University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

Preprint series, Vol. 47 (2009), 1066

DOMINATION GAME

Boštjan Brešar Sandi Klavžar
Douglas F. Rall

ISSN 1318-4865

January 9, 2009

Ljubljana, January 9, 2009

Domination game*

Boštjan Brešar[†]
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
bostjan.bresar@uni-mb.si

Sandi Klavžar[†]
Faculty of Mathematics and Physics
University of Ljubljana, Slovenia
and
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
sandi.klavzar@fmf.uni-lj.si

Douglas F. Rall
Department of Mathematics, Furman University
Greenville, SC, USA
doug.rall@furman.edu

January 8, 2009

Abstract

The domination game played on a graph G consists of two players, Dominator and Staller who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen by either player, at least one additional vertex is dominated. Dominator wishes to dominate the graph in as few steps as possible and Staller wishes to delay the process as much as possible. The game domination number $\gamma_g(G)$ is the number of vertices chosen when Dominator starts the game and the Staller-start game domination number $\gamma_g'(G)$ when Staller starts the game. It is proved that for any graph G, $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$, and that all possible values can be realized. It is also proved that for any graph G, $\gamma_g(G) - 1 \leq \gamma_g'(G) \leq \gamma_g(G) + 2$, and that most of the possibilities for mutual values of $\gamma_g(G)$ and $\gamma_g'(G)$ can be realized. A connection with Vizing's conjecture is established and several problems and conjectures stated.

*The paper was initiated during the visit of the third author to the IMFM supported in part by the Slovenia-USA bilateral grant BI-US/08-10-004.

[†]Supported by the Ministry of Science of Slovenia under the grant P1-0297. The author is also with the Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana.

Keywords: domination, domination game, game domination number, Vizing's conjecture

AMS subject classification (2000): 05C69, 91A43

1 Introduction

The game chromatic number was described for the first time in 1981 [9] and remained more or less unnoticed for many years. However, in the last several years the topic has received an astonishing amount of attention. See, for instance, [4, 7, 12, 15, 16], and some papers on different variations of the game chromatic number [2, 3, 5, 13, 14]. On the other hand, it seems natural to study an analogous game with respect to domination, the problem brought to our attention by Mike Henning [10]. In fact, we find it quite surprising that this game has been ignored so far and hope to initiate its research here. Just like in the coloring variant, we have one player who wishes to dominate a graph in as few steps as possible and another player who wishes to delay the process as much as possible. As far as we know, the paper [1] is until now the only paper dealing with game domination; however, the concept from [1] is different from ours.

We describe two games played on a finite graph $G = (V, E)$. In Game 1 two players, Dominator and Staller, alternate—with Dominator going first—taking turns choosing a vertex from G. We let $d_1, d_2, \ldots$ denote the sequence of vertices chosen by Dominator and $s_1, s_2, \ldots$ the sequence chosen by Staller. (Note that in some cases it will be convenient to speak about the 0-th step of some player.) These vertices must be chosen in such a way that whenever a vertex is chosen by either player, at least one additional vertex of the graph G is dominated that was not dominated by the vertices previously chosen. That is, for each i,

- $N[d_i] \setminus \cup_{j=1}^{i-1} N[\{d_j, s_j\}] \neq \emptyset$; and
- $N[s_i] \setminus \left(\cup_{j=1}^{i-1} N[\{d_j, s_j\}] \cup N[d_i] \right) \neq \emptyset$.

In Game 2 the players alternate choosing vertices as in Game 1, except that Staller begins. In this game we denote the two sequences of vertices by $s_1', s_2', \ldots$ and $d_1', d_2', \ldots$. As in Game 1, we also require that each chosen vertex strictly enlarges the closed neighborhood of the set of chosen vertices.

Since the graph G is finite, each of these games will end in some finite number of moves regardless of how the vertices are chosen. In both of the games Dominator chooses vertices using a strategy that will force the game to end in the fewest number of moves, and Staller uses a strategy that will prolong the game as long as possible. We define the *game domination number* of G to be the total number of vertices chosen when Dominator and Staller play Game 1 on graph G using optimal strategies, and we denote this value by $\gamma_g(G)$. The *Staller-start game domination number* of G, denoted by $\gamma_g'(G)$, is the cardinality of the set of vertices chosen when Game 2 is played on G.

