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CONVEX EXCESS AND EULER-TYPE INEQUALITY FOR PARTIAL CUBES

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Convex excess and Euler-type inequality for partial cubes

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Abstract

The convex excess $e(G)$ of a graph G is introduced as $\sum(|C|-4)/2$ where the summation goes over all convex cycles of G. It is proved that for a partial cube G with n vertices, m edges, and isometric dimension $i(G)$, $2n-m-i(G)-e(G) \leq 2$. Moreover, the equality holds if and only if the so-called zone graphs of G are trees. This answers the question from [2] whether partial cubes admit an Euler-type inequality. It is also shown that a suggestion for an Euler-type inequality from [2] does not hold.

Key words: Partial cube; Hypercube; Euler-type inequality; Convex excess; zone graph

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1 Introduction

Partial cubes present one of the central and most studied classes of graphs in all of the metric graph theory. They were introduced by Graham and Pollak [11] as a model

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for interconnection networks but found many additional applications afterwards. For instance, they form the central graph class in media theory, see the recent book [9].

For classical results on partial cubes we refer to the books [6, 12]. Among numerous recent results on partial cubes we mention the following. A lot of effort was made in order to classify cubic partial cubes, see [7, 15]. Polat wrote a series of papers on a highly interesting subclass of partial cubes, called netlike partial cubes, a class that forms a common generalization of median graphs and even cycles, see [19, 20]. Subdivisions of graphs and other classes of graphs give some interesting partial cubes, see [1, 5]. Several derived graphs from partial cubes were studied as well, the papers [3, 13] give the flavor of this aspect. For two more very recent aspects of partial cubes see [10, 18]. Finally, Eppstein [8] gives a quadratic algorithm for recognizing partial cubes.

Among subclasses of partial cubes, median graphs deserve special attention. (In fact, they were studied as much as partial cubes.) It was proved in [14] that for a median graph G with n vertices and m edges,

$$2n - m - i(G) \leq 2\,, \tag{1}$$

where $i(G)$ stands for the so-called isometric dimension of G. Moreover, the equality holds if and only if G is a cube-free median graph. This theorem was extended in [2] to those partial cubes that can be obtained by means of a connected expansion procedure. But the problem whether there is an Euler-type relation for all partial cubes remained open.

For a graph G let

$$\mathcal{C}(G) = \{C \mid C \text{ is a convex cycle of } G\}$$

and set

$$e(G) = \sum_{C \in \mathcal{C}(G)} \frac{|C| - 4}{2}\,.$$

We call $e(G)$ the *convex excess* of G.

Let F be a Θ-class of a partial cube G. Then the *F-zone graph*, Z_F, is the graph with $V(Z_F) = F$, vertices f and f' being adjacent in Z_F if they belong to a common convex cycle of G.

With the above two new concepts we can give the following Euler-type relation.

Theorem 1.1 *For a partial cube G with n vertices and m edges,*

$$2n - m - i(G) - e(G) \leq 2\,. \tag{2}$$

Moreover the equality holds if and only if all zone graphs of G are trees.

Let $C = C_{2r}$ be an isometric cycle of a median graph G. Then the convex closure of C is Q_r. Therefore all convex cycles of median graphs have length four and consequently $e(G) = 0$. Thus Theorem 1.1 immediately implies (1).

It was further asked in [2] if it is true that $2n - m - 2i(G) \leq 0$ holds for partial cubes with more than two vertices. We next show that the answer is negative. With Theorem 1.1 in hand it seems plausible to search for possible counterexamples among graphs with convex excess bigger than their isometric dimension. Such examples can indeed be constructed, for instance, as follows.

Let $P(r, s)$, $1 \leq s \leq r$, be the parallelogram hexagonal graph, see Fig. 1 for $P(5, 3)$.

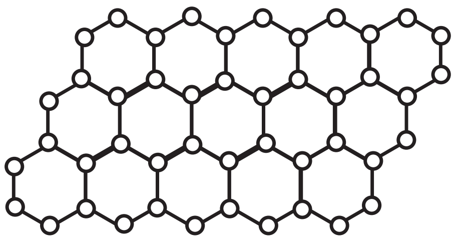

Figure 1: The parallelogram hexagonal graph $P(5, 3)$

Then $P(r, s)$ has $n = (r+1)(2s+2)-2$ vertices, $m = (r+1)(2s+1)-2+r(s+1)$ edges, while $i(P(r, s)) = 2r + 2s - 1$. Consequently,

$$2n - m - 2i(P(r, s)) = rs - 2(r + s) + 3\,.$$

In particular, for $r = s$ this reduces to $r^2 - 4r + 3$ which is strictly positive for any $r \geq 4$ and in fact arbitrarily large.

