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LOWER BOUNDS FOR DOMINATION AND TOTAL DOMINATION NUMBER OF DIRECT PRODUCTS GRAPHS

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Lower bounds for domination and total domination number of direct products graphs

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Abstract

An exact lower bound for the domination number and the total domination number of the direct product of finitely many complete graphs is given: $\gamma(\times_{i=1}^{t} K_{n_i}) \geq t+1$, $t \geq 3$. Sharpness is established in the case when the factors are large enough in comparison to the number of factors. The main result gives a lower bound for the domination (and the total domination) number of the direct product of two arbitrary graphs: $\gamma(G\times H) \geq \gamma(G)+\gamma(H)-1$. Infinite families of graphs that attain the bound are presented. For these graphs it also holds $\gamma_t(G \times H) = \gamma(G) + \gamma(H) - 1$. Some additional parallels with the total domination number are made.

Key words: dominating set; domination number; total dominating set; total domination number; direct product graphs; complete graph

1 Introduction

The study of domination number in product graphs has a long history. Back in 1963 Vizing [15] posed a still opened conjecture concerning domination number of Cartesian product graphs

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H).$$

In 1995 Gravier and Khelladi [4] posed an analogous conjecture for direct product graphs, that is

$$\gamma(G \times H) \geq \gamma(G)\gamma(H).$$

A year later Nowakowski and Rall [12] gave a counterexample for the latter conjecture. The same year Klavžar and Zmazek [9] found an infinite series of counterexamples. Moreover, using these graphs they showed that the difference

$$\gamma(G)\gamma(H) - \gamma(G \times H)$$

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can be arbitrary large. A decade later Brešar, Klavžar and Rall [1] found an upper bound

$$\gamma(G \times H) \leq 3\gamma(G)\gamma(H),$$

where G and H are graphs with no isolated vertices. In the same paper it is shown that with some modifications (subdividing each edge with two vertices and attaching pendant vertices) of two arbitrary connected graphs we can achieve the upper bound. It is necessary to mention the bound from [12]

$$\gamma(G \times H) \geq \max\{\rho(G)\gamma(H), \rho(H)\gamma(G)\} \quad (1)$$

and the improved bound from [13]

$$\gamma(G \times H) \geq \max\{\rho(G)\gamma_t(H), \rho(H)\gamma_t(G)\}. \quad (2)$$

These two bounds will come in handy later.

Some exact values of the domination number of direct products of certain graphs can be, for instance, found in [4, 10, 11]. These results involve products of two paths, product of a path and a complement of a path, product of K_2 and a tree, bipartite graph, odd cycle, ... Jha ([6, 7]) worked on perfect r-domination of the direct product of two and three cycles. Klavžar, Špacapan and Žerovnik ([8, 16]) continued with the study of r-perfect codes in the product of finitely many cycles. The size of a 1-perfect code (if there exists one) is equal to the domination number, hence in these papers (indirectly) the domination number of the product of certain cycles is given.

As mentioned, the lower bound for the domination number of direct product is not multiplicative. In Section 3 we will prove that it is additive:

$$\gamma(G \times H) \geq \gamma(G) + \gamma(H) - 1$$

and that this bound is sharp.

Before we get to the latter we will study the domination number of the direct product of finitely many complete graphs. In Section 2 we will prove that

$$\gamma(\times_{i=1}^{t} K_{n_i}) \geq t + 1$$

for $t \geq 3$ and $n_i \geq 2$. The latter inequality (which is for these graphs quite more accurate than the general one) becomes equality if $n_i \geq t + 1$. These results will motivate us for a general result on the lower bound but are not needed in the following proofs. After the general result we will take a look in Section 4 in what circumstances the lower bound is achieved. For arbitrary $d \in \mathbb{N}$ we will provide graphs G and H which give the lower bound with $d = \gamma(G \times H)$. For $d \in \{1, 2\}$ there are only trivial cases and for a fixed $d \geq 3$ we will show that for every pair $1 < d_1, d_2 < d$ with $d = d_1 + d_2 - 1$ there are graphs G and H with $\gamma(G) = d_1$

and $\gamma(H) = d_2$ that give the lower bound. The cases of the lower bound with a complete graph as one of the two factors will turn out to be trivial. These results will give us some information also for the total domination number.

In the rest of the section we introduce the notations that are used in this paper.

The *direct product* $G \times H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$ and edges $(g_1, h_1)(g_2, h_2)$, where $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. This graph product is commutative and associative, hence it extends naturally to more than two factors. The direct product of graphs $G_1, \ldots, G_n$ will be denoted $\times_{i=1}^{n} G_i$.