While there is no general upper bound on the game chromatic number of a graph in terms of a function of its chromatic number, the game version of domination number is bounded by two times the ordinary domination number. Nevertheless in establishing results of this paper we found more inspiration in game coloring theory than in domination theory. In particular, the main idea of our proofs is similar to that from [4, 15] in the sense that an 'imagination strategy' is used. Let us explain this idea briefly. When proving bounds on some type of game domination number of a graph, one of the players (either Dominator or Staller) imagines another game is played at the same time, usually on the same graph. For the imagined game the number of moves k is known, and hence if Dominator (resp. Staller) is the player who imagines the parallel game, then he has a strategy that ensures the imagined game takes at most (at least) k moves. The basic procedure of his strategy in the real game is simply to copy each move of the opponent to the imagined game, respond in the imagined game by an optimal move, and finally copy back this move to the real game. Two problems are possible: some of his moves that are legal in the imagined game need not be legal in the real game, and some of the moves of the opponent in the real game need not be legal in the imagined game. This is, of course, the main problem and such cases are handled in different ways with respect to a given situation. The overall aim is to ensure that the number of moves in the real game is bounded by the number of moves in the imagined game (usually these numbers are the same or differ by at most one), which gives the bound on the corresponding game domination number of the graph.

We first concentrate on Game 1, and among other results establish a connection between the game domination number and Vizing's conjecture and show that $\gamma_g(G)$ is bounded with $\gamma(G)$ below and with $2\gamma(G)-1$ above. Then, in Section 3, we show that for any positive integer k and any nonnegative integer $r \leq k-1$ there exists a graph G such that $\gamma(G) = k$ and $\gamma_g(G) = k + r$. In the second part of the paper we study the relation between Game 1 and Game 2. In Section 4 we prove that for any graph G, $\gamma_g(G) - 1 \leq \gamma_g'(G) \leq \gamma_g(G) + 2$. In the subsequent section we study possible pairs (k, ℓ) such that there exists a graph G with $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. We conclude the paper with several open problems and conjectures.

2 Basic properties of γ_g and Vizing's conjecture

When Game 1 is played on a graph G, the set of vertices chosen by Dominator and Staller together is a dominating set of G. Thus we easily have the bound $\gamma_g(G) \geq \gamma(G)$ for every G. On the other hand, Dominator can order the vertices in a minimum dominating set $A = \{d_1, d_2, \ldots, d_{\gamma(G)}\}$ and play according to this list. Because of the way that Staller selects vertices, it is possible that Dominator may not be able to use some of the vertices in this list. However, when Dominator exhausts the sequence in A the graph is dominated, and hence no more moves are legal. Together these prove the following bounds relating the ordinary domination number and the game domination number.

Theorem 1 *For any graph G,*

$$\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1\,.$$

So the game domination number is nicely bounded by the domination number, a property that doesn't hold for the game chromatic number.

It is well-known that the chromatic analog of γ_g is not monotone, even with respect to spanning subgraphs (a typical case is the complete bipartite graph $K_{n,n}$ whose game chromatic number is 3, while its spanning subgraph, obtained by removing a perfect matching has game chromatic number n). Somewhat surprisingly, γ_g does not behave on spanning subgraphs in a manner consistent with the ordinary domination number γ. For example, if H is the connected spanning subgraph of $K_2 \,\square\, K_3$ obtained by deleting two K_2-fiber's edges, then

$$\gamma_g(K_2 \,\square\, K_3) = 3 > 2 = \gamma_g(H)\,.$$

We next connect γ_g with Vizing's conjecture, the main open problem in graph domination. This conjecture states that for all graphs G and H,

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H)\,,$$

where $G \,\square\, H$ denotes the Cartesian product of graphs G and H [11]. One of the finest results so far, related to this problem, is due to Clark and Suen [6] who proved that for all graphs G and H,

$$\gamma(G \,\square\, H) \geq \frac{1}{2}\gamma(G)\gamma(H).$$

We note that if Vizing's conjecture is true, then, for every pair of graphs G and H, we can use Theorem 1 to get the following inequalities:

$$\gamma_g(G \,\square\, H) \geq \gamma(G \,\square\, H) \geq \gamma(G)\gamma(H) \geq \frac{1}{4}\gamma_g(G)\gamma_g(H)\,.$$

Thus, if one could find graphs G and H such that $\gamma_g(G)\gamma_g(H) > 4\,\gamma_g(G \,\square\, H)$, then this shows that Vizing's conjecture is false.

