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ON THE FIBONACCI DIMENSION OF PARTIAL CUBES

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Abstract

The Fibonacci dimension $\mathrm{fdim}(G)$ of a graph G was introduced in [7] as the smallest integer d such that G admits an isometric embedding into Q_d, the d-dimensional Fibonacci cube. A somewhat new combinatorial characterization of the Fibonacci dimension is given, which enables more comfortable proofs of some previously known results. In the second part of the paper the Fibonacci dimension of the resonance graphs of catacondensed benzenoid systems is studied. This study is inspired by the fact, that the Fibonacci cubes are precisely the resonance graphs of a subclass of the catacondensed benzenoid systems. The main result shows that the Fibonacci dimension of the resonance graph of a catacondensed benzenoid system G depends on the inner dual of G. Moreover, we show that computing the Fibonacci dimension can be done in linear time for a graph of this class.

1 Introduction and preliminaries

The *hypercube* of order d and denoted Q_d is the graph $G = (V, E)$ where the vertex set $V(G)$ is the set of all binary strings $u^{(1)}u^{(2)}\ldots u^{(d)}$, $u^{(i)} \in \{0,1\}$. Two vertices $x, y \in V(G)$ are adjacent in Q_d if and only if x and y differ in precisely one place.

Let $G = (V, E)$ be a connected graph and $u, v \in V$. Then the *distance* $d_G(u, v)$ between u and v is the number of edges on a shortest u,v-path. A subgraph H of a graph G is *isometric* if for any vertices u and v of H holds $d_H(u, v) = d_G(u, v)$. Isometric subgraphs of hypercubes are called *partial cubes*. The *isometric dimension* of a graph G, $\mathrm{idim}(G)$, is the smallest integer d such that G isometrically and irredundantly embeds into the d-dimensional cube.

We will use $[n]$ for the $\{1, \ldots, n\}$ and $\bar{G}$ for the complement graph of G in this paper.

Fibonacci string of length d is a binary string $u^{(1)}u^{(2)}\ldots u^{(d)}$ with $u^{(i)} \cdot u^{(i+1)} = 0$ for $i \in [d-1]$. In other words, a Fibonacci string is a binary string without two consecutive ones. The Fibonacci cube Γ_d, $d \geq 1$, is the subgraph of Q_d induced by the Fibonacci strings of length d.

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Fibonacci cubes were introduced as a model for interconnection networks in [4]. It is well known that Fibonacci cubes are partial cubes [6], for some other results see e.g. [13] and the references therein.

Let $\beta : V(G) \to V(Q_k)$ be an isometric embedding. We will denote the i-th coordinate of β with $\beta^{(i)}$.

In the following we describe the concept introduced in [7]. Let G be a partial cube with idim(G) $= k$ and assume that we are given an isometric embedding of G into Q_k. Each pair $(i, \chi) \in [k] \times \{0, 1\}$ defines the semicube $W(i, \chi) = \{u \in V(G) | \beta^{(i)}(u) = \chi\}$. For any $i \in [k]$, we refer to $W_{(i,0)}, W_{(i,1)}$ as a *complementary pair of semicubes.*

Any isometric embedding of G with idim(G) into Q_k describes the same family of semicubes and pairs of complementary semicubes, but indexed in a different way. For a partial cube G and a complementary pair of semicubes $W_{(i,0)}, W_{(i,1)}$, the set of edges with one endvertex in $W_{(i,0)}$ and the other in $W_{(i,1)}$ forms a Θ-class of G. The Θ-classes of G constitute a partition of $E(G)$ and will be denoted with $1, \ldots, k$ in the sequel.

Let G be a connected graph. The *Fibonacci dimension*, fdim(G), is the smallest integer f such that G admits an isometric embedding into Γ_f.

Let G be a partial cube. In [7] the graph $X(G)$ was defined as follows. The nodes of $X(G)$ are the semicubes $W_{(i,\chi)}, (i, \chi) \in [k] \times \{0, 1\}$ of G, two semicubes being adjacent if they are disjoint.

A path P of $X(G)$ with the property that $|P \cap \{W_{(i,0)}, W_{(i,1)}\}| \leq 1$ for each complementary pair of semicubes $W_{(i,0)}, W_{(i,1)}$, is called a coordinating path. A set of paths $\mathcal{P}$ of $X(G)$ is called a system of coordinating paths provided that any $P \in \mathcal{P}$ is a coordinating path and for each complementary pair of semicubes $W_{(i,0)}, W_{(i,1)}$ there is exactly one $P \in \mathcal{P}$ such that $|P \cap \{W_{(i,0)}, W_{(i,1)}\}| = 1$.