$D \subseteq V(G)$ is a *dominating set* of a graph G if every vertex from $V(G) \setminus D$ is adjacent to some vertex from D. If each vertex of a dominating set D has a neighbor in D, then D is caled a *total dominating set*. The *domination number* (resp. *total domination number*) $\gamma(G)$ (resp. $\gamma_t(G)$) is the size of smallest dominating set (resp. total dominating set) of G. Naturally, when we will talk about the total domination number of a graph we will always presume that this graph has no isolated vertices.

For a set $A \subseteq V(G \times H)$ we denote

$$p_G(A) = \{g \in V(G) \mid (g, h) \in A \text{ for some } h \in V(H)\}$$

projection of A on G and

$$p_H(A) = \{h \in V(H) \mid (g, h) \in A \text{ for some } g \in V(G)\}$$

projection of A on H. In the case when A dominates $G \times H$ then $p_G(A)$ dominates G and $p_H(A)$ dominates H. This statement follows quickly from the fact that a projection of an edge in $G \times H$ on G (resp. H) is an edge in G (resp. H). This gives

$$|p_G(A)| \geq \gamma(G) \text{ and } |p_H(A)| \geq \gamma(H),$$

if A dominates $G \times H$. This simple fact will be very useful later.

For a vertex $g \in V(G)$ we denote $N[g] = \{g' \in V(G) \mid gg' \in E(G)\} \cup \{g\}$ the *closed neighborhood* of g. A set $A \subseteq V(G)$ is a 2-*packing* of G if for every two different $a_1, a_2 \in A$ follows that their closed neighborhoods are disjoint. The 2-*packing number* $\rho(G)$ is the size of largest 2-packing of G.

Let g and g' be vertices of a connected graph G. The *distance* $d(g, g')$ is the number of edges on some shortest path that starts in g and ends in g'. With $\operatorname{diam}(G) = \max\{d(u, v) | u, v \in V(G)\}$ we denote *diameter* of G. For a complete graph K_n, $n \geq 1$, we will always assume $V(K_n) = \{0, 1, \ldots, n-1\}$. The vertices $u = (u_1, \ldots, u_t)$, $v = (v_1, \ldots, v_t) \in V(\times_{i=1}^{t} K_{n_i})$ are adjacent if and only if $u_i \neq v_i$ for all i.

2 Products of complete graphs

We start with the direct product of finitely many complete graphs which will give us a motivation for the main result.

Theorem 2.1 *Let $G = \times_{i=1}^{t} K_{n_i}$, where $t \geq 3$ and $n_i \geq 2$ for all i. Then*

$$\gamma(G) \geq t + 1.$$

Proof. Assume on the contrary that

$$D = \{(x_1^{(1)}, x_2^{(1)}, \dots, x_t^{(1)}), (x_1^{(2)}, x_2^{(2)}, \dots, x_t^{(2)}), \dots, (x_1^{(t)}, x_2^{(t)}, \dots, x_t^{(t)})\}$$

is a dominating set of G (of size t). Let y_k, z_k be pairwise different elements from $V(K_{n_k})$ for $1 \leq k \leq t$.

Suppose there is a coordinate in which at least three vertices from D agree. Wlog the first coordinate has this property.

Suppose first wlog $x_1^{(1)} = x_1^{(2)} = x_1^{(3)} \neq x_1^{(4)}, \dots, x_1^{(t)}$. Then we take the 2^2 vertices of the form

$$(x_1^{(1)}, x_2^{(4)}, x_3^{(5)}, \dots, x_{t-2}^{(t)}, w_{t-1}, w_t),$$

where $w_k \in \{y_k, z_k\}$ (in the case $t = 3$ we take $(x_1^{(1)}, w_2, w_3)$). All these vertices agree in some coordinate with arbitrary vertex from D, hence they are not adjacent to any of the vertices from D. They can be equal only to the first three vertices from D and $2^2 > 3$, hence at least one of them is not contained in D, a contradiction.

Next case, suppose wlog $x_1^{(1)} = x_1^{(2)} = x_1^{(3)} = x_1^{(4)} \neq x_1^{(5)}, \dots, x_1^{(t)}$. Then there are 2^3 vertices of the form

$$(x_1^{(1)}, x_2^{(5)}, x_3^{(6)}, \dots, x_{t-3}^{(t)}, w_{t-2}, w_{t-1}, w_t),$$

where $w_k \in \{y_k, z_k\}$ (in the case $t = 4$ we take $(x_1^{(1)}, w_2, w_3, w_4)$). These vertices agree in at least one coordinate with the vertices from D and can be equal only to the first four vertices from D. As $2^3 > 4$ some of these vertices are not dominated, a contradiction.