Another connection between the game domination number and the inequality in Vizing's conjecture is the following. Suppose there is a constant $c > 0$ such that $c\,\gamma_g(G \,\square\, H) \geq \gamma_g(G)\gamma_g(H)$. Then

$$c \cdot 2\gamma(G \,\square\, H) \geq c \cdot \gamma_g(G \,\square\, H) \geq \gamma_g(G)\gamma_g(H) \geq \gamma(G)\gamma(H)\,.$$

That is, for all pairs G and H,

$$\gamma(G \,\square\, H) \geq \frac{1}{2c}\gamma(G)\gamma(H)\,.$$

Note that $c = 1$ gives the bound established by Clark and Suen while a proof that $c = 1/2$ would establish Vizing's conjecture.

3 Realization of game domination numbers

We have observed in Theorem 1 that $\gamma_g(G)$ lies between $\gamma(G)$ and $2\gamma(G)-1$. In this section we prove that all possible values for γ_g are eventually realizable.

If H is a graph, then we say that G is a *generalized corona* with *base* H if G is constructed from H by adding at least one leaf as a neighbor of each vertex of H.

Lemma 2 *Let G be a generalized corona. Then when Game 1 is played on G there is an optimal strategy for Dominator in which he selects only base vertices.*

Proof. Players are playing Game 1 on generalized corona G with base H. Dominator will imagine another game is played at the same time on the graph G, and he will play it according to an optimal strategy that forces the game to end in $\gamma_g(G)$ steps. In this imagined game it is possible that Dominator will make some of his moves on leaves. We need to show that in the real game there will be no more moves than in the imagined game. Whenever it is legal, Dominator will copy a move he makes in the imagined game to the real game, and copy the move made in response by Staller from the real to the imagined game. When in the imagined game an optimal move of Dominator is on a leaf d_k, then instead of copying the leaf to the real game, Dominator plays on the base vertex, d'_k, adjacent to that leaf. (Note that this is a legal move, since the leaf d_k was not dominated before that.) Hence the sequences of moves are $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d_k$ in the imagined game and $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d'_k$ in the real game. The game continues in the same way, and whenever a leaf d_i is played by Dominator in the imagined game, its neighboring base vertex d'_i is picked in the real game. All moves of Staller (made in the real game) are legal when copied to the imagined game since he is playing on the graph (of the real game) in which all vertices are dominated which are dominated in the graph of the imagined game. Suppose that in the strategy of Dominator (that is used in the imagined game) he plays on d'_k or one of the leaves attached to it (or some other base vertex whose leaf was previously chosen in the game). In this case Dominator imagines another move of Staller (on a leaf), and responds with the neighboring base vertex. His response is then copied also to the real game. Note that also after these moves, the set of vertices dominated in the real game contains the set of vertices dominated in the imagined game. Hence when the imagined game is finished the real game is also finished, and the number of moves in the real game is less than or equal to the number of moves in the imagined game. □

The above lemma tells us also that we may assume that in a generalized corona Dominator plays on base vertices. More precisely, given a generalized corona G, there is a game in G with $\gamma_g(G)$ moves in which Dominator plays only on base vertices and Staller uses an optimal strategy.

Lemma 3 *Let G be a generalized corona and let x be a vertex in G of degree 1. Then*

$$\gamma_g(G-x) \leq \gamma_g(G) \leq \gamma_g(G-x)+1.$$

Proof. Let $G' = G - x$. We first prove that $\gamma_g(G') \leq \gamma_g(G)$. Suppose Game 1 is played on G'. As in the proof of Lemma 2, Dominator will imagine another game is played at the same time on the graph G, and he will play it according to a strategy that forces the game to end in at most $\gamma_g(G)$ steps. He will copy each of his moves from the imagined game to the real game, and copy each of Staller's moves from the real to the imagined game. Since x cannot be selected by Staller (as he plays only on G'), we can assume that x will not be selected during the imagined game played on G because Dominator chooses only base vertices of G. Hence the moves in both games are the same all the time, and once the game on G is finished, the game on G' is also trivially finished.

We next prove that $\gamma_g(G) \leq \gamma_g(G') + 1$. Suppose Game 1 is played on G. Similarly as before, Dominator will imagine another game is played at the same time on the graph G', and he will play it according to a strategy that forces the game to end in at most $\gamma_g(G')$ steps. He will copy each of his moves from the imagined game to the real game, and copy each of Staller's moves from the real to the imagined game, as long as this is possible. Note that at some point in the game Staller can choose x which cannot be copied in the imagined game. In that case (when x is chosen) Dominator imagines another move of Staller, which he may assume is a leaf s_i, in the imagined game, and responds according to his optimal strategy (which is always copied to the real game). If Staller later makes a move that is illegal in the imagined game (by picking the leaf s_i or by picking the base neighbor of s_i whose only undominated neighbor at that point is s_i), then Dominator again imagines another move of Staller in the imagined game, and so on. At the end of the imagined game all vertices are also dominated in the real game, except perhaps one leaf. Hence, the last player to move must dominate that leaf and the game ends. The number of steps is at most one more in the real game. In the case when x is not chosen, the games have the same sequence of moves, according to the definition of the imagined game, and so after the imagined game is ended, x is the only vertex that could be left undominated in the real game. In that case, x must be dominated at the final step no matter whose turn it is.