Let $p(X(G))$ be the minimum size of a system of coordinating paths of $X(G)$.

Theorem 1. *[7] Let G be a partial cube. Then fdim(G) = idim(G) + p(X(G)) - 1.*

In the next we section we slightly modified the concept described above. This gives somewhat new combinatorial characterization of the Fibonacci dimension, which enables easier proofs of some previously known results. In Section 3 the Fibonacci dimension of the resonance graphs of catacondensed benzenoid systems is studied. This study is inspired by the fact, that the Fibonacci cubes are precisely the resonance graphs of zigzag hexagonal chains, which constitute a subclass of the catacondensed benzenoid systems. The main result of this section shows that the Fibonacci dimension of the resonance graph of a catacondensed benzenoid system G depends on the inner dual of G. This fact leads to the algorithm to compute the Fibonacci dimension for a graph of this class in linear time.

2 New graph, old results

Let G be a partial cube with idim(G)=k. We define the graph $X^{\tau}(G)$ as follows. Its nodes are Θ classes of G, two Θ-classes i and j are adjacent in $X^{\tau}(G)$ if there exist some $\chi, \chi' \in \{0, 1\}$ such that $W_{(i,\chi)}$ and $W_{(j,\chi')}$ are disjoint. Note that the set of vertices of $X^{\tau}(G)$ can be obtained from the set of vertices of $X(G)$, by merging every pair of complementary nodes $W_{(i,0)}, W_{(i,1)}$ of $X(G)$ into a single node; two merged nodes i and j

are adjacent in $X^\tau(G)$ if there exist an adjacent pair of complementary semicubes $W_{(i,\chi)}$ and $W_{(j,\chi')}$ in $X(G)$.

For a partial cube G its crossing graph $G^\#$ was introduced in [10] as follows. The vertices of $G^\#$ are the Θ-classes of G, two vertices being adjacent if the respective Θ-classes cross in G. More precisely, if $W_{(i,0)}, W_{(i,1)}$ and $W_{(j,0)}, W_{(j,1)}$ are pairs of complementary semicubes corresponding to Θ-classes i and j, then i and j cross if each semicube has a nonempty intersection with the semicubes from the other pair; that is, it holds that $W_{(i,0)} \cap W_{(j,0)}$, $W_{(i,0)} \cap W_{(j,1)}$, $W_{(i,1)} \cap W_{(j,0)}$, and $W_{(i,1)} \cap W_{(j,1)}$ are nonempty.

Proposition 1. *Let G be a partial cube. Then $X^\tau(G)$ is isomorphic to the complement of $G^\#$.*

Proof. Since both graphs admit the same set of vertices, we have to show the claim for the set of edges. Let i and j be two Θ-classes of G adjacent in $X^\tau(G)$. Thus there exist some $\chi, \chi' \in \{0,1\}$ such that $W_{(i,\chi)}$ and $W_{(j,\chi')}$ are disjoint. It follows that i and j cannot be adjacent in $G^\#$ and we get $E(X^\tau(G)) \subseteq E(\bar{G}^\#)$.

For the inverse note that $ij \in E(\bar{G}^\#)$ if at least one of $W_{(i,0)} \cap W_{(j,0)}$, $W_{(i,0)} \cap W_{(j,1)}$, $W_{(i,1)} \cap W_{(j,0)}$, and $W_{(i,1)} \cap W_{(j,1)}$ is empty. It follows that ij belongs to $E(X^\tau(G))$. Therefore $E(G^\#) \subseteq E(X^\tau(G))$ and the assertion follows. □

A set of vertex disjoint simple paths that together contain all the vertices of a graph G is called a *path cover*. The *path cover number* $pc(G)$ is the minimum number of paths in a path cover of G.

Lemma 1. *Let G be a partial cube. Then $pc(X^\tau(G)) = p(X(G))$.*

Proof. Let $\mathcal{P}$ be a system of coordinating paths of $X(G)$. For $P \in \mathcal{P}$ construct P' in $X^\tau(G)$ such that a Θ-class i belongs to P' if $W_{(i,0)}$ or $W_{(i,1)}$ belong P. Note that every edge in P induces an edge in $X^\tau(G)$. Vertices i and j are then adjacent in P' if for some $\chi, \chi' \in \{0,1\}$, $W_{(i,\chi)}$ and $W_{(j,\chi')}$ are disjoint in P. It follows that P' is a path in $X^\tau(G)$. Moreover, by the definition of a system of coordinating paths, for every i, either $W_{(i,0)}$ or $W_{(i,1)}$ is covered by a path of $\mathcal{P}$ exactly once. Thus, if we repeat the construction for every path of $\mathcal{P}$ we get a path cover of $X^\tau(G)$. Therefore $pc(X^\tau(G)) \leq p(X(G))$.