In the next section we define the concepts and introduce the techniques needed in this paper. The subsequent section contains the proof of Theorem 1.1. We conclude the paper with a couple of related remarks.

2 Preliminaries

The graph distance $d_G(u, v)$ between vertices u and v of a connected graph G is the usual shortest path distance. If H and H' are subgraphs of G, the distance between the subgraphs is defined as $d(H, H') = \min\{d(u, u') \mid u \in H, u' \in H'\}$. A shortest path between vertices u and v will be briefly called a *u, v-geodesic*.

A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. *Partial cubes* are isometric subgraphs of hypercubes. H is *convex*, if for all u, $v \in V(H)$, all shortest u, v paths from G belong to H. A convex subgraph is isometric but the converse need not be true. The *convex closure* of H is the smallest convex subgraph of G containing H.

A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u,v-path, shortest u,w-path, and shortest w,v-path.

The concept of expansion is due to Mulder [16, 17] in the context of median graphs and Chepoi [4] in the context of partial cubes (and partial Hamming graphs).

We say that two nonempty isometric subgraphs G_1 and G_2 form an *isometric cover* of a graph G provided that $G = G_1 \cup G_2$, by which we mean that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If G is connected then $G_1 \cap G_2 \neq \emptyset$ for every isometric cover G_1, G_2.

Suppose G_1, G_2 is an isometric cover of G. For $i = 1, 2$, let $\widetilde{G}_i$ be an isomorphic copy of G_i, and for a vertex $u \in G_1 \cap G_2$, let u_i be the corresponding vertex in $\widetilde{G}_i$. The *expansion* of G with respect to G_1, G_2 is the graph $\widetilde{G}$ obtained from the disjoint union of $\widetilde{G}_1$ and $\widetilde{G}_2$, where for each $u \in G_1 \cap G_2$ the vertices u_1 and u_2 are joined by a new edge in $\widetilde{G}$. We say that the expansion is *connected* provided that $G_1 \cap G_2$ is a connected graph. Chepoi [4] proved that a graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions.

Another important concept that yields a characterization of partial cubes is the Djoković-Winkler relation Θ defined on the edge set of a connected graph G as follows. Two edges $e = xy$ and $f = uv$ of G are in the relation Θ if $d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)$. Relation Θ is reflexive and symmetric. Winkler [21] proved that a connected bipartite graph is a partial cube if and only if Θ is also transitive. Hence Θ is an equivalence relation on the edge set $E(G)$ of a partial cube G. It partitions $E(G)$ into the so-called Θ-*classes*. The *isometric dimension* $i(G)$ of a partial cube G is the number of its Θ-classes. Equivalently, $i(G)$ equals the dimension of the smallest hypercube into which G embeds isometrically.

We now list several well-known lemmas about the relation Θ that are needed in the next section. For their proofs see [12].

Lemma 2.1 *Suppose P is a geodesic in a graph G. Then no two different edges of P are in relation Θ. In particular, if G is a partial cube, the edges of P belong to pairwise different Θ-classes.*

Lemma 2.2 *Let G be a bipartite graph and suppose that $xy\Theta uv$. Then the notation can be chosen so that $d(x,u) = d(y,v) = d(y,u) - 1 = d(x,v) - 1$.*

Lemma 2.3 *Suppose P is a walk connecting the endpoints of an edge e. Then P contains an edge $f \neq e$ with $e\Theta f$.*

Lemma 2.4 *Let C be an isometric cycle of a partial cube G and e an edge of C. Then e is in relation Θ to exactly one edge of C, namely, its antipodal edge.*

Θ-classes and expansions are related as follows.

Lemma 2.5 *Let G be a partial cube and let $\widetilde{G}$ be the expansion of G with respect to an isometric cover G_1, G_2. Then the edges between $\widetilde{G}_1$ and $\widetilde{G}_2$ form a Θ-class of $\widetilde{G}$, while the other Θ-classes of $\widetilde{G}$ are induced by those of G. In particular, $i(\widetilde{G}) = i(G) + 1$.*

The reverse operation to the expansion is defined as follows. Let G be a partial cube and E its Θ-class. Contracting every edge of E, a partial cube is obtained that is called the *contraction* of G (with respect to E).

