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CROSSING-CRITICAL GRAPHS WITH LARGE MAXIMUM DEGREE

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Crossing-critical graphs with large maximum degree

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Abstract

A conjecture of Richter and Salazar about graphs that are critical for a fixed crossing number k is that they have bounded bandwidth. A weaker well-known conjecture is that their maximum degree is bounded in terms of k. In this note we disprove these conjectures for every $k \geq 171$, by providing examples of k-crossing-critical graphs with arbitrarily large maximum degree.

A graph is k-*crossing-critical* (or simply k-*critical*) if its crossing number is at least k, but every proper subgraph has crossing number smaller than k. Using the Excluded Grid Theorem of Robertson and Seymour [8], it is not hard to argue that k-crossing-critical graphs have bounded tree-width [2]. However, all known constructions of crossing-critical graphs suggested that their structure is "path-like". Salazar and Thomas conjectured (cf. [2]) that they have bounded path-width. This problem was solved by Hliněný [3], who proved that the path-width of k-critical graphs is bounded above by $2^{f(k)}$, where $f(k) = (432 \log_2 k + 1488)k^3 + 1$.

In the late 1990's, two other conjectures were proposed (see [7] or [6]).

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Conjecture 1. *For every positive integer k, there exists an integer $D(k)$ such that every k-crossing-critical graph has maximum degree less than $D(k)$.*

The second conjecture was proposed as an open problem in the 1990's by Carsten Thomassen and formulated as a conjecture by Richter and Salazar [7].

Conjecture 2. *For every positive integer k, there exists an integer $B(k)$ such that every k-crossing-critical graph has bandwidth at most $B(k)$.*

Conjecture 2 would be a strengthening of Hliněný's theorem about bounded path-width and would also imply Conjecture 1.

Hliněný and Salazar [5] recently made a step towards Conjecture 1 by proving that k-crossing-critical graphs cannot contain a subdivision of $K_{2,N}$ with $N = 30k^2 + 200k$.

In this note we give examples of k-crossing-critical graphs of arbitrarily large maximum degree, thus disproving both Conjectures 1 and 2.

A *special graph* is a pair (G, T), where G is a graph and $T \subseteq E(G)$. The edges in the set T are called *thick edges* of the special graph. A *drawing* of a special graph (G, T) is a drawing of G such that the edges in T are not crossed. The crossing number $\mathrm{cr}(G, T)$ of a special graph is the minimum number of edge crossings in a drawing of (G, T) in the plane. (We set $\mathrm{cr}(G, T) = \infty$ if a thick edge is crossed in every drawing of G.) An edge $e \in E(G) \setminus T$ is *k-critical* if $\mathrm{cr}(G, T) \geq k$ and $\mathrm{cr}(G - e, T) < k$. Let $\mathrm{crit}_k(G, T)$ be the set of k-critical edges of (G, T). If $T = \emptyset$, then we write just $\mathrm{cr}(G)$ for the crossing number of G and $\mathrm{crit}_k(G)$ for the set of k-critical edges of G. Note that the graph G is k-critical if $\mathrm{crit}_k(G) = E(G)$.

A standard result (see, e.g., [1]) is that we can eliminate the thick edges by replacing them with sufficiently dense subgraphs. (In fact, one can replace every edge xy by $t = \mathrm{cr}(G, T) + 1$ parallel edges or by $K_{2,t}$ if multiple edges are not desired.)

Lemma 3. *For every special graph (G, T) with $\mathrm{cr}(G, T) < \infty$ and for any k, there exists a graph $\tilde{G} \supseteq G$ such that $\mathrm{cr}(G, T) = \mathrm{cr}(\tilde{G})$ and $\mathrm{crit}_k(G, T) \subseteq \mathrm{crit}_k(\tilde{G})$.*

Furthermore, note the following:

Lemma 4. *Let k be an integer. Any graph G with $\mathrm{cr}(G) \geq k$ contains a k-crossing-critical subgraph H such that $\mathrm{crit}_k(G) \subseteq E(H)$.*

Proof. For a contradiction, suppose that G is a smallest counterexample. If G were k-critical, then we would set $H = G$, hence G contains a non-k-critical edge e. It follows that $\mathrm{cr}(G - e) \geq k$. Let f be a k-critical edge in

