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ABSTRACT. In this note we show that the $(n-2)$-dimensional volumes of codimension 2 faces of an n-dimensional simplex are algebraically independent quantities of the volumes of its edge-lengths. The proof involves computation of the eigenvalues of Kneser graphs.

INTRODUCTION

Let $\mathcal{T}_n$ be the set of congruence classes of n-simplices in Euclidean space $\mathbb{E}^n$. The set $\mathcal{T}_n$ is an open manifold (also a semi-algebraic set) of dimension $\binom{n+1}{2}$. Coincidentally, a simplex $T \in \mathcal{T}_n$ is determined by the $\binom{n+1}{2}$ lengths of its edges. Furthermore, the square of the volume of $T \in \mathcal{T}_n$ is a polynomial in the squares of the edge-lengths $\ell_{ij} = \|v_i - v_j\|_2$ $(1 \le i < j \le n+1)$, where $v_1, \dots v_{n+1}$ are the vertices of T. This polynomial is given by the Cayley-Menger determinant formula (cf., e.g., [5] or [2]):

$$V^2(T) = \frac{(-1)^{n+1}}{2^n (n!)^2} \det C, \tag{1}$$

where C is the *Cayley-Menger matrix* of dimension $(n+2)\times(n+2)$, whose rows and columns are indexed by $\{0, 1, \dots, n+1\}$ and whose entries are defined as follows:

$$C_{ij} = \begin{cases} 0, & i = j \\ 1, & \text{if } i = 0 \text{ or } j = 0, \text{ and } i \neq j \\ \ell_{ij}^2, & \text{otherwise.} \end{cases}$$

Note that an n-simplex has $\binom{n+1}{2}$ edges and the same number of $(n-2)$-dimensional faces, and so the following question is natural:

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Question 1. Is the congruence class of every n-simplex determined by the $(n-2)$-dimensional volumes of its $(n-2)$-faces?

Question 1 must be classical, but the earliest reference stating it that we are aware of is Warren Smith's PhD thesis [10].

At the AIM workshop on Rigidity and Polyhedral Combinatorics, Bob Connelly (who was unaware of the reference [10]) raised the following:

Question 2. Is the *volume* of every n-simplex determined by the $(n-2)$-dimensional volumes of its $(n-2)$-faces?

In fact, Connelly stated Question 2 for $n = 4$, which is the first case where the question is open. For $n = 3$ the answer is trivially "Yes", since $3-2 = 1$, and we are simply asking if the volume of the simplex is determined by its edge-lengths. In dimension 2, the answer is trivially "No", since $2-2 = 0$, and the volume of codimension-2 faces of a triangle carries no information.

Clearly, the affirmative answer to Question 1 would imply an affirmative answer to Question 2. In this paper we first show that the answer to Question 2, and hence also to Question 1 is negative for every $n \geq 4$. We actually found out that this has been answered previously for $n = 4$ in [1], where an example is given and attributed to Philip Tuckey; see also [3].

Our examples are given in a separate section. Several reasons suggest that the following question may still have an affirmative answer:

Question 3. Is it true that for every choice of $\binom{n+1}{2}$ positive real numbers, there are only finitely many congruence classes of n-simplices whose $(n-2)$-dimensional volumes of the $(n-2)$-faces are equal to these numbers?

In this note we show that a weaker statement holds:

Theorem 4. *The* $\binom{n+1}{2}$ $(n-2)$*-dimensional volumes of the* $(n-2)$*-faces of an* n*-simplex are algebraically independent over* $\mathbb{C}[\ell_{ij};\ 1 \leq i < j \leq n+1]$.

Theorem 4 is clearly a necessary step in the direction of resolving Question 3, but is far from sufficient. To show it, consider the map of $\mathbb{R}^{(n+1)n/2}$ to $\mathbb{R}^{(n+1)n/2}$, which sends the vector ℓ of edge-lengths of an n-simplex to the vector Y of volumes of $(n-2)$-dimensional faces. To show Theorem 4, it is enough to check that the Jacobian $J(\ell) = \partial Y/\partial \ell$ is non-singular at *one* point. We will use the most obvious point p_1, the one corresponding to a regular simplex with all edge-lengths equal to 1. By symmetry considerations, the Jacobian $J(p_1)$ can be written as $J(p_1) = cM$, where c is a constant and M is

$$M_{e,F} = \begin{cases} 1, & \text{if the edge } e \text{ is incident with the } (n-2)\text{-face } F \\ 0, & \text{otherwise.} \end{cases}$$

