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ON THE FUNCTIONAL INEQUALITY FOR THE SPECTRAL RADIUS OF COMPACT OPERATORS

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Abstract. In [6], Elsner, Hershkowitz and Pinkus characterized functions $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfying

$$r(F(A_1, \ldots, A_n)) \le F(r(A_1), \ldots, r(A_n))$$

for all non-negative matrices A_1, …, A_n of the same order, where r denotes the spectral radius. We generalize this result to the setting of infinite non-negative matrices that define compact operators on a Banach sequence space.

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1. Introduction

Throughout the paper, let R denote the set $\{1, \ldots, N\}$ for some $N \in \mathbb{N}$ or the set $\mathbb{N}$ of all natural numbers. Let $S(R)$ be the vector lattice of all complex sequences $(x_i)_{i\in R}$. Given $x, y \in S(R)$ we write $x \le y$ if $x_i \le y_i$ for all $i \in R$. A Banach space $L \subseteq S(R)$ is called a *Banach sequence space* if $x \in S(R)$, $y \in L$ and $|x| \le |y|$ imply that $x \in L$ and $\|x\|_L \le \|y\|_L$. Note that in the literature such a space L is usually called a Banach function space over a measure space (R, μ), where μ denotes the counting measure on the set R. The cone of non-negative elements in L is denoted by L_+.

A matrix $K = [k_{ij}]_{i,j\in R}$ is called *non-negative* if $k_{ij} \ge 0$ for all $i, j \in R$. For notational convenience, we sometimes write $k(i, j)$ instead of k_{ij}. Given matrices K and H, we write $K \le H$ if the matrix $H - K$ is non-negative.

By an *operator* on a Banach sequence space L we always mean a linear operator on L. We say that a non-negative matrix K defines an operator on L if $Kx \in L$ for all $x \in L$, where $(Kx)_i = \sum_{j\in R} k_{ij} x_j$. Then $Kx \in L_+$ for all $x \in L_+$ and so K defines a *positive* operator on L. Recall that this operator is always bounded, i.e., its operator norm

$$\|K\| = \sup\{\|Kx\|_L : x \in L_+, \|x\|_L \le 1\}$$

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is finite. Also, its spectral radius $r(K)$ is always contained in the spectrum. An operator K is called *compact* if $\overline{K(B_L)}$ is a compact set, where B_L denotes the closed unit disc of the space L, i.e.,

$$B_L = \{x \in L : \|x\|_L \le 1\}.$$

If $\{x_n\}_{n\in\mathbb{N}} \subset S(R)$ is a decreasing sequence and $x = \inf\{x_n \in S(R) : n \in \mathbb{N}\}$, then we write $x_n \downarrow x$. Similarly we define $x_n \uparrow x$. A Banach sequence space L has an *order continuous norm*, if $0 \le x_n \downarrow 0$ implies $\|x_n\|_L \to 0$ as $n \to \infty$. It is well known that spaces l^p, $1 \le p < \infty$, have order continuous norm. Moreover, the norm of any reflexive Banach sequence space is order continuous. However, the norm $\|x\|_\infty = \sup\{|x_i| : i \in \mathbb{N}\}$ on the space l^∞ of all bounded sequences or on the space c of all convergent sequences, is not order continuous. Indeed, let $\{x_n\}_{n=1}^\infty \subset c$, where $x_n = \{a_{m,n}\}$ and $a_{m,n} = 0$ for $m \le n$ and $a_{m,n} = 1$ for $m > n$. Then $x_n \downarrow 0$, but $\|x_n\|_\infty = 1$ for all $n \in \mathbb{N}$. On the other hand, the space $L = c_0$ (equipped with the norm $\|\cdot\|_\infty$) of all null convergent sequences is an example of a non-reflexive Banach sequence space, such that L and its Banach dual space $L^* = l^1$ have order continuous norms. For the theory of Banach function spaces, Banach lattices and positive operators we refer the reader to the books [19], [12] and [1].

Similarly as in [5] and [4] let us denote by $\mathcal{L}$ the collection of all Banach sequence spaces L satisfying the property that $e_n = \chi_{\{n\}} \in L$ and $\|e_n\|_L = 1$ for all $n \in R$. All Banach sequence spaces mentioned above belong to the set $\mathcal{L}$.