We can continue with this procedure until the $(t-2)$nd step, where we assume $x_1^{(1)} = x_1^{(2)} = \cdots = x_1^{(t)}$ and take 2^{t-1} vertices of the form

$$(x_1^{(1)}, w_2, w_3, \dots, w_t),$$

where $w_k \in \{y_k, z_k\}$. These vertices are not adjacent to any of the vertices from D and $2^{t-1} > t$, hence at least one of them is not dominated.

Hence we may now assume that there is no coordinate in which three vertices from D agree. We are now left with two cases that need to be considered. The following conclusions are very similar as those in the upper lines.

First case, wlog $x_1^{(1)} = x_1^{(2)} \neq x_1^{(3)}, \ldots, x_1^{(t)}$. Then we take the two vertices

$$(x_1^{(1)}, x_2^{(3)}, x_3^{(4)}, \ldots, x_{t-1}^{(t)}, y_t),$$
$$(x_1^{(1)}, x_2^{(3)}, x_3^{(4)}, \ldots, x_{t-1}^{(t)}, z_t).$$

These two vertices agree in at least one coordinate with all of the vertices from D, hence no vertex from D is adjacent to any of them. Observing the first coordinate we get that they can be equal only to the first two vertices from D. If this is the case then $x_2^{(3)} = x_2^{(2)} = x_2^{(1)}$, a contradiction with the assumption that no such coordinate exists. Hence one of these two vertices is not contained in D. Again, it follows that D is not a dominating set.

The second and the last case, assume that all the elements in an arbitrary fixed coordinate are pairwise different. Then the vertex

$$(x_1^{(1)}, x_2^{(2)}, \ldots, x_t^{(t)})$$

differ from all the vertices from D and clearly is not adjacent to any of them, the final contradiction. $\square$

The bound given in Theorem 2.1 is sharp and it remains sharp also for the total domination number:

Corollary 2.2 *Let $G = \times_{i=1}^{t} K_{n_i}$, where $t \geq 3$ and $n_i \geq t+1$ for all i. Then*

$$\gamma(G) = t + 1 = \gamma_t(G).$$

Proof. Consider the set

$$D = \big\{(0,0,\ldots,0), (1,1,\ldots,1), \ldots, (t,t,\ldots,t)\big\}.$$

Let $x = (x_1, x_2, \ldots, x_t) \in V(G) \setminus D$. Suppose that x is not adjacent to any of the vertices from D, thus x must agree in at least one coordinate with every vertex from D. Hence every of $t+1$ elements from $\{0, 1, \ldots, t\}$ must appear on some coordinate of x which is not possible as x has only t coordinates available. Additionally, D induces a complete graph on $t+1$ vertices. $\square$

Let us now mention probably more or less known results concerning the direct product of less than four complete graphs. In the case of three factors we give a more general result as in Corollary 2.2.

For all $n_1, n_2, n_3 \in \mathbb{N}$, $n_i \geq 2$, is

$$\begin{array}{lrcl} \text{(i)} & \gamma(K_{n_1} \times K_{n_2}) & = & \left\{ \begin{array}{ll} 2, & n_i = 2 \text{ for some } i \in \{1,2\} \\ 3, & \text{other} \end{array} \right. . \\ \text{(ii)} & \gamma(K_{n_1} \times K_{n_2} \times K_{n_3}) & = & 4. \end{array} \quad (3)$$

(i) Wlog $n_1 = 2$. The vertices $(0,0), (1,0)$ clearly dominate $K_{n_1} \times K_{n_2}$ and one vertex is not enough. Now let $n_1, n_2 > 2$. Suppose that the vertices $(x,y), (z,w)$ dominate $K_{n_1} \times K_{n_2}$. Wlog $y \neq w$. If $x = z$ then the vertex (x,v) is not dominated for $v \in K_{n_2} \setminus \{y,w\}$ and if $x \neq z$ then the vertex (x,w) is not adjacent (or equal) to any of the two starting vertices. Hence, $\gamma(K_{n_1} \times K_{n_2}) > 2$. By the same arguments as in the proof of Corollary 2.2 follows that the set $\{(0,0),(1,1),(2,2)\}$ dominates $K_{n_1} \times K_{n_2}$. (ii) Theorem 2.1 gives $\gamma(K_{n_1} \times K_{n_2} \times K_{n_3}) \geq 4$. In [14] it is proven that the set

$$\{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$$

dominates the direct product of three complete graphs.