The first observation is that the constant c above is not equal to 0:

Lemma 5. $J(p_1) = \dfrac{1}{(n-2)!\,(n-1)^{1/2}\,2^{(n-4)/2}}\,M.$

Proof. Let $\nu = \dfrac{(n-1)^{1/2}}{(n-2)!\,2^{(n-2)/2}}$ denote the $(n-2)$-dimensional volume of the regular $(n-2)$-simplex with all edge-lengths 1. Let us observe that the volume of a k-dimensional simplex is a *homogeneous* function of degree k of the edge-lengths. An application of Euler's Homogeneous Function Theorem shows that at p_1,

$$\frac{\partial Y_F}{\partial \ell_e} = \begin{cases} \frac{2}{n-1}\,\nu, & \text{if the edge } e \text{ is incident with the } (n-2)\text{-face } F \\ 0, & \text{otherwise.} \end{cases}$$

This implies that $c = \frac{2}{n-1}\,\nu$ and completes the proof. □

The eigenvalues of M

As shown above, Theorem 4 reduces to the assertion that the determinant of the matrix M is not zero. We will actually be able to compute all eigenvalues of M, which is of interest in its own right.

Theorem 6. *Eigenvalues of M are $\lambda_1 = \binom{n-1}{2}$ (simple eigenvalue), $\lambda_2 = 1$ with multiplicity $\frac{1}{2}(n+1)(n-2)$, and $\lambda_3 = 2-n$ with multiplicity n.*

Corollary 7. *The absolute value of the determinant of M equals*

$$\tfrac{1}{2}(n-2)^{n+1}(n-1) \neq 0,$$

for $n > 2$.

To prove Theorem 6, let us first observe that the $\binom{n+1}{2}$ rows of M are indexed by the 2-element subsets of the set $R = \{1, \ldots, n+1\}$, and its columns are indexed by the $(n-1)$-subsets F of R. By replacing each column index F with its complement $R \setminus F$, then the columns are indexed by the same set as the rows. After this convention, the matrix M becomes a symmetric matrix with zero diagonal since $M_{e,f} = 1$ if and only if $e \subseteq R \setminus f$, which is equivalent to $f \subseteq R \setminus e$. Therefore, M is the adjacency matrix of a graph G_n whose vertices are the 2-element subsets of R, and two of them are adjacent if and only if they are disjoint. Thus, the complement $\overline{G_n}$ of G_n is isomorphic to the line graph $L(K_{n+1})$ of the complete graph K_{n+1} on $n+1$ vertices.

The eigenvalues of $L(K_{n+1})$ are (see [4, p. 19]): $t_1 = 2n-2$, $t_2 = -2$, and $t_3 = n-3$, with the same multiplicities (respectively) as claimed above for the eigenvalues of M. Since the graph $L(K_{n+1})$ is regular, it is an easy exercise to see that its adjacency matrix A and the adjacency matrix M of its complement have the same set of eigenvectors. By using the fact that $A + M + I = x^t \cdot x$, where $x = (1, \dots, 1)^t$ is the eigenvector of A and M corresponding to the dominant eigenvalues of these matrices, we conclude that the eigenvalues of M are $\lambda_1 = \binom{n+1}{2} - t_1 - 1$ and $\lambda_i = -t_i - 1$ for $i = 2, 3$ (preserving multiplicities). Thus, $\lambda_1 = \binom{n+1}{2} - 2n + 1 = \binom{n-1}{2}$, $\lambda_2 = 1$, and $\lambda_3 = 2 - n$, respectively.

Singular examples

Let us consider the n-simplex in $\mathbb{R}^n$ with vertices $v_0, v_1, \dots, v_n$ given as follows. The vertex v_0 has the first $n-2$ coordinates equal to $((n-1)^{1/2} + 1)/(2^{1/2}(n-2))$, while its last two coordinates are 0. For $i = 1, 2, \dots, n-2$, the vertex v_i has ith coordinate equal to $2^{-1/2}$ and all other coordinates 0. These $n-1$ vertices form a regular $(n-2)$-simplex contained in $\mathbb{R}^{n-2} \subset \mathbb{R}^n$ with all side lengths 1. Let $a = \frac{1}{n-1} \sum_{i=0}^{n-2} v_i$ be its barycenter, and let $c := \|v_0 - a\|_2$ denote the distance from a to the vertices v_i. A short calculation shows that $c^2 = \frac{1}{2} - \frac{1}{2(n-1)}$. Now, let v_{n-1} be obtained from a by changing its last two coordinates to be real numbers p and q satisfying $p^2 + q^2 = 1 - c^2$. Similarly, let v_n be obtained in the same way by choosing another pair r, s of numbers satisfying $r^2 + s^2 = 1 - c^2$. This gives rise to an n-simplex whose all sides are equal to 1 except for the side $v_{n-1}v_n$ whose square length is $t := (p - r)^2 + (q - s)^2$. By fixing t, this simplex is determined up to congruence, and we denote it by $T(t)$. Observe that t may take any value between 0 and $4(1 - c^2)$, by selecting p, q, r, s appropriately.