G, i.e., $\mathrm{cr}(G - f) < k$. As $\mathrm{cr}((G - e) - f) \leq \mathrm{cr}(G - f) < k$, f is a k-critical edge in $G - e$. Therefore, $\mathrm{crit}_k(G) \subseteq \mathrm{crit}_k(G - e)$. Since G is the smallest counterexample, $G-e$ has a k-critical subgraph H with $\mathrm{crit}_k(G-e) \subseteq E(H)$. However, $H \subseteq G$ and $\mathrm{crit}_k(G) \subseteq E(H)$, which is a contradiction. □

Let us now proceed with the main result. Two paths P_1 and P_2 in a special graph are *almost edge-disjoint* if all the edges in $E(P_1) \cap E(P_2)$ are thick.

Lemma 5. *For any d, there exists a special graph (G, T) such that* $\mathrm{crit}_{171}(G, T)$ *contains at least d edges incident with one of the vertices of G.*

Proof. Let (G, T) be the special graph drawn as follows: we start with $d+1$ thick cycles $C_0, C_1, \ldots, C_d$ intersecting in a vertex v, i.e., $C_i \cap C_j = \{v\}$ for $0 \leq i < j \leq d$. Their lengths are $|C_0| = 28$, $|C_d| = 24$ and $|C_i| = 7$ for $1 \leq i < d$. They are drawn in the plane so that all their vertices are incident with the unbounded face and their clockwise order around v is $C_0, C_1, \ldots, C_d$. See Figure 1 illustrating the case $d = 5$. Let $C_0 = va_1a_2 \ldots a_{19}b_1b_2b_3c_1^0c_2^0 \ldots c_5^0$, $C_d = vt^db_3'b_2'b_1'a_1'a_2' \ldots a_{19}'$ and $C_i = vt^ic_1^ic_2^i \ldots c_5^i$ for $1 \leq i < d$. Furthermore, add d vertices $s^1, \ldots, s^d$ adjacent to v. The clockwise cyclic order of the neighbors of v is $a_1, c_5^0, s^1, t^1, c_5^1, s^2, t^2, c_5^2, \ldots, s^{d-1}, t^{d-1}, c_5^{d-1}, s^d, t^d, a_{19}'$. For $1 \leq i \leq d$, add thick cycles K_i whose vertices in the clockwise order are t^i, s^i, and five new vertices $\tilde{c}_5^{i-1}, \tilde{c}_4^{i-1}, \ldots, \tilde{c}_1^{i-1}$. Finally, add the following edges: $c_j^i\tilde{c}_j^i$ for $0 \leq i < d$ and $1 \leq j \leq 5$, a_ia_i' for $1 \leq i \leq 19$ and b_ib_i' for $1 \leq i \leq 3$. As described, $T = \bigcup_{i=0}^{d} E(C_i) \cup \bigcup_{i=1}^{d} E(K_i)$. Let $M = \{a_1a_1', a_2a_2', \ldots, a_{19}a_{19}', b_1b_1', b_2b_2', b_3b_3'\}$.

This drawing $\mathcal{G}$ of (G, T) has $\binom{19}{2} = 171$ crossings, as the edges a_ia_i' and a_ja_j' intersect for each $1 \leq i < j \leq 19$, and there are no other crossings. Let us show that $\mathrm{cr}(G, T) = 171$. Let $\mathcal{G}'$ be an arbitrary drawing of (G, T), and for a contradiction assume that it has less than 171 crossings. Let us first observe that every thick cycle C_i and K_j is an induced nonseparating cycle of G. Therefore it bounds a face of $\mathcal{G}'$. Consider the cyclic clockwise order of the neighbors of v according to the drawing $\mathcal{G}'$. For each cycle C_i ($0 \leq i \leq d$), the two edges of C_i incident with v are consecutive in this order, since C_i bounds a face. Without loss on generality, we assume that each cycle C_i bounds a face distinct from the unbounded one. If the cyclic order of the vertices around the face C_i is the same as in the drawing $\mathcal{G}$, we say that C_i is drawn *clockwise*, otherwise it is drawn *anti-clockwise*. We may assume that C_0 is drawn clockwise. If C_d were drawn clockwise as well, then each pair of edges a_ia_i' and a_ja_j' with $1 \leq i < j \leq 19$ would intersect, and the drawing $\mathcal{G}'$ would have at least 171 crossings. Therefore, C_d is drawn anti-clockwise. It