In both cases the total number of vertices selected in the real game is at most one more than in the imagined game. Thus, $\gamma_g(G) \leq \gamma_g(G') + 1$. □

We can now prove the announced result of this section.

Theorem 4 *For any $k \geq 1$ and any $r \in \{0, 1, \ldots, k-1\}$ there exists a graph G such that $\gamma(G) = k$ and $\gamma_g(G) = k + r$.*

Proof. Consider an arbitrary graph G with $\gamma(G) = 1$ to see that the statement holds for $k = 1$. Select now $k \geq 2$ and fix it. Let G_k be the generalized corona with base K_k and exactly one leaf attached everywhere. Then it is clear that $\gamma(G_k) = \gamma_g(G_k) = k$. Let H_k be the generalized corona with base K_k and enough, say k, leaves attached everywhere. Again, $\gamma(H_k) = k$. By Lemma 2, we may assume that when Game 1 is played on H_k, Dominator selects only base vertices. If Staller plays only on leaves of H_k, then no matter how they play, before Dominator selects the last base vertex, Staller has at least one leaf available to select. It follows that Dominator needs k steps to finish the game and consequently, $\gamma_g(H_k) = 2k - 1$.

The generalized corona H_k can be obtained from the corona G_k by attaching leaves one by one until all the base vertices have k leaves attached. In this way there is a sequence of graphs

$$G_k = X_0, X_1, \ldots, X_{k(k-1)} = H_k \,,$$

such that $\gamma_g(G_k) = k$, $\gamma_g(H_k) = 2k - 1$, and $\gamma_g(X_{i-1}) \le \gamma_g(X_i) \le \gamma_g(X_{i-1}) + 1$ for any $1 \le i \le k(k-1)$. Therefore, for any $r \in \{0, 1, \ldots, k-1\}$ there exists an index j, such that $\gamma_g(X_j) = k + r$. Since $\gamma(X_j) = k$, the proof is complete. □

We note that in the above proof a smaller number of leaves in H_k would also do the job. If $2^s \le k < 2^{s+1}$, then attaching t leaves at each vertex of K_k, where $t > \log_2 k$, already suffices.

4 Dominator-start versus Staller-start game

In this section we compare the game domination number with the Staller-start game domination number. We first note that, a bit surprisingly, for some graphs the latter invariant is strictly smaller than the game domination number. For example, $\gamma_g'(C_6) = 2$ while $\gamma_g(C_6) = 3$. When Dominator moves first on the 6-cycle, Staller responds by choosing a neighbor of d_1 and thus forces the game to last a total of 3 moves. However, when Game 2 is played on C_6, Dominator can always choose b_1' which together with s_1' forms a minimum dominating set of C_6. On the other hand, the two invariants are not far apart. More precisely, in this section we prove the following.

Theorem 5 *For any graph G,*

$$\gamma_g(G) - 1 \le \gamma_g'(G) \le \gamma_g(G) + 2.$$

The main proof idea below is that Dominator will be imagining another appropriate game and playing it according to an optimal strategy. The imagined game will be dependent on a concrete situation.

In order to prove our theorem, the following variant of the domination game will be useful. Consider the game in which Dominator starts, and Staller is allowed, but not obligated, to skip exactly one move. That is, at some point in the game, instead of picking a vertex, Staller may decide to pass, and it is Dominator's turn again. Afterwards the vertices are picked alternatingly again until the end. The number of moves in such a game, where both players are playing optimally, is denoted by $\gamma_g^{sp}(G)$. We call this game the *Staller-pass game.*