Next we observe that the volumes of the $(n-2)$-faces of $T(t)$ take only two values. If an $(n-2)$-face does not contain both v_{n-1} and v_n, then it is a regular simplex, whose volume is independent of t. On the other hand if an $(n-2)$-simplex contains v_{n-1} and v_n, its volume $w = w(t)$ is uniquely determined by t. In fact, if we put the square distances in the Cayley-Menger determinant, we conclude that $w(t)^2$ is a quadratic polynomial in t, $w(t)^2 = \alpha t^2 + \beta t + \gamma$. If $t = 0$, the volume is 0, so $\gamma = 0$. For $t = 1$ we have the regular $(n-2)$-simplex, so $\alpha + \beta = \frac{n-1}{2^{n-2}((n-2)!)^2}$. Finally, using (1) (with the value of n being replaced by $n-2$) and looking at the Cayley-Menger determinant expansion term with t^2, we conclude that

$$\alpha = -\frac{(-1)^{n-1}}{2^{n-2}((n-2)!)^2} \det(J_{n-2} - I_{n-2}),$$

where J_{n-2} is the all-1-matrix and I_{n-2} is the identity matrix of order $n-2$. Since $\det(J_{n-2} - I_{n-2}) = (-1)^{n-3}(n-3)$, we conclude that $\alpha = -(n-3)2^{2-n}/((n-2)!)^2$ and $\beta = (n-2)2^{1-n}/((n-2)!)^2$. In particular,

$$w(t)^2 = \frac{1}{2^{n-2}((n-2)!)^2} ((3-n)t^2 + (2n-4)t).$$

This function is symmetric around the point $t_0 = \frac{n-2}{n-3}$. Consequently, the non-congruent n-simplices $T(t_0 - x)$ and $T(t_0 + x)$ have the same $(n-2)$-volumes of their $(n-2)$-faces for each admissible value of x, i.e. for $0 < x < \frac{n-2-4/(n-1)}{n-3}$. These examples thus show that Questions 1 and 2 have negative answers.

Concluding remarks

One can ask the same question as above for other dimension-complementary volumes, i.e. about the volumes of the $(k-1)$-faces and the $(n-k)$-faces of an n-simplex, where $2 \le k \le n/2$. If one would compare, similarly as in the case $k = 2$ above, the dependence of $(n-k)$-volumes of an $(n-k)$-face Q on the $(k-1)$-volumes of the $(k-1)$-faces $F \subset Q$, the corresponding "Jacobian" would again be a constant multiple of a symmetric matrix M, whose entries are indexed by the k-subsets of the set $R = \{1, \dots, n+1\}$ (after the column indices pass to the complementary subsets), and

$$M_{E,F} = \begin{cases} 1, & \text{if the } E \cap F = \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The graph whose adjacency matrix is M is known as the *Kneser graph* $K(n+1, k)$. Its eigenvalues can be comuted using the methods from the theory of association schemes and can be found, for example, in [8, Section 9.4].

Theorem 8. *Let n and k be integers, where $2 \le k \le n/2$, and let M be the matrix of order $\binom{n+1}{k} \times \binom{n+1}{k}$ whose entries are determined by (2). The eigenvalues of M are the integers*

$$\lambda_i = (-1)^i \binom{n-k-i+1}{k-i}, \qquad i = 0, 1, \dots, k.$$

Since $2 \le k \le n/2$, none of the eigenvalues in Theorem 8 is zero. This raises the question whether there is an analogy with Theorem 4 for $2 \le k \le n/2$, between the collection of the $\binom{n+1}{k}$ $(n-k)$-dimensional volumes of the $(n-k)$-faces of an n-simplex and the collection of all $(k-1)$-dimensional volumes of its $(k-1)$-faces.

As a final remark, we would like to point out that our original approach to this problem [9] used results about *divisors* [6] (also known as *equitable partitions* [8]) combined with the representation theory of the symmetric group and the notion of *Gelfand pairs* as developed in [7].

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