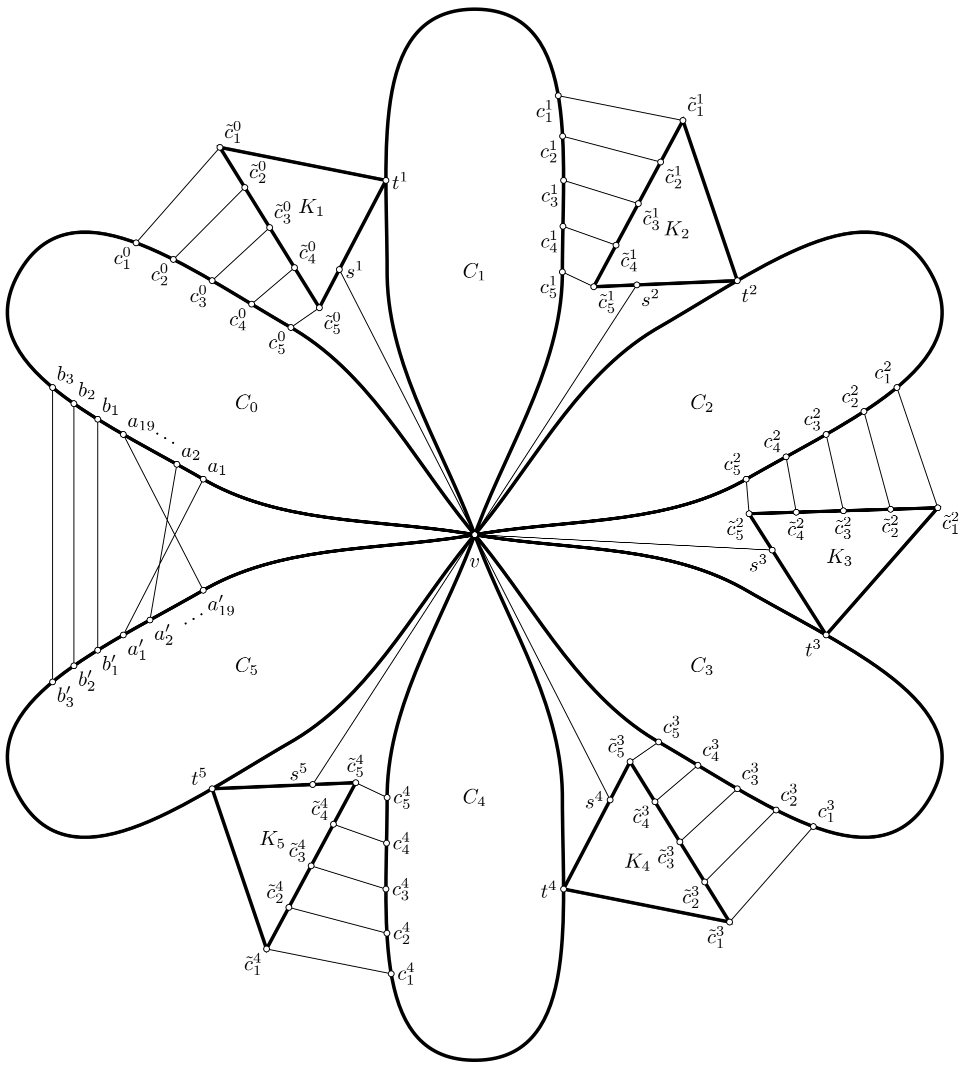

Figure 1: A special graph with critical edges vs^i

follows that the edges $a_ia'_i$ and $b_jb'_j$ intersect for $1 \le i \le 19$ and $1 \le j \le 3$, and the edges $b_ib'_i$ and $b_jb'_j$ intersect for $1 \le i < j \le 3$, giving 60 crossings. For $1 \le i \le 5$, let P_i be the path $c_i^0\tilde{c}_i^0\tilde{c}_{i-1}^0 \dots \tilde{c}_1^0 t^1 c_1^1 c_2^1 \dots c_i^1 \tilde{c}_i^1 \dots \tilde{c}_1^1 t^2 \dots t^d$. These paths are mutually almost edge-disjoint and each of them intersects all edges of M in the drawing $\mathcal{G}'$, thus contributing at least 110 crossings all together. Therefore, the drawing $\mathcal{G}'$ has at least 170 crossings. Since we assume that this drawing has less than 171 crossings, we conclude that there are no other crossings.