3 Proof of Theorem 1.1

We begin our proof by two lemmas concerning convex cycles in partial cubes.

Lemma 3.1 *Let G_1, G_2 be an isometric cover of a partial cube G and let C be a convex cycle of G. Then C is either completely contained in one of the G_i's or C meets $G_1 \cap G_2$ in two antipodal vertices.*

Proof. Let $G_0 = G_1 \cap G_2$. Suppose C meets $G_1 - G_2$ as well as $G_2 - G_1$. Let y be a vertex of C that lies in $G_1 - G_2$ and z a vertex of C from $G_2 - G_1$. Let u be the first vertex from $C \cap G_0$ on one of the two y, z-paths along C and let v be the first vertex from $C \cap G_0$ on the other y, z-path. Let P be the u, v-path that goes along C and contains y. Let Q be the other u, v-path along C, that is, the one that contains z. Since C is convex and hence $d(u, v) = d_C(u, v)$, at least one of P and Q is a geodesic. Furthermore, since G_1 and G_2 are isometric and hence $d = d_{G_1}(u, v) = d_{G_2}(u, v)$, and because C is convex, we infer that $|P| = |Q| = d$. Therefore u and v are antipodal vertices of C. Suppose there is a vertex $w \in C \cap G_0$, $w \neq u, v$. Then $w \in Q$ and $d(w, v) < d$. Since $z \notin G_1$ and G_1 is isometric, there must be a w, v-geodesic contained in G_1. But this is not possible as C is convex. Hence we conclude that $C \cap G_0 = \{u, v\}$. □

A cycle that meets the intersection of an isometric cover in two antipodal vertices will be called a *cross cycle*.

Let G be a partial cube and let $\widetilde{G}$ be the expansion of G with respect to an isometric cover G_1, G_2. Suppose C is a convex cycle of G. By Lemma 3.1, C is either completely contained in one of the G_i's or C is a cross cycle. In the first case C expands to an identical cycle $\widetilde{C}$ in $\widetilde{G}$. Suppose C is a cross cycle and let y and z be its (antipodal) vertices from $G_1 \cap G_2$. Then C naturally expands to a cycle $\widetilde{C}$ of $\widetilde{G}$ that consists of the copy of the y, z-subpath of C in $\widetilde{G}_1$, the copy of the z, y-subpath of C in $\widetilde{G}_2$, and the edges $y_1 y_2$ and $z_1 z_2$. Note that $|\widetilde{C}| = |C| + 2$.

Lemma 3.2 *Let G be a partial cube and let $\widetilde{G}$ be the expansion of G with respect to an isometric cover G_1, G_2. Suppose C is a convex cycle of G. Then $\widetilde{C}$ is a convex cycle of $\widetilde{G}$.*

Proof. The statement is clearly true if C is contained in one of the G_i's, hence let us assume that C meets $G_0 = G_1 \cap G_2$ in antipodal vertices y and z.

Suppose $\widetilde{C}$ is not convex. Then there are vertices $u, v \in \widetilde{C}$ and a u, v-geodesic P such that P is not contained in $\widetilde{C}$. Assume that $d(u, v)$ is as small as possible so that P meets $\widetilde{C}$ only in u and v. Suppose $u, v \in \widetilde{G}_i$ for some i. Since the distance function d_{G_i} coincides with the distance function $d_{\widetilde{G}_i}$ we infer that C is not convex. Hence we may assume that $u \in \widetilde{G}_1$ and $v \in \widetilde{G}_2$. But now P necessarily contains an edge x_1x_2 where $x_1 \in \widetilde{G}_1$ and $x_2 \in \widetilde{G}_2$. Since the contraction shorten the length of P we infer again that C is not convex. $\square$

We are now ready for the key lemma of our proof.

Lemma 3.3 *Let G_1, G_2 be an isometric cover of a partial cube G and let $\widetilde{G}$ be the expansion of G with respect to G_1, G_2. Suppose that $G_0 = G_1 \cap G_2$ consists of t connected components. Then G contains $t-1$ convex cycles $C^1, \ldots, C^{t-1}$ each of them intersecting G_0 in two antipodal vertices. (That is, C^i, $i \leq 1 \leq t-1$, is a cross cycle.)*

Proof. There is nothing to be proved for $t = 1$, hence assume in the reminder $t \geq 2$. We will construct the cycles $C^1, \ldots, C^{t-1}$ inductively.