Lemma 6 *For any graph G, $\gamma_g^{sp}(G) \le \gamma_g(G) + 1$.*

Proof. Let the players play the Staller-pass game. The strategy of Dominator is that he will be imagining another game is being played at the same time—an ordinary domination game—and he will be playing it according to an optimal strategy (hence the length of this game will not be greater than $\gamma_g(G)$). So two games are played at the same time: the real (Staller-pass) game, and the (ordinary) game, imagined by Dominator. In the first part of the game Dominator will just copy each move of Staller to his imagined game, and respond in the imagined game with an optimal move from the ordinary domination game. Each of these moves of Dominator is then also copied to the real game. This will continue until Staller decides to pass a move in the real game (if he decides that at all; but if he does not decide to pass any move, then the two games are the same and thus both have $\gamma_g(G)$ moves). Up to that point the moves in both games are the same, and they form the sequence $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d_k$. Then in the real game it is Dominator's turn, but in the imagined game it is Staller's turn. Hence in the imagined game Dominator imagines that Staller made a (legal) move, say s_k, and he responds in this game in an optimal way by picking, say d_{k+1}. Then he also picks the same vertex in the real game. So the current sequence of the real game is $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d_k, d_{k+1}$. The game next continues in the same way as before. Note that all moves of Dominator are chosen (in an optimal way) with respect to the imagined game, so all his moves will be legal also in the real game. On the other hand, Staller chooses his moves with respect to the real game, but in the imagined game the copy of this move may not be legal. Suppose that some move of Staller, say s_m, is not legal in the imagined game. Note that this can only happen for $m > k$. Denote by

$$N[C] = \bigcup_{i=1}^{m} N[d_i] \cup \bigcup_{\substack{i=1 \\ i \neq k}}^{m-1} N[s_i] \,.$$

Note that the move of Staller is illegal in the imagined game precisely when

$$N[s_m] \setminus N[C] \subseteq N[s_k] \setminus N[C] \,.$$

Indeed, the move s_m is legal in the real game, so its closed neighborhood is not included in $N[C]$, and it must be that all vertices from $N[s_m] \setminus N[C]$ (and such vertices exist) are in the neighborhood of s_k.

We distinguish two cases. First suppose that $N[s_m] \setminus N[C]$ is a proper subset of $N[s_k] \setminus N[C]$. Then Dominator makes a non standard move—he picks s_k in the real game—and he does nothing in the imagined game. Note that s_k is a legal move in the real game, and after that move of Dominator the sets of dominated vertices are the same in both games, and it is Staller's turn in both games. Moreover, the number of moves in the real game is one more than the number of moves in the game imagined by Dominator. Since Dominator can play until the end by the same strategy as in the beginning of the game (since Staller already used his pass), the total number of moves in the real (Staller-pass) game is at most $\gamma_g(G)+1$.

The second possibility is that $N[s_m] \setminus N[C] = N[s_k] \setminus N[C]$. Hence, at that time (after Staller's move s_m) both games already have the same sets of dominated vertices. Hence, instead of this move Dominator imagines another (legal) move of Staller in the imagined game. He again responds to that move optimally, and copies the same move in the real game where it is his turn. (Note that, at that time, the number of moves in the real game is one less than the number of moves in the imagined game.) The game continues in the same way as in the beginning, until either the imagined game is ended, or some move of Staller is again not legal in the imagined game. In the latter case again one of the two cases appear, which Dominator can resolve in the way we have explained, and the game goes on. In the former case (which eventually must happen since the graph is finite), when the imagined game is finished, the real game may not be finished, since one of the vertices (the last imagined move of Staller) was not picked in the real game. As soon as it is Dominator's turn in the real game, he picks that vertex and the game ends. In the worst case, the imagined game ended with the move of Dominator. Hence the real game ends after at most two additional moves (the move of Staller and the final move of Dominator). Since before that the real game had one less move, we get that the (real) Staller-pass game has at most one move more than the imagined game, yielding the inequality of the lemma. □

Lemma 7 *For any graph G, $\gamma_g'(G) \le \gamma_g^{sp}(G)+1$.*

Proof. Let the players play Game 2 on G. This time Dominator will be imagining a Staller-pass domination game started by Dominator. Dominator will be playing the imagined game according to an optimal strategy, hence the number of moves in this game will not be greater than $\gamma_g^{sp}(G)$. Let us denote the first move in the real game by s_0. In the imagined game, this move is ignored, and it is Dominator's turn to play. His first move d_1 is played according to his strategy in the Staller-pass game, by which he can ensure that there are at most $\gamma_g^{sp}(G)$ moves altogether. In the first part of the game each of his moves is copied to the real game, hence d_1 is also his first move in the real game (unless already the second part of the game began, which we will explain soon). Staller responds in the real game by s_1, and this move is copied also to the imagined game, and so on. In the first part of the game the sequence of moves in the real game is $s_0, d_1, s_1, \ldots, d_{k-1}, s_{k-1}$, and in the imagined game it is $d_1, s_1, \ldots, d_{k-1}, s_{k-1}$. Note that every move which is legal in the real game is also legal in the imagined game, hence all moves of Staller will result in a legal move of Staller in the imagined game. On the other hand, moves of Dominator are copied to the real game, and it can happen at some point that such a move, say d_k, where $k \ge 1$, is not legal in the real game. The first part of the game is then ended, and we come to the second part. If this situation never happens during the game, there is clearly nothing to prove, since then the real game is ended in at most $\gamma_g^{sp}(G)+1$ steps, as we claim.