The paper is organized as follows. In the second section we state some of the known results that will be used in our proofs. In section 3 we state the known characterization of those functions $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfying

$$r(F(A_1, \ldots, A_n)) \le F(r(A_1), \ldots, r(A_n))$$

for all non-negative matrices $A_1, \ldots, A_n$ of order m and for all $m \in \mathbb{N}$. It is known, that a straightforward generalization of this characterization to the setting of infinite non-negative matrices that define operators on a Banach sequence space, is not true. However, the main purpose of this paper is to show, that a straightforward generalization to the setting of compact operators is possible.

2. Preliminaries

In this section we state some of the known results, which we will need in our proofs. The following result is a special case of [16, Theorem 1] (see also [18, Theorem 4.1]).

Theorem 2.1. *Let $L \in \mathcal{L}$ such that L has an order continuous norm and K, H non-negative matrices that satisfy $K \le H$. If the matrix H defines a compact operator on L, then also the matrix K defines a compact operator on L.*

Remark 2.2. The assumption on order continuity in Theorem 2.1 is necessary (see e.g. [16], [18], [1]). For example, let $L = l^1 \times l^\infty \in \mathcal{L}$ equipped with the norm $\|(x, y)\|_L =$

$\max\{\|x\|_1, \|y\|_\infty\}$. Then L does not have an order continuous norm. Define positive operators $K_1, H_1 : l^1 \to l^\infty$ by

$$K_1(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) \quad \text{and} \quad H_1(x_1, x_2, x_3, \ldots) = (\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \ldots).$$

We have $K_1 \le H_1$, H_1 is compact (rank 1) operator, while K_1 is not compact. By $K(x, y) = (0, K_1 x)$ and $H(x, y) = (0, H_1 x)$ for $x \in l^1, y \in l^\infty$, define positive operators K and H on L. Again we have $K \le H$, where H is compact, while K is not.

Given non-negative matrices $K = [k_{ij}]_{i,j\in R}$ and $H = [h_{ij}]_{i,j\in R}$, let $K \circ H = [k_{ij} h_{ij}]_{i,j\in R}$ be the *Hadamard (or Schur) product* of K and H and let $K^{(t)} = [k_{ij}^t]_{i,j\in R}$ be the *Hadamard (or Schur) power* of K for $t \ge 0$. Similarly, the *Hadamard product* of vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}_+^n$ is defined by $a \circ b = (a_1 b_1, \ldots, a_n b_n)$. For $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ and $t \in \mathbb{R}_+$ we also let $a^t = (a_1^t, \ldots, a_n^t)$. Here we use the convention $0^0 = 1$. The following result was proved in [15] and [4].

Theorem 2.3. *Given L in $\mathcal{L}$, let $K, K_1, \ldots, K_n$ be non-negative matrices that define operators on L, $t \ge 1$ and α_1, α_2,..., α_n positive numbers such that $\sum_{i=1}^n \alpha_i \ge 1$.*

Then the non-negative matrices $K^{(t)}$ and $K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)}$ also define operators on L and the following inequalities hold

$$k_{ij} \le \|K\| \ \textit{for all}\ i, j \in R, \tag{1}$$

$$K^{(t)} \le \|K\|^{t-1} K, \tag{2}$$

$$r(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)}) \le r(K_1)^{\alpha_1}\, r(K_2)^{\alpha_2} \cdots r(K_n)^{\alpha_n}. \tag{3}$$

Remark 2.4. In the finite-dimensional case the inequality (3) goes back to Kingman [11] implicitly, and it was later considered by several authors ([8], [2], [7], [10], [14], [3], [9], [5]) using different methods.

Corollary 2.5. *Given $L \in \mathcal{L}$ with an order continuous norm, let K and H be non-negative matrices that define operators on L and let $t \ge 1$. If the matrix K defines a compact operator on L, then the matrices $K^{(t)}$ and $K \circ H$ define compact operators on L.*

Proof. The result follows from Theorem 2.1, the inequality (2) and the inequality

$$K \circ H \le \|H\|_\infty K,$$

where $\|H\|_\infty := \sup_{i,j\in R} h_{ij} \le \|H\|$ by (1). □

For non-negative matrices $K, K_m, m \in \mathbb{N}$, that define operators on L, we denote $K_m \uparrow K$ if for all $x \in L_+$, $K_m x \uparrow Kx$ on L. The following result on the continuity of spectral radius is a special case of [13, Theorem 2.4] and it will be the main tool in the proof of Theorem 3.8.