Let $H_0, \ldots, H_{t-1}$ be the connected components of G_0. We may without loss of generality assume that $d = d(H_0, H_1) = \min\{d(H_i, H_j) \mid 0 \leq i < j \leq t-1\}$. Select vertices $x_0 \in H_0$ and $x_1 \in H_1$ such that $d_G(x_0, x_1) = d$. Clearly, $d \geq 2$. As G_1 is isometric in G, there is an x_0, x_1-geodesic P that lies completely in G_1. Analogously there exists an x_0, x_1-geodesic Q that lies completely in G_2. Note that $|P| = |Q|$. By the minimality assumption, all inner vertices of P lie in $G_1 - G_2$ and the inner vertices of Q lie in $G_2 - G_1$. Let C^1 be the cycle obtained by amalgamating P and Q.

Suppose we have constructed cycles $C^1, \ldots, C^r$, where $1 \leq r \leq t-2$ and that these cycles intersect G_0 in vertices from $\cup_{i=0}^{r} H_r$. Select now indices $0 \leq i \leq r$ and $r+1 \leq j \leq t-1$ such that $d(H_i, H_j)$ is as small as possible. Let $y \in H_i$ and $z \in H_j$ be vertices such that $d(y, z) = d(H_i, H_j)$. As above, there is a y, z-geodesic such that all of its inner vertices are in $G_1 - G_2$ and a y, z-geodesic such that its inner vertices lie in $G_2 - G_1$. Combining these two geodesics yields the cycle C^{r+1}. Assume without loss of generality that $j = r+1$. In this way we have constructed $r+1$ cycles $C^1, \ldots, C^{r+1}$ that intersect G_0 in vertices from $\cup_{i=0}^{r+1} H_i$. Proceeding in this way cycles $C^1, \ldots, C^{t-1}$ are obtained.

To complete the proof we need to show that every C^r is convex. So let $C^r = yPzQy$, where $y \in H_i$, $z \in H_r$, $0 \leq i < r$, P is contained in G_1 and Q is contained in G_2.

We first show isometry. Suppose by contradiction that C^r is not isometric. Then there are vertices $u, v \in C^r$ such that $d_G(u,v) < d_{C^r}(u,v)$. By the construction of C^r neither can both u and v lie in G_1 nor can they both lie in G_2. So assume $u \in G_1 - G_2$ and $v \in G_2 - G_1$. Let u and v be selected so that $d(u,v)$ is as small as possible among all pairs of vertices that violate isometricity of C^r. Let R be a u,v-geodesic. By the minimality assumption, R is internally disjoint with C^r. Let x be a vertex of R that lies in G_0. Then $d(u,v) = d(u,x) + d(x,v)$. See Fig. 2 where the described situation is depicted.

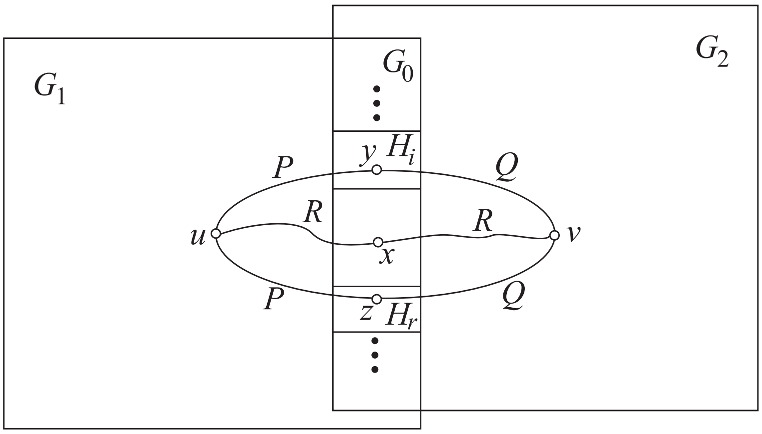

Figure 2: Situation from the proof of isometry

Claim: $d(z,x) < d(z,y)$ and $d(y,x) < d(z,y)$.