Set

$$N[C]=\bigcup_{i=1}^{k-1}(N[s_i]\cup N[d_i])$$

and similarly as in the proof of Lemma 6 note that the move d_k is illegal in the real game precisely when $N[d_k]\setminus N[C]\subseteq N[s_0]\setminus N[C]$.

First suppose that $N[d_k]\setminus N[C]$ is a proper subset of $N[s_0]\setminus N[C]$. Then after Dominator plays d_k in the imagined game, he imagines Staller plays s_0 in this game (which is legal in this case). In the second case when $N[d_k]\setminus N[C]=N[s_0]\setminus N[C]$, Dominator imagines that Staller skips a move. Recall that the Staller-pass game is imagined, hence this is the first and the only time a move of the Staller is skipped. In both cases, after that point the set of vertices that are dominated coincide in both games, and it is Dominator's turn in both games. Hence the game is played as in the beginning (Dominator following the optimal strategy of the Staller-pass game), but now there are no problems with legality of his moves anymore. When the imagined game ends, also the real game ends. In the first case, the number of moves in the imagined game is one more than in the real game, while in the second case, the number of moves in both games is equal. □

From Lemmas 6 and 7 the upper bound of Theorem 5 follows immediately.

The lower bound of Theorem 5 (or, equivalently, an upper bound for $\gamma_g(G)$ with respect to $\gamma_g'(G)$) can be obtained directly. We do this in the rest of the section.

Let the players play Game 1 on G. Dominator will be imagining that Game 2 is being played on another copy of G and playing it according to an optimal strategy. Hence the number of moves in this game will not be greater than $\gamma_g'(G)$. So two games are played at the same time: the real Game 1, and Game 2, imagined by Dominator. He imagines the first move of Staller (arbitrarily), say s_0. He responds to that move in the imagined game by d_1 and copies this move to the real game. This move and all further moves of Dominator in the imagined game are played according to his strategy in Game 2, by which he can ensure that there are at most $\gamma_g'(G)$ moves altogether in the imagined game. After d_1 it is Staller's turn in the real game, and his move s_1 is copied in the imagined game. The moves in the first part of the game create the following sequences: $s_0,d_1,s_1,\ldots,d_{k-1},s_{k-1},d_k$ in the imagined game and $d_1,s_1,\ldots,d_{k-1},s_{k-1},d_k$ in the real game. Note that every move which is legal in the imagined game is also legal in the real game. Hence all moves of Dominator will result in a legal move of Dominator in the real game. On the other hand, moves of Staller are copied to the imagined game, and it can happen at some point that such a move, say s_k, where $k\geq 1$, is not legal in the imagined game. The first part of the game is then ended, and we come to the second part. If this situation never happens during the game, there is clearly nothing to prove, since then the real game is ended in at most $\gamma_g'(G)+1$ steps, as we claim.

Denote by

$$N[C] = \bigcup_{i=1}^{k} N[d_i] \cup \bigcup_{i=1}^{k-1} N[s_i]$$

and note that s_k is illegal in the imagined game precisely when $N[s_k] \setminus N[C] \subseteq N[s_0] \setminus N[C]$.

Suppose that $N[s_k] \setminus N[C]$ is a proper subset of $N[s_0] \setminus N[C]$. In this case Dominator plays (an unusual move) s_0 in the real game, and it is Staller's turn again in both games. Now, note as well that the sets of vertices that are dominated so far are equal in both games. Hence the game continues until the end with the same sequence of moves in both games, with Dominator ensuring that there are at most $\gamma_g(G)$ moves played in the imagined game. Since in the real game there is one more move, the claim is proven in this case. In the second case when $N[s_k] \setminus N[C] = N[s_0] \setminus N[C]$, the sets of dominated vertices are equal in both games already then. But in the real game it is Dominator's move, while in the imagined game it is Staller's move. Then Dominator imagines another (legal) move of Staller in the imagined game, say s_{k+1}. Again at that time there is one more move in the imagined game than in the real game, and the situation is similar as in the beginning (only that the role of s_0 is taken by s_{k+1}). Now the game goes on by the same strategy of Dominator as in the beginning, and the situation can be repeated several times if the second case appears, and Dominator solves it in the same way as we explained (if the first case appears, then the conclusion of the game is trivial, as explained before). Hence the remaining situation is that the imagined game stops at some point. Then it may happen that in the real game not everything is dominated, and such undominated vertices are all in the neighborhood of the last vertex s_ℓ, imagined by Dominator (s_ℓ can also be s_0 or s_{k+1}). Then, after the imagined game is finished there are at most two moves until the end of the real game, because Dominator can play s_ℓ once it his turn. Since before the last move of the imagined game there is one more move in the imagined game than in the real game, we conclude that at the end the number of moves in the real game is at most one more than in the imagined game. The proof of the lower bound of Theorem 5 is thus complete.