Let $\mathcal{C}$ be an optimal path cover of $X^\tau(G)$. For a $P' \in \mathcal{C}$ construct the coordinating path P in $X(G)$ as follows: if i and j are adjacent in P', then let $W_{(i,\chi)}$ and $W_{(j,\chi')}$, $\chi, \chi' \in \{0,1\}$, be adjacent in P for every pair of semicubes $W_{(i,\chi)}$, $W_{(j,\chi')}$ adjacent in $X(G)$. Note that if i and j are adjacent in P', then $W_{(i,\chi)}$, $\chi \in \{0,1\}$, has to be adjacent to exactly one of $W_{(j,0)}$ and $W_{(j,1)}$ in $X^\tau(G)$. Moreover, the sets $W_{(i,0)}$ and $W_{(i,1)}$ partition $V(G)$. Therefore, if $W_{(i,\chi)}$ and $W_{(j,\chi')}$ are adjacent (disjoint) in $X(G)$, then their complementary semicubes cannot be disjoint in $X(G)$. It follows that there can be exactly one pair χ, χ', such that $W_{(i,\chi)}$ and $W_{(j,\chi')}$ are adjacent in $X(G)$ and thus P is a coordinating path in $X^\tau(G)$. Moreover, if we repeat the construction for every path of $\mathcal{C}$, we obtain a system of coordinating paths of $X(G)$. Therefore $p(X(G)) \leq pc(X^\tau(G))$ which completes the proof.

□

From Lemma 1, Theorem 1, and Proposition 1 we immediately get the following theorem.

Theorem 2. *Let G be a partial cube. Then*
$$fdim(G) = idim(G) + pc(X^{\tau}(G)) - 1 = idim(G) + pc(\bar{G}^{\#}) - 1.$$

Theorem 2 also leads to the alternative proofs of the following two results which have been already proven in [7].

Theorem 3. *Let G be a partial cube with idim(G) $= k$. Then fdim(G) $= 2k - 1$ if and only if $G^{\#} = K_k$.*

Proof. It idim(G) $= k$, then from Theorem 2 it follows that fdim(G) $= 2k - 1$ if and only if $pc(\bar{G}^{\#}) = k$. □

Theorem 4. *It is NP-complete to decide if idim(G) $=$ fdim(G) for a given graph G.*

Proof. From Theorem 2 it follows that fdim(G) $=$ idim(G) if and only if $pc(\bar{G}^{\#}) = 1$. In other words, the Fibonacci dimension is equal to the isometric dimension of G if $\bar{G}^{\#}$ admits a Hamilton path. It is shown in [10] that every graph is the crossing graph of some median graph (median graphs form a subclass of partial cubes). Moreover, since it is NP-hard to find a Hamilton graph in an arbitrary graph, the assertion follows. □

3 Resonance graphs

A *benzenoid system* or a *benzenoid graph* is a finite connected plane graph with no cut vertices in which every interior region is bounded by a regular hexagon of a side length one. A benzenoid system G is *catacondensed* if any triple of hexagons of G has empty intersection. It is well known that benzenoid graphs possess very natural chemical background. In particular, the skeleton of carbon atoms in a benzenoid hydrocarbon is a benzenoid graph. The interested reader is invited to consult the books [1, 3] dedicated to these class of graphs.

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$.

Let H be a fixed subgraph of a graph G, $H \subseteq G$. The *peripheral expansion* pe($G; H$) of G with respect to H is the graph obtained from the disjoint union of G and an isomorphic copy of H, in which every vertex of the copy of H is joined by an edge with the corresponding vertex of $H \subseteq G$. Note that the ends of the newly introduced edges induce a subgraph of pe($G; H$) isomorphic to $H \Box K_2$.

Let e be an edge of a benzenoid graph G. Then the *cut* C_e corresponding to e is the set of edges so that with every edge e' of C_e also the opposite edge with respect to a hexagon containing e' belongs to C_e.

A *matching* of a graph G is a set of pairwise independent edges. A matching is a *1-factor*, if it covers all the vertices of G. For a graph G, let $\mathcal{F}(G)$ be the set of its 1-factors. In addition, if $e_1, e_2, \ldots, e_n$ are fixed edges of G, let $\mathcal{F}(G; e_1, e_2, \ldots, e_n)$ denotes the set of those 1-factors of G that contain the fixed edges.