Suppose on the contrary that $d(z,x) \geq d(z,y)$. Considering the geodesic P we get

$$d(z,u) + d(u,y) = d(z,y) \leq d(z,x) \leq d(z,u) + d(u,x)\,.$$

Consequently $d(u,y) \leq d(u,x)$. On the other hand, since Q is a geodesic we infer that

$$d(z,v) + d(v,y) = d(z,y) \leq d(z,x) \leq d(z,v) + d(v,x)\,.$$

Therefore $d(v,y) \leq d(v,x)$. It follows that

$$d_{C^r}(u,v) \leq d(u,y) + d(y,v) \leq d(u,x) + d(x,v) = d_G(u,v)\,.$$

This contradiction proves the first claimed inequality. The second inequality follows by symmetry.

Let $x \in H_k$. Recall that the cycle C^r is constructed in such a way that $d(y,z) = d(H_i, H_r)$ is as small as possible where $i < r$. Now, if $k < r$ then since $d(z,x) < d(z,y)$, x would be a better selection than y. And if $k \geq r$ then because $d(y,x) < d(z,y)$, x would be a better selection than z. We conclude that C^r is isometric.

Assume now that C^r is not convex. Let u, v be vertices of C^r for which there exists a u, v-geodesic R such that $R \not\subset C^r$. Let u and v be selected as close as possible. Then R is internally disjoint from C^r. Let T be the u, v-geodesic along C^r and let $T' = C^r - T$.

Suppose $u, v \in G_1$. Then $T \subseteq G_1$ or $\{u, v\} = \{y, z\}$ and T can be chosen to be in G_1. Let u' be the neighbor of u on R and u'' the neighbor of u on T. Note that $u' \neq u''$. (See the left-hand side of Fig. 3.) Since R is internally disjoint from C^r, $R \cup T'$ forms a cycle, denote it with C'. The internal vertices of R do not meet G_0. Indeed, if $x \in R \cap G_0$ would be such a vertex then $d(z, x) < d(z, y)$ would hold, hence as above we would have a contradiction with the selection of y and z. Therefore C' is a cross cycle just as C^r, thus with the same argument as for C^r we find out that C' is isometric. Let f be the antipodal edge to uu' on C'. (See the left-hand side of Fig. 3 again.) Then by Lemma 2.4, uu' is in relation Θ with f. On the other hand, f is also in relation Θ with its antipodal edge on C^r. This is the edge uu''. Since G is a partial cube and hence Θ is transitive, it follows that $uu' \Theta uu''$. But this is not possible since $u' \neq u''$.

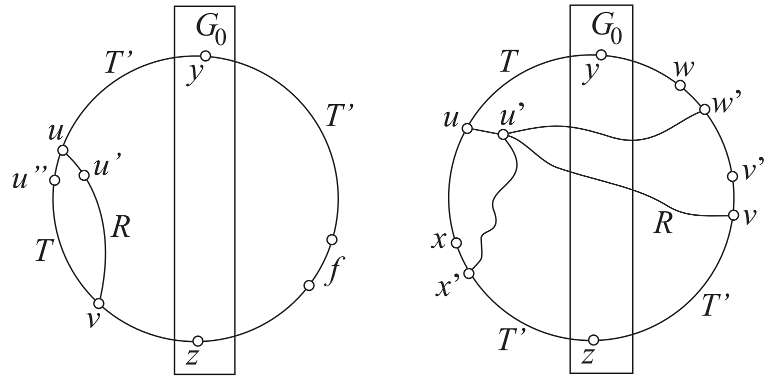

Figure 3: Situations from the proof of convexity

So suppose $u \in G_1 - G_2$ and $v \in G_2 - G_1$. We may assume that a u, v-geodesic T along C^r contains y. Recall that R is internally disjoint from T. Let u' be the neighbor of u on R and v' the neighbor of v on T. By Lemma 2.3, the edge $e = uu'$ is in relation Θ to at least one edge $f = ww'$ on $R \cup T$ and by Lemma 2.1 $f \notin R$. Assume without loss of generality $d(u, w) < d(u, w')$. (See the right-hand side of Fig. 3.) Then Lemma 2.2 implies that $d(u', w') = d(u, w)$ and $d(u, w') = 1 + d(u', w')$. By the minimality assumption it then follows that $f = v'v$. Since C^r is isometric, vv' is in relation Θ to its antipodal edge on C^r, say xx'. Clearly, $xx' \in G_1$. By the transitivity of Θ, $uu' \Theta xx'$ and by Lemma 2.2 we may assume that $d(u, x) = d(u', x')$. Let X be a u', x'-geodesic. Then X does not meet G_0 for otherwise we would find a vertex in G_0 that is closer to z than to y. Then $uu'X$ is a u, x'-geodesic that lies in G_1. But this has been ruled out above. □

We are now in a position to complete the proof of Theorem 1.1.