We end this section with a simple infinite series of counterexamples of the Vizing-like conjecture. Consider the case when $G = H = \times_{i=1}^{n} K_{2n+1}$, $n \geq 3$. By Corollary 2.2 is $\gamma(G) = \gamma(H) = n + 1$ and $\gamma(G \times H) = 2n + 1$. Hence the difference

$$\gamma(G)\gamma(H) - \gamma(G \times H) = n^2 \quad (4)$$

can be arbitrary large.

3 A lower bound

The example (4) of the previous section gives us a motivation for the next main result of this paper.

Theorem 3.1 *Let G and H be arbitrary graphs. Then*

$$\gamma(G \times H) \geq \gamma(G) + \gamma(H) - 1.$$

Proof. Suppose that $D \subseteq V(G \times H)$ dominates $G \times H$ and that $|D| = \gamma(G) + \gamma(H) - 2$. As already mentioned, the projection on G (resp. H) of a dominating set in $G \times H$ is a dominating set in G (resp. H), that is $|p_G(D)| \geq \gamma(G)$ and $|p_H(D)| \geq \gamma(H)$. If $\gamma(G) = 1$ then $|D| = \gamma(H) - 1$ and $p_H(D)$ is a dominating set in H with size less then $\gamma(H)$, a contradiction. Hence we may assume that $\gamma(G), \gamma(H) \geq 2$. As $|p_G(D)| \geq \gamma(G) \geq 2$ we can find a subset

$$D_0 = \left\{(p_1,q_1),(p_2,q_2),\ldots,(p_{\gamma(G)-1},q_{\gamma(G)-1})\right\}$$

of the set D with $p_i \neq p_j$ for all $i \neq j$. As $|p_G(D_0)| = \gamma(G) - 1 < \gamma(G)$ there exists a vertex $g_1 \in V(G)$ which is neither adjacent to any of the vertices in $p_G(D_0)$ nor contained in $p_G(D_0)$. Notice that $|D \setminus D_0| = \gamma(H) - 1$ and $|p_H(D \setminus D_0)| \leq \gamma(H) - 1 < \gamma(H)$. Clearly, there exists a vertex $h_1 \in V(H)$ which is neither adjacent to any of the vertices in $p_H(D \setminus D_0)$ nor contained in $p_H(D \setminus D_0)$. Observe that the vertex $(g_1, h_1) \in V(G \times H)$ is not adjacent to any of the vertices in D. To avoid a contradiction with the dominancy of D we must have $(g_1, h_1) \in D$. As $g_1 \notin p_G(D_0)$ it follows $(g_1, h_1) \in D \setminus D_0$. But $h_1 \notin p_H(D \setminus D_0)$, a contradiction. □

A stronger result than Theorem 3.1 holds for graphs with diameter greater than 2.

Proposition 3.2 *Let G and H be graphs with $\gamma(H) \geq \gamma(G)$ or with $\gamma_t(H) \geq \gamma_t(G)$. If $\rho(G) \geq 2$ then*

$$\gamma(G \times H) \geq \gamma(G) + \gamma(H).$$

Proof. Assume first that $\gamma(H) \geq \gamma(G)$. Using bound (1) we get

$$\gamma(G \times H) - \gamma(H) \geq \rho(G)\gamma(H) - \gamma(H) \geq \gamma(H) \geq \gamma(G)$$

and the result follows.

Assume next that $\gamma_t(H) \geq \gamma_t(G)$. Taking in notice bound (2) we have

$$\gamma(G \times H) - \gamma_t(H) \geq \rho(G)\gamma_t(H) - \gamma_t(H) \geq \gamma_t(H) \geq \gamma_t(G) \geq \gamma(G),$$

hence

$$\gamma(G \times H) \geq \gamma(G) + \gamma_t(H) \geq \gamma(G) + \gamma(H),$$

which completes the proof. □

In the case when a graph G is connected from $\rho(G) \geq 2$ it clearly follows $\mathrm{diam}(G) > 2$. We note also that, in general, there is no correlation between the two inequalities $\gamma(H) \geq \gamma(G)$ and $\gamma_t(H) \geq \gamma_t(G)$.

4 Families attaining equality

Now we will take a look what properties must have the two graphs and the minimal dominating set of their direct product in the case when lower bound from Theorem 3.1 is achieved.