Theorem 2.6. *Let L be a Banach sequence space and let $K, K_m, m \in \mathbb{N}$, be non-negative matrices, such that K defines a compact operator on L and that $K_m \uparrow K$ on L. Then $r(K_m) \uparrow r(K)$ as $m \to \infty$.*

For a positive integer m let P_m denote a positive operator on $L \in \mathcal{L}$ (called an *initial projection* or *truncation*) defined by $(P_m x)_i = x_i$ if $i \leq m$ and $(P_m x)_i = 0$ if $i > m$. If L is a finite dimensional space with dimension N, then we define P_m to be the identity operator on L for all $m > N$.

Corollary 2.7. *Let $L \in \mathcal{L}$ with an order continuous norm and let K be a non-negative matrix that defines a compact operator on L. Then $r(P_m K P_m) \uparrow r(K)$ as $m \to \infty$.*

Proof. We prove, that for all $x \in L_+$ we have

$$\|P_m K P_m x - Kx\|_L \to 0 \tag{4}$$

as $m \to \infty$. This implies $P_m K P_m \uparrow K$ and thus we have $r(P_m K P_m) \uparrow r(K)$ by Theorem 2.6.

For the proof of (4) let $x \in L_+$ and $\varepsilon > 0$. We may assume that $\|K\| > 0$. Since $P_m x \uparrow x$, $P_m K x \uparrow Kx$ and since L has an order continuous norm, we have $\|P_m x - x\|_L \to 0$ and $\|P_m Kx - Kx\|_L \to 0$ as $m \to \infty$. Now choose m_0 such that $m \geq m_0$ implies

$$\|P_m x - x\|_L < \frac{\varepsilon}{2\|K\|} \quad \text{and} \quad \|P_m Kx - Kx\|_L < \frac{\varepsilon}{2}.$$

For these m we now have

$$\|P_m K P_m x - Kx\|_L = \|P_m K P_m x - P_m Kx + P_m Kx - Kx\|_L \leq$$

$$\|P_m K(P_m x - x)\|_L + \|P_m Kx - Kx\|_L < \varepsilon,$$

since $\|P_m\| = 1$. This proves (4), which completes the proof. □

From the proof of (4) we can conclude also the following result (see also the proof of [17, Proposition 3.1.(i)], where the result is stated for $L = l^p, 1 \leq p < \infty$).

Proposition 2.8. *Let $L \in \mathcal{L}$ with an order continuous norm and let K be a positive operator on L. Then $\|P_m K P_m\| \uparrow \|K\|$ as $m \to \infty$.*

Remark 2.9. In Proposition 2.8 the operator K need not be compact. On the other hand, the compactness of K in Corollary 2.7 is necessary. Indeed, if we consider the right shift operator S on l^2, we have $P_m S P_m \uparrow S$, $r(P_m S P_m) = 0$ for all $m \in \mathbb{N}$, but $r(S) = 1$ (see also [13, Example 2.6]).

3. The inequality $r(F(K_1, \ldots, K_n)) \leq F(r(K_1), \ldots, r(K_n))$

Let $K_1 = [k_1(i,j)]_{i,j \in R}, \ldots, K_n = [k_n(i,j)]_{i,j \in R}$ be non-negative matrices and $F : \mathbb{R}_+^n \to \mathbb{R}_+$ a given function (where $\mathbb{R}_+ = [0, \infty)$). We define the matrix $F(K_1, \ldots, K_n) =$

$[m(i,j)]_{i,j\in R}$ by $m(i,j) = F(k_1(i,j),\dots,k_n(i,j))$ for all $i,j \in R$. In [6], partially motivated by the inequality (3), the inequality

$$r(F(A_1,\dots,A_n)) \le F(r(A_1),\dots,r(A_n)) \tag{5}$$

was considered for non-negative square matrices $A_1,\dots,A_n$. The following theorem was proved in ([6, Theorem 2.1]).

Theorem 3.1. *A function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies (5) for all non-negative matrices $A_1,\dots,A_n$ of order m and all $m \in \mathbb{N}$ if and only if*

$$F(a) + F(b) \le F(a+b) \tag{6}$$

and

$$\prod_{k=1}^{s} F^{1/s}(a_k) \le F((a_1 \circ \cdots \circ a_s)^{1/s}) \tag{7}$$

for all $a,b,a_1,\dots,a_s \in \mathbb{R}_+^n$ and $s = 2,3,\dots$. Furthermore, if F is continuous, or $n=1$, then it suffices to take only $s=2$ in (7), i.e., the property (7) may be replaced by the property

$$\sqrt{F(a)F(b)} \le F(\sqrt{a\circ b}) \tag{8}$$

for all $a,b \in \mathbb{R}_+^n$.