The result clearly holds for the smallest partial cubes K_1 and K_2. We proceed by induction on the number of expansion steps. Let G be a partial cube with n vertices and m edges and suppose that $2n - m - i(G) - e(G) \leq 2$ holds. Let G_1, G_2 be an isometric cover of G. Let $G_0 = G_1 \cap G_2$, $n_0 = |V(G_0)|$, and $m_0 = |E(G_0)|$. Let $\widetilde{G}$ be the expansion of G with respect to G_1, G_2. Setting $\widetilde{n} = |V(\widetilde{G})|$ and $\widetilde{m} = |E(\widetilde{G})|$ we have

$$\widetilde{n} = n + n_0 \quad \text{and} \quad \widetilde{m} = m + n_0 + m_0\,.$$

From Lemma 2.5 we also know that

$$i(\widetilde{G}) = i(G) + 1\,.$$

Assume that G_0 consists of t connected components. Construct cycles C^r, $1 \leq r \leq t-1$, as in Lemma 3.3. Then $\widetilde{C^r}$ is well-defined and $|\widetilde{C^r}| = |C^r| + 2$ holds. Moreover, each cycle C^r is convex by Lemma 3.3, therefore $\widetilde{C^r}$ is convex by Lemma 3.2. It follows that $\widetilde{C^r}$ contributes one more to $e(\widetilde{G})$ than C^r contributes to $e(G)$. Since by Lemma 3.2 any convex cycle C of G that contributes to $e(G)$ expends to a convex cycle $\widetilde{C}$ of $\widetilde{G}$ that contributes at least $(|C| - 4)/2$ to $e(\widetilde{G})$, we conclude that

$$e(\widetilde{G}) \geq e(G) + t - 1\,.$$

Note also that $m_0 \geq n_0 - t$. Having all these relations in mind we obtain:

$$\begin{aligned}
&2\widetilde{n} - \widetilde{m} - i(\widetilde{G}) - e(\widetilde{G}) \\
&\leq 2(n + n_0) - (m + n_0 + m_0) - (i(G) + 1) - (e(G) + t - 1) \\
&= (2n - m - i(G) - e(G)) + (n_0 - m_0 - t) \\
&\leq 2 + (n_0 - (n_0 - t) - t) \\
&= 2\,.
\end{aligned}$$

□

It remains to prove the equality part of (2).

We first claim that if for a partial cube the equality holds in (2) then its every zone graph is a tree.

Suppose on the contrary that there exists a partial cube $\widetilde{G}$ on $\widetilde{n}$ vertices and $\widetilde{m}$ edges such that $2\widetilde{n} - \widetilde{m} - i(\widetilde{G}) - e(\widetilde{G}) = 2$ such that for some Θ-class F, the zone graph Z_F of $\widetilde{G}$ is not a tree. Assume in addition that $i(\widetilde{G})$ is as small as possible among partial cubes that fulfill (i) and (ii).

Clearly, $i(\widetilde{G}) \geq 3$. Let G be the contraction of $\widetilde{G}$ with respect to F. Let $\widetilde{G}$ be obtained be the expansion of G with respect to the isometric cover G_1, G_2 of G and set $G_0 = G_1 \cap G_2$. Let n, m be the number of vertices of G, and let n_0 and t be the number of vertices and connected components of G_0.

Then $\widetilde{n} = n + n_0$, $\widetilde{m} \geq m + n_0 + (n_0 - t)$, and $i(\widetilde{G}) = i(G) + 1$. Moreover, since Z_F is connected and not a tree, we infer (having in mind Lemma 3.2) that $e(\widetilde{G}) \geq e(G) + t$. Also, since G is a partial cube, $2n - m - i(G) - e(G) \leq 2$. From all the above we get:

$$\begin{aligned}
2 &= 2\widetilde{n} - \widetilde{m} - i(\widetilde{G}) - e(\widetilde{G}) \\
&\leq 2(n + n_0) - (m + n_0 + (n_0 - t)) - (i(G) + 1) - (e(G) + t) \\
&= (2n - m - i(G) - e(G)) - 1 \\
&\leq 2 - 1\,.
\end{aligned}$$

This contradiction proves the claim.