Proposition 4.1 *Let G and H be graphs with $\gamma(G \times H) = \gamma(G) + \gamma(H) - 1$ and let $\gamma(G) = 1$. Then $G = K_1$ and H is an edgeless graph.*

Proof. Let D be a dominating set for $G \times H$ with $|D| = \gamma(H)$.

First case, suppose there is a vertex $(g_0, h_0) \in D$ such that $(g_1, h_0) \notin D$ for some $g_1 \in V(G)$. The set $D' = D \setminus \{(g_0, h_0)\}$ dominates $\{g_0\} \times V(H)$ except maybe the vertex (g_0, h_0). Hence $p_H(D')$ dominates H except maybe the vertex h_0. On the otherside, there must be a vertex from D' that is adjacent to (g_1, h_0), thus the projection of this vertex on H is adjacent to h_0. Hence $p_H(D')$ dominates H with $|p_H(D')| \leq \gamma(H) - 1$, a contradiction.

Second case, for every $(g, h) \in D$ follows $(g', h) \in D$ for all $g' \in V(G)$. Suppose $|V(G)| \geq 2$. We have now (at least) two vertices from D with the same H-coordinate, hence $|p_H(D)| \leq \gamma(H) - 1$, again a contradiction (taking in mind that $p_H(D)$ dominates H). Thus $G = K_1$. But then $G \times H$ is an edgeless graph with $\gamma(H)$ vertices and $|V(H)| = \gamma(H)$, implying H is an edgeless graph. □

Proposition 3.2 tells us that the lower bound can be achieved only when the 2-packing number of a graph G (the one with smaller domination number or with smaller total domination number) is equal 1 (by the way, this means that G is connected). The latter is clearly equivalent to $\mathrm{diam}(G) \leq 2$. As $\mathrm{diam}(G) = 1$ implies that G is a complete graph and from this follows $G = K_1$ (if G and H give the lower bound), the more interesting cases that give the lower bound are those with the property $\mathrm{diam}(G) = 2$, where $\gamma(H) \geq \gamma(G)$ or $\gamma_t(H) \geq \gamma_t(G)$.

Eliminating the trivial cases (involving at least one K_1) we can now prove:

Proposition 4.2 *Let D dominate $G \times H$ with $|D| = \gamma(G) + \gamma(H) - 1$ and let G and H be both different from K_1. Then*

$$|p_G(D)| = |p_H(D)| = \gamma(G) + \gamma(H) - 1.$$

Proof. Applying Proposition 4.1 we have $\gamma(G), \gamma(H) > 1$. Suppose that $|p_G(D)| \leq \gamma(G) + \gamma(H) - 2$. Then there exist two vertices $(g_1, h_0), (g_1, h_1) \in D$ with the same G-coordinate. $\gamma(G) \leq |p_G(D)|$ thus we can find a set

$$D_1 = \big\{(g_1, h_1), (p_2, q_2), \ldots, (p_{\gamma(G)-1}, q_{\gamma(G)-1})\big\} \subseteq D$$

with $|p_G(D_1)| = \gamma(G) - 1$. Hence there exist $g_2 \in V(G)$ and $h_2 \in V(H)$, where g_2 is not dominated by the set $p_G(D_1)$ and h_2 is not dominated by the set $p_H\big(D \setminus (D_1 \cup \{(g_1, h_0)\})\big)$ (taking in mind that $|p_H\big(D \setminus (D_1 \cup \{(g_1, h_0)\})\big)| \leq \gamma(H) - 1$). Observe that (g_2, h_2) is not adjacent to any of the vertices from D (including the vertex (g_1, h_0)) so it must belong to D. From $g_2 \notin p_G(D_1)$ follows

$$(g_2, h_2) \in D \setminus D_1 = \big(D \setminus (D_1 \cup \{(g_1, h_0)\})\big) \cup \{(g_1, h_0)\}.$$

But $h_2 \notin p_H\big(D \setminus (D_1 \cup \{(g_1, h_0)\})\big)$, hence $(g_2, h_2) = (g_1, h_0)$, a contradiction with $g_1 \neq g_2$. □

Proposition 4.2 (and 4.1) helps us to construct pairs of graphs which give the lower bound for the domination number starting from the direct product. From now on suppose that G and H are graphs with

$$\gamma(G \times H) = \gamma(G) + \gamma(H) - 1.$$

Let us denote $d = \gamma(G \times H)$ and find graphs G and H with $d = n$ for $n \in \mathbb{N}$. The easiest way to get a pair of such graphs with fixed $d \in \mathbb{N}$ is to take K_1 and an edgeless graph on d vertices. According to the example(s) from (4) there are more interesting pairs of graphs. For $d = 2n + 1$, $n \geq 3$, we can take $G = H = K_{2n+1}^n$ but also for $d = 5$ we can take $G = H = K_5^2$ (taking into account (3) and Corollary 2.2). For $d = 2n$, $n \geq 3$, the graphs $G = K_{2n}^n$ and $H = K_{2n}^{n-1}$ give the lower bound. With this examples cases for $d \geq 5$ are found. There certainly are other nontrivial cases, for $d \geq 3$ see below, but for $d \leq 2$ we have no other options besides the trivial ones.