Remarks 3.2. (i) In general the property (7) can not be replaced by the property (8) in Theorem 3.1, since there exist functions that satisfy (6) and (8), but do not satisfy (5) (see [6, Example p.116-117]).

(ii) The property (6) implies that F is a coordinatewise increasing function, which satisfies $F(0)=0$ and $2F(a) \le F(2a)$ for all $a \in \mathbb{R}_+^n$.

The following result is obtained by a precise inspection of the proofs of [6, Proposition 2.4] and [4, Theorem 3.7].

Lemma 3.3. *Let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a coordinatewise increasing function satisfying (8) for all $a,b \in \mathbb{R}_+^n$ and let D denote the set $D = \{a \in \mathbb{R}_+^n : F(a) > 0\}$. Then the following properties hold:*

(i) *If $r = (r_1,\dots,r_n) \in \mathrm{int}D$, then there exists $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}_+^n$ such that*

$$F(a) \le F(r) r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} a_1^{\alpha_1} \cdots a_n^{\alpha_n} \tag{9}$$

for all $a \in \mathbb{R}_+^n$. Also $\alpha_i = 0$, if $r_i = 0$.

If, in addition, the function F satisfies

$$2F(a) \le F(2a) \tag{10}$$

for all $a \in \mathbb{R}_+^n$, then $F(0)=0$ and $\sum_{i=1}^n \alpha_i \ge 1$.

(ii) *If the set D is open, then the function F is lower-semicontinuous, i.e., the set $\{a \in \mathbb{R}_+^n : F(a) \le c\}$ is closed for all $c \ge 0$.*

By considering (non-compact) weighted shifts on $L = l^2$ we obtain the following result (see the proof of [4, Theorem 3.7]).

Proposition 3.4. *If the function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies*

$$r(F(K_1, \ldots, K_n)) \leq F(r(K_1), \ldots, r(K_n))$$

for all non-negative matrices $K_1, \ldots, K_n$ that define operators on l^2, then the function F is upper-semicontinuous, i.e., the set $\{a \in \mathbb{R}_+^n : F(a) \geq c\}$ is closed for all $c \geq 0$.

If, in addition, the set $D = \{a \in \mathbb{R}_+^n : F(a) > 0\}$ is open, then the function F is continuous.

By applying the last three results and the inequality (3) in the proof of [4, Theorem 3.7] we obtain the following improvement of it (and of Theorem 3.1).

Theorem 3.5. *The following assertions are equivalent for a given function $F : \mathbb{R}_+^n \to \mathbb{R}_+$.*

(i) The set $D = \{a \in \mathbb{R}_+^n : F(a) > 0\}$ is open and

$$r(F(K_1, \ldots, K_n)) \leq F(r(K_1), \ldots, r(K_n)) \tag{11}$$

for all $L \in \mathcal{L}$ and all non-negative matrices $K_1, \ldots, K_n$ that define operators on L.

(ii) F is a continuous function satisfying (6) *and* (8).

(iii) F a coordinatewise increasing continuous function satisfying (8) *and $2F(a) \leq F(2a)$ for all $a, b \in \mathbb{R}^n$.*

As noted in [4] a straightforward generalization of Theorem 3.1 to the setting of Theorem 3.5 does not hold, since there are functions that satisfy (6) and (7), but are not upper semi-continuous (for example $F(x) = x \cdot \chi_{(1,\infty)}(x)$ for $x \in \mathbb{R}_+$). However, as we will see bellow (Theorem 3.8), a straightforward generalization of Theorem 3.1 to the setting of compact operators is possible. The following two lemmas will be needed.

Lemma 3.6. *A function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies the conditions* (6) *and* (7) *for all $a, b, a_1, \ldots, a_s \in \mathbb{R}_+^n$ and $s = 2, 3, \ldots$, if and only if*

$$r(P_m F(K_1, \ldots, K_n) P_m) \leq F(r(K_1), \ldots, r(K_n)) \tag{12}$$

for all $L \in \mathcal{L}$ and all non-negative matrices $K_1, \ldots, K_n$ that define operators on L and for all $m \in \mathbb{N}$.

Proof. $(\Leftarrow)$: For each $m \in \mathbb{N}$ take $L = \mathbb{C}^m$ and apply Theorem 3.1.