The cycle $va_1a'_1a'_2 \dots a'_{19}$ splits the plane into two regions R_1 and R_2, such that R_1 contains the face bounded by C_0 and R_2 contains the face bounded by C_d. For $j = 1, 2$, let A_j be the set of cycles C_i ($0 \le i \le d$) such that the face bounded by C_i lies in the region R_j. As P_1 intersects the edge $a_1a'_1$ only once, $A_1 = \{C_0, C_1, \dots, C_{k-1}\}$ and $A_2 = \{C_k, C_{k+1}, \dots, C_d\}$ for some k with $1 \le k \le d$. As the path P_1 does not intersect itself, all cycles in A_1 are drawn clockwise and their clockwise order around v is C_0, C_1, …, C_{k-1}. Similarly, all cycles in A_2 are drawn anti-clockwise and their clockwise order around v is C_d, C_{d-1}, …, C_k.

Let us now consider the cycle K_k. Since the edges $c_4^{k-1}\tilde{c}_4^{k-1}$ and $c_5^{k-1}\tilde{c}_5^{k-1}$ do not intersect, the thick path $c_5^{k-1}vt^ks^k\tilde{c}_5^{k-1}$ is not intersected, and C_{k-1} is drawn clockwise, K_k is drawn clockwise as well. Since C_k lies in the region R_2, the vertex t^k and thus the whole thick cycle K_k lie in R_2. However, that means that the edge s^kv intersects either the path P_1 or the edge $a_1a'_1$, which is a contradiction. We conclude that $\mathrm{cr}(G, T) = 171$.

On the other hand, $\mathrm{cr}(G - vs^k, T) < 171$, for $1 \le k \le d$ (in fact, $\mathrm{cr}(G - vs^k, T) = 170$). To see that, consider the drawing of $(G - vs^k, T)$ in which the cycles C_0, C_1, …, C_{k-1} are drawn clockwise, the cycles C_k, C_{k+1}, …, C_d are drawn anti-clockwise, and the cyclic order of the neighbors of v is $a_1c_5^0s^1t^1c_5^1 \dots s^{k-1}t^{k-1}c_5^{k-1}a'_{19}t^dc_5^{d-1}s^{d-1}t^{d-1} \dots c_5^kt^k$. The intersections of this drawing are of edges $a_ia'_i$ with $b_jb'_j$ for $1 \le i \le 19$ and $1 \le j \le 3$, the edges $b_ib'_i$ with $b_jb'_j$ for $1 \le i < j \le 3$, and the edges $c_i^{k-1}\tilde{c}_i^{k-1}$ with all edges of M for $1 \le i \le 5$. Therefore, the edge vs^k is 171-critical for each k, so v is incident with d critical edges. □

We are ready for our main result.

Theorem 6. *For every $k \ge 171$ and every d, there exists a k-crossing-critical graph H containing a vertex of degree at least d.*

Proof. Let (G, T) be the special graph constructed in Lemma 5. By Lemma 3, there exists a graph $H' \supseteq G$ such that $\mathrm{cr}(H') = \mathrm{cr}(G, T) \ge 171$ and

$\mathrm{crit}_{171}(G,T) \subseteq \mathrm{crit}_{171}(H')$. Let H be the 171-critical subgraph of H' obtained by Lemma 4. As $\mathrm{crit}_{171}(G,T) \subseteq \mathrm{crit}_{171}(H') \subseteq E(H)$, H contains at least d edges incident with one vertex, hence $\Delta(H) \geq d$. For $k > 171$ we add to H $k-171$ copies of the graph K_5 in order to get a k-crossing-critical graph. □

Actually, in the proof of Theorem 6, we can take $t = \lfloor \frac{k}{171} \rfloor$ copies of the graph H and $k - 171t$ copies of K_5. This gives rise to a k-critical graph with $t = \Omega(k)$ vertices of (arbitrarily) large degree. We conjecture that this is best possible in the following sense:

Conjecture 7. *For every positive integer k there exists an integer $D = D(k)$ such that every k-crossing-critical graph contains at most k vertices whose degree is larger than D.*

It is not even obvious if there exist k-crossing-critical graphs with arbitrarily many vertices of degree more than 6. Surprisingly, such examples have been constructed recently by Hliněný [4]. His examples may contain arbitrarily many vertices of any even degree smaller than $2k-1$.

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