If $d = 1$ then $G = H = K_1$ and the only pair of graphs with $d = 2$ is K_1 and an edgeless graph on two vertices (Proposition 4.1).

For $d = 3$ we can construct a family of graphs which in a direct product give domination number equal 3. The base for the construction is the Hajós graph (Figure 1).

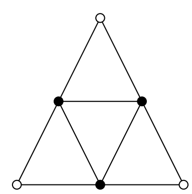

Figure 1: Hajós graph

The three filled vertices form a dominating set of the product of two Hajós's graphs, see Figure 2. The numbers beside a vertex are suggesting which of the filled vertices is adjacent to the vertex. It can be argumented that this is the minimal nontrivial case of the lower bound. It is straightforward to see that in nontrivial cases every vertex from $V(G)$ (resp. $V(H)$) is adjacent to at least two vertices from $p_G(D)$ (resp. $p_H(D)$). Otherwise it is easy to find a vertex in the product that is not dominated. Among others, this means that $p_G(D)$ and $p_H(D)$ in G and H induce K_3. To get the factor's domination number equal 2 with the minimal number of added vertices and edges to K_3 the Hajós graph follows.

Adding arbitrary many vertices to Hajós graph and connecting every one of them to (at least) two of the filled vertices in Figure 1 we get a family of graphs with domination number equal 2 having the property that the product of any two of them give the lower bound. If G and H are such graphs and $(g, h) \in G \times H$, then

g (resp. h) has at least two neighbors in $p_G(D)$ (resp. $p_H(D)$) and consequently (g,h) has at least one neighbor in D, where D is the set of the three filled vertices in Figure 2.

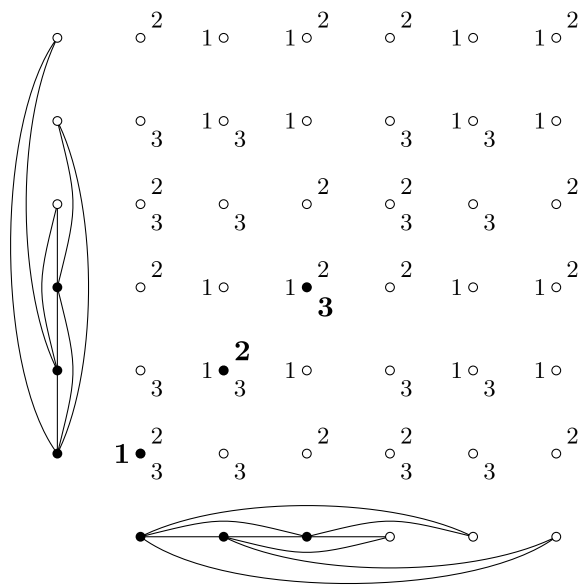

Figure 2: Product of two Hajós's graphs

Notice also that we can add edges to any graph of described family as long as domination number remains equal 2. That is, any graph with domination number equal 2 that contains a spanning subgraph of the form described above can easily be used to get the lower bound.

For $d = 4$ consider the graphs shown in Figure 3. Then $G \times H$ give the lower bound as shown in Figure 4. Similar as above, we can construct an infinite series of graphs such that any two of them in product give the lower bound equal 4. We can add arbitrary many vertices to graph G (resp. H) and connect each one of them to (at least) two (resp. three) filled vertices in Figure 1. The domination number remains in both cases unchanged. By the same arguments as above, every vertex of the product of such two graphs is adjacent to at least one of the filled vertices in Figure 4. Again, the same thing holds for all graphs with specific domination number (2 or 3) and with these graphs as spanning subgraphs.