5 Realizable pairs

For a pair of positive integers k and ℓ we say the pair (k, ℓ) is *realizable* if there exists a graph G such that $\gamma_g(G) = k$ and $\gamma'_g(G) = \ell$. From Theorem 5 we know that $k - 1 \leq \ell \leq k + 2$ holds for a realizable pair (k, ℓ).

Proposition 8 *For any positive integer k, each of the pairs (k, k), $(k, k+1)$ and $(2k+1, 2k)$ is realizable.*

Proof. For a positive integer k, let P'_k denote the tree of order $2k$ obtained from the path of order k by adding a vertex of degree 1 adjacent to each vertex of P_k. It is straightforward to show that if either Game 1 or Game 2 is played on P'_k, then the game will last exactly k steps. Thus, $\gamma_g(P'_k) = k = \gamma'_g(P'_k)$, and thus P'_k realizes (k, k).

Let T be the tree constructed from the path of order 4 by adding two new leaves adjacent to one of the vertices of degree 2. For an integer $k \geq 2$, let F_k be the graph $T \cup (k-2)P_2$. When Game 1 is played on F_k, Dominator can begin by selecting the vertex of degree 4 in T. It is clear that exactly $k-1$ more moves will be made in this game, and since $\gamma(F_k) = k$, Theorem 1 implies that $\gamma_g(F_k) = k$. However, Staller can select a leaf adjacent to the vertex of degree 4 in T on his first move in Game 2. Dominator can now force the game to end in at most k, and no fewer, additional steps. It is easy to check that Staller can do no better in his first move, and hence $\gamma'_g(F_k) = k+1$. Therefore, F_k realizes $(k, k+1)$. The star $K_{1,2}$ realizes $(1, 2)$, and so each pair $(k, k+1)$ can be realized by some graph.

For $k \geq 1$, let $G_k = C_6 \cup (2k-2)P_2$. In Game 1 played on G_k, Dominator can be forced to be the first player to select a vertex from C_6. By then selecting a vertex on the 6-cycle adjacent to the one chosen by Dominator, Staller can make the game last until exactly $2k+1$ vertices are chosen in total. Dominator would never allow more than 3 vertices to be chosen from the cycle, and since exactly one vertex must be selected from each P_2, it follows that $\gamma_g(G_k) = 2k+1$. On the other hand, Staller can be required to select the first vertex from the 6-cycle in a play of Game 2 on G_k. By then selecting the vertex on the 6-cycle diametrically opposite the one chosen by Staller, Dominator can force the game to end in exactly $2k$ moves. Hence $\gamma'_g(G_k) \leq 2k$ and by Theorem 5 and the above fact that $\gamma_g(G_k) = 2k+1$ we conclude that $\gamma'_g(G_k) = 2k$. Thus G_k realizes $(2k+1, 2k)$. □

Suppose G is a graph that realizes $(k, k+2)$ for some k. Then it is easy to see that $G \cup C_4$ realizes $(k+2, k+4)$. In Game 1 (resp. Game 2) Dominator (Staller) plays first on G, and all of his subsequent moves are also in G, except when his opponent plays on C_4, in which case he responds in C_4 as well. In fact, the strategies on $G \cup C_4$ only show that $\gamma_g(G \cup C_4) \leq k+2$ and $\gamma'_g(G \cup C_4) \geq k+4$, but we know by Theorem 5 that $\gamma'_g(G \cup C_4) - \gamma_g(G \cup C_4)$ is not greater than 2 which implies that $(k+2, k+4)$ is really the pair realized by $G \cup C_4$.

From the definitions it is clear that no graph realizes $(2, 1)$. We think that also for $k \geq 2$, there is no graph G such that $\gamma_g(G) = 2k$ and $\gamma'_g(G) = 2k-1$. It is also obvious that $(1, 3)$ cannot be a realized pair, since graphs with $\gamma_g(G) = 1$ are precisely those with $\gamma(G) = 1$. In addition let us show that $(2, 4)$ cannot be realized.

