begin{bmatrix} 9 & 5 & 2 & -1 & -8 \\ -5 & 0 & 5 & 8 & -2 \\ 2 & -9 & 8 & 8 & 6 \\ 0 & 6 & 4 & -1 & -9 \\ 7 & -1 & -6 & 7 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & -9 & -1 & 6 & 0 \\ -6 & 4 & 6 & -9 & 4 \\ 2 & -1 & 0 & 3 & -1 \\ -4 & 8 & -5 & -2 & -3 \\ -6 & 0 & 3 & 6 & -6 \end{bmatrix},$$

and

$$C = \begin{bmatrix} -6 & 3 & 0 & 3 & 4 \\ 3 & -2 & 7 & -3 & -3 \\ -3 & 7 & 6 & -4 & 6 \\ 0 & 7 & 2 & -3 & 1 \\ -6 & 1 & 6 & 0 & -2 \end{bmatrix}.$$

We are looking for parameters λ *and* μ *such that two eigenvalues of the matrix* $A+\lambda B+\mu C$ *are* $\sigma_1 = 2$ *and* $\sigma_2 = 3$*. If we write this as a two-parameter eigenvalue problem (9) and apply Algorithm 3 we obtain* 20 *suitable pairs* (λ, μ)*. The closest solution to* $(0, 0)$*, which corresponds to the smallest perturbation of* A*, is* $(0.2593, 0.0067)$*.*

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A Linearization of two-parameter matrix polynomials

Let us recall the definition of the weak linearization for one-parameter matrix polynomial.

Definition 22 *An $ln \times ln$ linear matrix pencil $A - \lambda B$ is a weak linearization [10] (of order ln) of a matrix polynomial $L(\lambda)$ if there exist unimodular matrix* *polynomials $E(\lambda)$ and $F(\lambda)$ such that*

$$\begin{bmatrix} L(\lambda) & 0 \\ 0 & I_{l(n-1)} \end{bmatrix} = E(\lambda)(A - \lambda B)F(\lambda).$$

We can generalize this idea to the general two-parameter polynomial eigenvalue problem.

Theorem 23 *Let*

$$P(\lambda, \mu) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \lambda^i \mu^j A_{ij}$$

be a two-parameter matrix polynomial, where A_{ij} is an $n \times n$ matrix for each i, j. Let us define

$$\begin{aligned} K_{ij}(\lambda, \mu) &= A_{ij}, \quad i + j < k - 1, \\ K_{ij}(\lambda, \mu) &= A_{ij} + \lambda A_{i+1,j}, \quad i + j = k - 1, \ i \neq 0, \\ K_{0,k-1}(\lambda, \mu) &= A_{0,k-1} + \lambda A_{1,k-1} + \mu A_{0,k}. \end{aligned}$$

The linear matrix polynomial

$$L(\lambda, \mu) = \begin{array}{c} \\ n \\ 2n \\ 3n \\ \vdots \\ kn \end{array} \begin{array}{c} \begin{array}{ccccc} n & 2n & 3n & \dots & kn \end{array} \\ \begin{bmatrix} K_0 & K_1 & K_2 & \cdots & K_k \\ T_1 & -I_{2n} & & & \\ & T_2 & -I_{3n} & & \\ & & \ddots & \ddots & \\ & & & T_k & -I_{kn} \end{bmatrix} \end{array},$$

where K_r is an $n \times (r+1)n$ matrix in block form

$$K_r = \begin{bmatrix} K_{r0} & K_{r-1,1} & \cdots & K_{0r} \end{bmatrix}$$

and

$$T_r = \begin{bmatrix} \lambda I_n & & \\ & \ddots & \\ & & \lambda I_n \\ & & \mu I_n \end{bmatrix},$$

$r = 1, \ldots, k$, is a weak linearization of $P(\lambda,\mu)$. There exist unimodular two-parameter polynomial matrices

$$E(\lambda,\mu) = \begin{array}{c} \\ n \\ 2n \\ 3n \\ \vdots \\ kn \end{array} \begin{array}{c} \begin{array}{ccccc} n & 2n & 3n & \ldots & kn \end{array} \\ \begin{bmatrix} I_n & H_1 & H_2 & \cdots & H_k \\ & -I_{2n} & & & \\ & & -I_{3n} & & \\ & & \ddots & \ddots & \\ & & & & -I_{kn} \end{bmatrix} \end{array}$$

and

$$F(\lambda,\mu) = \begin{array}{c} \\ n \\ 2n \\ 3n \\ \vdots \\ kn \end{array} \begin{array}{c} \begin{array}{ccccc} n & 2n & 3n & \ldots & kn \end{array} \\ \begin{bmatrix} I_n & & & & \\ T_1 & I_{2n} & & & \\ & T_2 & I_{3n} & & \\ & & \ddots & \ddots & \\ & & & T_k & I_{kn} \end{bmatrix} \end{array}$$

such that

$$E(\lambda,\mu)L(\lambda,\mu)F(\lambda,\mu) = \begin{bmatrix} P(\lambda,\mu) & \\ 0 & I_{(k-2)(k+1)n/2} \end{bmatrix}. \tag{A.1}$$

Proof. Let us define a vector of the r-th powers of λ, μ multiplied by x as

$$\mathbf{x_r} = \begin{bmatrix} \lambda^r & \lambda^{r-1}\mu & \cdots & \lambda\mu^{r-1} & \mu^r \end{bmatrix}^T \otimes x.$$

It is easy to check that

$$\mathbf{x_{r+1}} = \begin{bmatrix} \lambda \mathbf{x_r} \\ \mu^{r+1} x \end{bmatrix} = T_r \mathbf{x_r}.$$

We define the vector $\mathbf{x^k} = \begin{bmatrix} \mathbf{x_0}^T & \mathbf{x_1}^T & \cdots & \mathbf{x_{k-1}}^T \end{bmatrix}^T$ which is composed of up to $(k-1)$-th powers of λ, μ.

If we write polynomial $P(\lambda,\mu)$ as

$$P(\lambda,\mu) = \sum_{i+j<k} \lambda^i \mu^j A_{ij} + \sum_{\substack{i+j=k \\ i\neq 0}} \lambda(\lambda^{i-1}\mu^j A_{ij}) + \mu^k A_{0k}$$

then it is easy to see that

$$P(\lambda,\mu) = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \lambda^i \mu^j K_{ij}.$$

It follows that

$$L(\lambda,\mu)\mathbf{x^k} = \begin{bmatrix} P(\lambda,\mu) \\ 0 \end{bmatrix}.$$

If we multiply $L(\lambda,\mu)$ and $F(\lambda,\mu)$, we obtain

$$L(\lambda,\mu)F(\lambda,\mu) = \begin{array}{c} \\ {\scriptstyle n} \\ {\scriptstyle 2n} \\ {\scriptstyle 3n} \\ \vdots \\ {\scriptstyle kn} \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} {\scriptstyle n} & {\scriptstyle 2n} & {\scriptstyle 3n} & {\scriptstyle \ldots} & {\scriptstyle kn} \end{array} \\ \begin{bmatrix} P(\lambda,\mu) & H_1 & H_2 & \cdots & H_{k-1} \\ & -I_{2n} & & & \\ & & -I_{3n} & & \\ & & \ddots & \ddots & \\ & & & & -I_{kn} \end{bmatrix} \end{array}$$

for some matrices $H_1,\ldots,H_{k-1}$ which we use in $E(\lambda,\mu)$. Equation (A.1) now follows readily. $\square$