Another fact that is obvious is that the domination set of the product in Figure 2 as in Figure 4 is a total domination set. As $\gamma(G) \leq \gamma_t(G)$ trivially holds, the bound from Theorem 3.1 is also an exact lower bound for the total

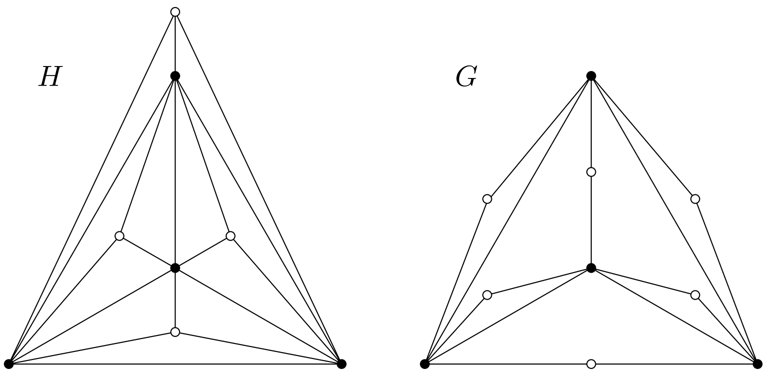

Figure 3: Graphs with $\gamma(G) = 3$, $\gamma(H) = 2$

domination number. That is,

$$\gamma(G \times H) = \gamma_t(G \times H) = \gamma(G) + \gamma(H) - 1$$

holds for infinite series of graphs G and H.

Observe that in graph G we could have taken additional two edges missing between the filled vertices in Figures 3 and 4. Then, as in the case of $d = 3$, the domination set of the product would induce a complete subgraph. This gives the motivation for the following result.

Let $K_d(k)$ denote a graph obtained from K_d by adding a vertex for every set of k vertices of K_d and adding k edges between this vertex and these k vertices. For instance, $K_3(2)$ is the Hajós graph.

Theorem 4.3 *Let $d \in \mathbb{N}$, $d \geq 3$, and $1 < k_1, k_2 < d$ with $k_1 + k_2 = d + 1$. Set $G = K_d(k_1)$ and $H = K_d(k_2)$. Then*

$$d = \gamma(G \times H) = \gamma_t(G \times H) = \gamma(G) + \gamma(H) - 1 = \gamma_t(G) + \gamma_t(H) - 1.$$

For the proof of the theorem we need the following lemma.

Lemma 4.4 *Let $d \geq 3$ and $1 < k < d$ and let $G = K_d(k)$. Then $\gamma(G) = d - k + 1 = \gamma_t(G)$.*

Proof. Every set of k vertices of K_d contains an element from a set of arbitrary $d - k + 1$ fixed vertices of K_d (the complement in K_d of this set is the size of $k - 1 < k$). Hence these $d - k + 1$ vertices dominate all of the $\binom{d}{k}$ added vertices and trivially they dominate the remaining $k - 1$ starting vertices. It follows $\gamma_t(G) \leq d - k + 1$.

Suppose that D dominates G with $|D| = d - k$.