Suppose now that G is a partial cube on n vertices and m edges such that every zone graph of G is a tree. We are going to prove that $2n - m - i(G) - e(G) = 2$ by induction on $i(G)$. The statement is clearly true for $i(G) \leq 2$.

Select an arbitrary Θ-class F of G and let G_1 and G_2 be the connected components of $G \setminus F$. Then G_1 and G_2 are partial cubes whose zone graphs are trees hence by the induction hypothesis $2n_1 - m_1 - i(G_1) - e(G_1) = 2$ and $2n_2 - m_2 - i(G_2) - e(G_2) = 2$, where n_1, n_2, m_1, m_2 are the number of vertices and edges of G_1 and G_2. Let G_{10} be the subgraph of G_1 induced with vertices that have a neighbor in G_2 and let G_{20} be the corresponding (isomorphic) subgraph of G_2. Let $n_0 = |V(G_{10}|$.

Note first that G_0 is a forest. Indeed, if C would be a cycle of G_0, then C and its isomorphic copy C' induce a ladder $C \,\square\, K_2$. Since partial cubes are bipartite and $K_{2,3}$-free, any 4-cycle is convex hence the ladder would give a cycle in Z_F.

Let t be the number of connected components of G_{10} (and of G_{20}). Since Z_F is a tree, there exist $t-1$ convex cycles $C^{(1)}, \ldots, C^{(t-1)}$, each connecting two different connected components of G_{10} (and of G_{20}). Note that each of these cycles contains two edges of F while its other edges lie in $G_1 \setminus G_{10}$ and in $G_2 \setminus G_{20}$. Now we have:

$$\begin{aligned}
&n = n_1 + n_2, \\
&m = m_1 + m_2 + n_0, \\
&i(G) = i(G_1) + i(G_2) - (n_0 - t) - \textstyle\sum_{j=1}^{t-1}(e(C^{(j)}) + 1) + 1, \\
&e(G) = e(G_1) + e(G_2) + \textstyle\sum_{j=1}^{t-1}(e(C^{(j)})).
\end{aligned}$$

Inserting these equalities into $2n - m - i(G) - e(G)$ we obtain to be equal

$$(2n_1 - m_1 - i(G_1) - e(G_1)) + (2n_2 - m_2 - i(G_2) - e(G_2)) - 2 = 2 + 2 - 2 = 2\,.$$

4 Concluding remarks

In Lemma 3.2 we have proved that during an expansion step a convex cycle of a partial cube lifts to a convex cycle. This property is not true for isometric cycles, see Fig. 4 for an example. This fact might be one of the reasons why an Euler-type inequality for partial cubes was so elusive.

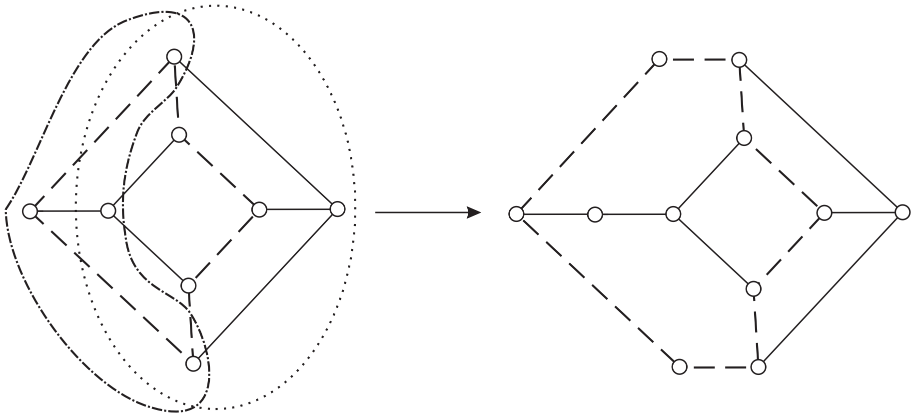

Figure 4: An isometric cycles need not expand to an isometric cycle

To characterize the equality in Theorem 1.1 we have introduced zone graphs and proved that their characteristic property for the equality in (2) is to be connected and with no cycles. In fact, using the arguments from the proof of Lemma 3.3 it can be proved that the zone graphs of partial cubes are always connected and have no multiple edges. It seems an interesting project to further study these graphs and their relation to partial cubes.

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