$(\Rightarrow)$: Let $m \in \mathbb{N}$ and let a function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfy (6) and (7). Let $L \in \mathcal{L}$ and let $K_1, \ldots, K_n$ be non-negative matrices that define operators on L. For each $l = 1, \ldots, n$ let us define a $m \times m$ non-negative matrix $K_l^{[m]}$ by $K_l^{[m]} = [k_l(i,j)]_{i,j=1}^m$. Then $r(K_l^{[m]}) = r(P_m K_l P_m) \leq r(K_l)$ and so we have by Theorem 3.1

$$r(F(K_1^{[m]}, \ldots, K_n^{[m]})) \leq F(r(K_1^{[m]}), \ldots, r(K_n^{[m]})) \leq F(r(K_1), \ldots, r(K_n)), \tag{13}$$

since the function F is coordinatewise increasing.

It is easy to see, that for the matrix $H := P_m F(K_1, \ldots, K_n) P_m$ the matrix $H^{[m]}$ equals the matrix $F(K_1^{[m]}, \ldots, K_n^{[m]})$. This together with (13) implies

$$r(P_m F(K_1, \ldots, K_n) P_m) = r(F(K_1^{[m]}, \ldots, K_n^{[m]})) \leq F(r(K_1), \ldots, r(K_n)),$$

which proves (12). □

Lemma 3.7. *Given $L \in \mathcal{L}$, let $K_1, \ldots, K_n$ be non-negative matrices that define operators on L. If $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is a coordinatewise increasing function that satisfies* (8) *and* (10)*, then the matrix $F(K_1, \ldots, K_n)$ also defines an operator on L.*

If, in addition, the matrices $K_1, \ldots, K_n$ define compact operators on L and if L has an order continuous norm, then also the matrix $F(K_1, \ldots, K_n)$ defines a compact operator on L.

Proof. If $F \equiv 0$, then there is nothing to prove. Assume that this is not the case. Then there exists some $r = (r_1, \ldots, r_n) \in intD$, where $D = \{a \in \mathbb{R}_+^n : F(a) > 0\}$. By Lemma 3.3(i) there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$, such that $t = \sum_{i=1}^n \alpha_i \geq 1$ and

$$\begin{aligned} F(K_1, \cdots, K_n) &\leq F(r) r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)} \\ &= F(r) r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} \left(K_1^{(\gamma_1)} \circ \cdots \circ K_n^{(\gamma_n)} \right)^{(t)}, \end{aligned}$$

where $\gamma_i = \alpha_i / t$ for $i = 1, \ldots, n$. Then

$$F(K_1, \cdots, K_n) \leq F(r) r_1^{-\alpha_1} \ldots r_n^{-\alpha_n} \left(\gamma_1 K_1 + \cdots + \gamma_n K_n \right)^{(t)},$$

since $\sum_{i=1}^m \gamma_i = 1$. By (2) we have $F(K_1, \cdots, K_n) \leq H$, where

$$H = F(r) r_1^{-\alpha_1} \ldots r_n^{-\alpha_n} \|K\|^{t-1} K \quad \text{and} \quad K = \gamma_1 K_1 + \cdots + \gamma_n K_n.$$

Clearly, the matrix H defines an operator on L and therefore so does the matrix $F(K_1, \cdots, K_n)$.

If the matrices $K_1, \ldots, K_n$ define compact operators on L, then so does the matrix H. If, in addition, L has an order continuous norm, then also the matrix $F(K_1, \ldots, K_n)$ defines a compact operator on L by Theorem 2.1. □

Having all the preliminaries prepared, it is now easy to prove the following result.

Theorem 3.8. *A function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies* (6) *and* (7) *for all $a, b, a_1, \ldots, a_s \in \mathbb{R}_+^n$ and $s = 2, 3, \ldots$, if and only if F satisfies* (11) *for all $L \in \mathcal{L}$ with an order continuous norm and for all non-negative matrices $K_1, \ldots, K_n$ that define compact operators on L.*

Proof. ($\Leftarrow$) : By Theorem 3.1.

($\Rightarrow$) : Let $L \in \mathcal{L}$ with an order continuous norm and let $K_1, \ldots, K_n$ be non-negative matrices that define compact operators on L. If a function F satisfies (6) and (7), then we have $r(P_m F(K_1, \ldots, K_n) P_m) \leq F(r(K_1), \ldots, r(K_n))$ for all $m \in \mathbb{N}$ by Lemma 3.6. By Lemma 3.7 the matrix $K := F(K_1, \ldots, K_n)$ defines a compact operator on L and so we have $r(P_m K P_m) \to r(K)$ as $m \to \infty$ by Corollary 2.7. This implies $r(F(K_1, \ldots, K_n)) \leq F(r(K_1), \ldots, r(K_n))$, which completes the proof. □

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