Let $\gamma_g(G) = 2$ and d_1, s_1 be a sequence of moves in Game 1. Then the set $A = V(G) \setminus N[d_1]$ induces a complete graph, otherwise Staller could enforce the game last longer by playing on a vertex in A that is not adjacent to some other vertex of A. Moreover, by the same argument, any vertex that is adjacent to a vertex of A is adjacent to all vertices of A. Now, let Game 2 be played on G, and let s'_1 be the first move of Staller. If s'_1 is such a move that $d'_1 = d_1$ is not legal, this means that $N[d_1]$ is already dominated. But then $\gamma'_g(G) = 2$ since Dominator can end the game by playing on a vertex of A. Hence, we may assume that d_1 is legal, and so after that only a subset of A remains undominated, which implies $\gamma'_g(G) \leq 3$.

6 Some problems and conjectures

We start with a question that could be very useful in proving results about the game domination number.

Problem 1 *Let G be a graph, and let A, B be subsets of $V(G)$ such that $A \subset B$. Suppose that Game 1 (or Game 2) is played on two copies of G. On the first copy of G the set of vertices that are already dominated is A, and on the second copy it is B. In both games the same player (Dominator or Staller) has the next move. Is it then true that the number of moves until the end on the first copy is at least as big as the number of moves on the second copy?*

We suspect that the above problem has an affirmative answer, but could find no argument.

Note that the bound in Lemma 6 is sharp (for example on $G = K_{1,2} \cup C_6$), but for Lemma 7 we do not know. As we have seen there exist graphs G for which $\gamma'_g(G) = \gamma_g(G) + 1$. We don't know whether the upper bound of Theorem 5 can be lowered by 1 and hence pose:

Conjecture 1 *Pairs $(k, k+2)$ for $k \geq 1$ are not realizable. That is, there exists no graph G, such that $\gamma_g(G) = k$ and $\gamma'_g(G) = k + 2$.*

We also suspect that the remaining pairs that satisfy the conditions of Theorem 5 are not realizable. More precisely:

Conjecture 2 *Pairs $(2k, 2k-1)$ for $k \geq 1$ are not realizable. That is, there exists no graph G, such that $\gamma_g(G) = 2k$ and $\gamma'_g(G) = 2k - 1$.*

References

[1] N. Alon, J. Balogh, B. Bollobás, T. Szabó, Game domination number, Discrete Math. 256 (2002) 23–33.

[2] S. D. Andres, The incidence game chromatic number, Discrete Appl. Math., in press. doi:10.1016/j.dam.2007.10.021

[3] T. Bartnicki, J. Grytczuk, H. A. Kierstead, The game of arboricity, Discrete Math. 308 (2008) 1388–1393.

[4] T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, I. Peterin, Game chromatic number of Cartesian product graphs, Electron. J. Combin. 15 (2008) #R72, 13 pp.

[5] A. Beveridge, T. Bohman, A. Frieze, O. Pikhurko, Game chromatic index of graphs with given restrictions on degrees, Theoret. Comput. Sci. 407 (2008) 242–249.

[6] W. E. Clark, S. Suen, An inequality related to Vizing's conjecture, Electron. J. Combin. 7 (2000), #N4, 3 pp.

[7] T. Dinski, X. Zhu, Game chromatic number of graphs, Discrete Math. 196 (1999) 109–115.

[8] A. S. Fraenkel, Combinatorial games: selected bibliography with a succinct gourmet introduction, Electron. J. Combin., Dynamic Survey 2 (2007) 78pp.

[9] M. Gardner, Mathematical games, Scientific American 244 (1981) 18–26.

[10] M. Henning, personal communication, 2003.

[11] W. Imrich, S. Klavžar, D. F. Rall, *Topics in Graph Theory: Graphs and Their Cartesian Product*, A K Peters, Wellesley, MA, 2008.

[12] H. A. Kierstead, T. Trotter, Planar graph coloring with an uncooperative partner, J. Graph Theory 18 (1994) 569–584.

[13] H. A. Kierstead, T. Trotter, Competitive colorings of oriented graphs, Electron. J. Combin. 8 (2001) #R12, 15pp.

[14] J. Nešetřil, E. Sopena, On the oriented game chromatic number, Electron. J. Combin. 8 (2001) R#14, 13pp.

[15] X. Zhu, Game coloring the Cartesian product of graphs, J. Graph Theory 59 (2008) 261–278.

[16] X. Zhu, Refined activation strategy for the marking game, J. Combin. Theory ser. B 98 (2008) 1–18.