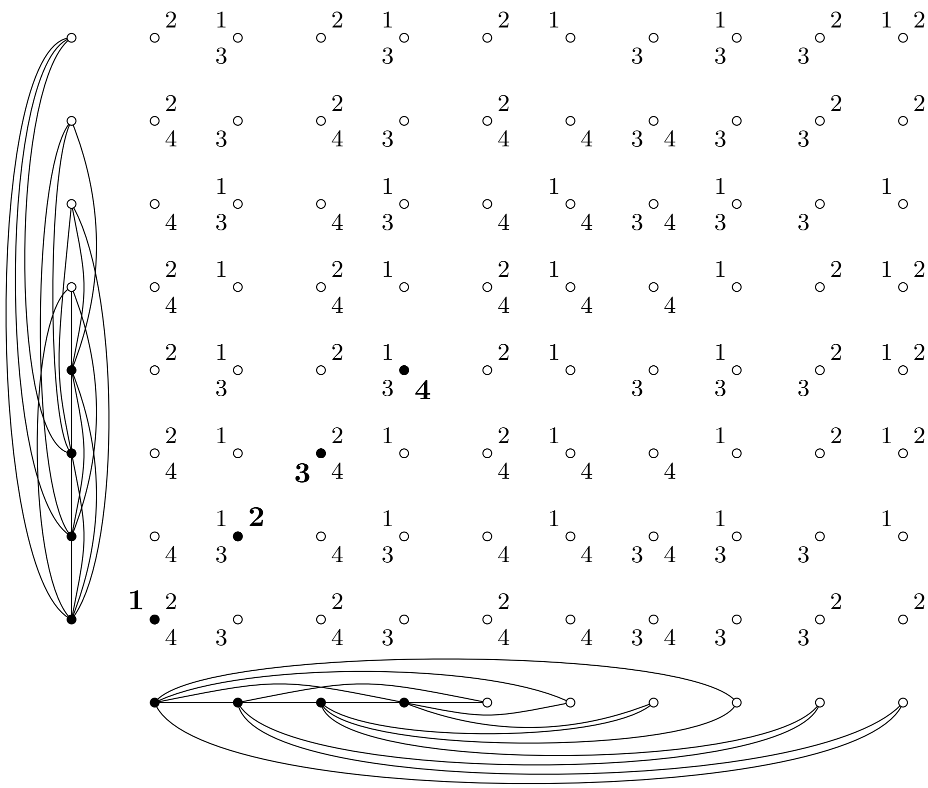

Figure 4: $G \times H$

Assume first that D contains only the vertices from the starting graph K_d. Then the added vertex adjacent to the remaining k vertices of K_d is not adjacent to any of the vertices from D, a contradiction.

Suppose next that D contains p added vertices, $1 \le p \le \min\{d-k, \binom{d}{k}\}$. If $k \ne d-1$ then

$$\binom{d}{k} \ge \binom{d}{2} = \frac{d}{2}(d-1) > \frac{d}{2}(d-k) > d-k$$

and for $k = d-1$ is $\binom{d}{d-1} = d > d-k$. That is, $1 \le p \le d-k$. The latter is not really needed as $k+p \le d$ follows already from the start. In D we have $d-k-p$ vertices from the starting graph K_d and as no two added vertices are adjacent, all of the $\binom{k+p}{k}$ added vertices adjacent only to the $k+p$ vertices from $V(K_d) \setminus D$ must belong to D. Thus $\binom{k+p}{k} \le p$. But

$$\binom{k+p}{k} \ge \binom{k+p}{1} = k+p > p,$$

the final contradiction. □

Proof of Theorem 4.3. By Lemma 4.4 we have $\gamma(G) = d - k_1 + 1 = \gamma_t(G)(= k_2)$ and $\gamma(H) = d - k_2 + 1 = \gamma_t(H)(= k_1)$, hence $\gamma(G) + \gamma(H) - 1 = d$. Let D be the set of arbitrary d vertices from a subgraph $K_d \times K_d \subset G \times H$ such that $|p_G(D)| = |p_H(D)| = d$. Every vertex $(g, h) \in V(K_d \times K_d) \setminus D$ is adjacent to $d-2$ vertices from D which does not project on G (resp. H) into g (resp. h). As $k_2 > 1$ every vertex $(g, h) \in V\big(K_d \times (H \setminus K_d)\big)$ is adjacent to at least one of the $d-1$ vertices from D which does not project on G in g. Analogously, arbitrary vertex from $V\big((G \setminus K_d) \times K_d\big)$ is adjacent to some vertex from D. Finally, let $(g, h) \in V\big((G \setminus K_d) \times (H \setminus K_d)\big)$. As g (resp. h) is adjacent to k_1 (resp. k_2) vertices in $V(K_d) = p_G(D) \subset V(G)$ (resp. $V(K_d) = p_H(D) \subset V(H)$) and $k_1 + k_2 = d + 1$ at least one of the vertices from D is adjacent to (g, h). Moreover, D induces a complete graph in $G \times H$. That is, D is a total dominating set. □

5 Concluding remarks

The Hajós graph was used in [2] for establishing the lower bound for the domination number given by the maximum of the $\{2\}$-domination number of the factors. In the same paper the graph G from Figure 3 (with additional two edges between the filled vertices) was used for construction of infinite series of graphs which give the lower bound of

$$\gamma_t(G \times H) \geq \max\left\{ \frac{|V(G)|}{\Delta(G)}\gamma_t(H), \frac{|V(H)|}{\Delta(H)}\gamma_t(G) \right\}. \tag{5}$$

The latter lower bound was presented in [3]. It is obvious that in all cases arising from Theorem 4.3 it makes no difference if we write γ or γ_t. Let $d \geq 3$, $G = K_d(d-1)$ and $H = K_d(2)$. As

$$\frac{d + \binom{d}{d-1}}{d - 1 + \binom{d-1}{(d-1)-1}}(d - 2 + 1) = d = \gamma_t(G \times H)$$

this two graphs give the lower bound from (5). That is, for every $d \geq 3$ we have found graphs with

$$\max\left\{ \frac{|V(G)|}{\Delta(G)}\gamma_t(H), \frac{|V(H)|}{\Delta(H)}\gamma_t(G) \right\} = \gamma_t(G \times H) = \gamma(G \times H) = \gamma(G) + \gamma(H) - 1.$$

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