Let G be a benzenoid graph. Then the vertex set of the *resonance graph* $R(G)$ of G consists of the 1-factors of G, two 1-factors being adjacent whenever their symmetric difference forms the edge set of a hexagon of G. The concept of a resonance graph has been introduced in chemistry and later introduced in mathematics under the name

Z-transformation graphs. An extensive survey on resonance graphs of plane bipartite graphs was presented by Zhang [15].

A *hexagonal chain* is a catacondensed benzenoid graph such that it has no hexagon with more than two adjacent hexagons. Our work is motivated by the fact, that the Fibonacci cubes are precisely the resonance graphs of zigzag hexagonal chains [11].

We now recall some notations concerning catacondensed benzenoid graph introduced in [8].

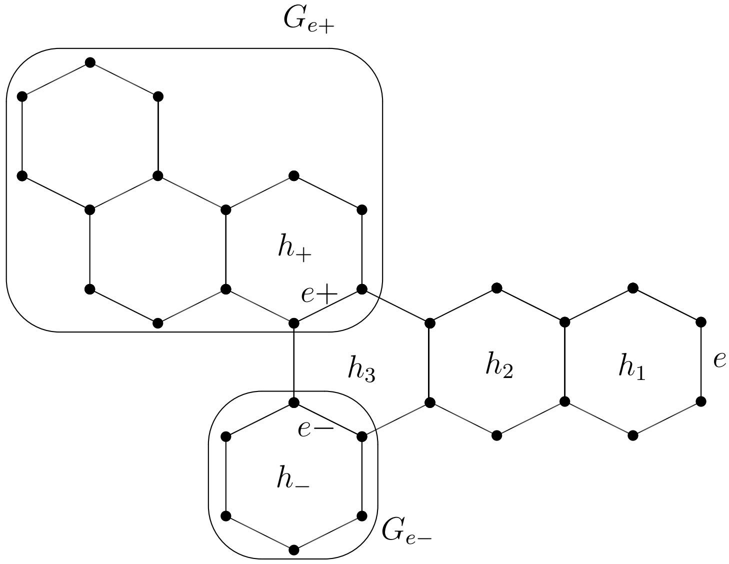

Figure 1: Catacondensed benzenoid graph G.

Let G be a catacondensed benzenoid graph and e an edge of G with ends of degree two. Let $e = e_0, e_1, \ldots, e_n$ be the edges of the cut C_e, and let $h_1 = h, h_2, \ldots, h_n$ be the corresponding hexagons. Let $e+$ and $e-$ be the edges of h_n incident to e_n, where $e+$ is the right edge looking from $e = e_0$ to e_n while $e-$ is the left edge, and let h_+ and h_- be the corresponding hexagons. Remove now from G the hexagons $h_1, \ldots, h_n$, except $e+$ and $e-$. Then the remaining graph consists of two connected components G_{e+} and G_{e-}, where $e+ \in G_{e+}$ and $e- \in G_{e-}$. Note that any of G_{e+} and G_{e-} is either a catacondensed benzenoid graph or a K_2. These notations are illustrated in Fig. 1. If G_{e+} is a catacondensed benzenoid graph, we repeat the described construction on G_{e+}, where the construction begins with $e+$. In this way we obtain two connected subgraph of G denoted G_{e++} and G_{e+-}. Similarly, if G_{e-} is a catacondensed benzenoid graph, then we repeat the construction on G_{e-}, starting with $e-$, to obtain connected subgraphs G_{e-+} and G_{e--}. In the case that $G_{e+} = K_2$ we set $G_{e++} = K_1$ and $G_{e+-} = K_1$, and if $G_{e-} = K_2$ we set $G_{e-+} = K_1$ and $G_{e--} = K_1$.

It was showed in [12] that the resonance graph $R(G)$ of a catacondensed benzenoid graph G can be isometrically embedded into the k-dimensional hypercube Q_k, where k is the number of hexagons of G. Thus, the resonance graph $R(G)$ of a catacondensed benzenoid graph G is a partial cube where the number of Θ-classes corresponds to the number of hexagons of G.

The structure of the resonance graphs of benzenoid graphs was closely examined in [8], where the following decomposition theorem is proven.

Theorem 5. *Let G be a catacondensed benzenoid graph and e the edge with ends of degree two with $|C_e| = n+1$, where $n \geq 1$. Let $Y = R(G)[\mathcal{F}(G;e)]$, $X = R(G)[\mathcal{F}(G;e,e+,e-)]$, and X_1 the copy of X in the first Y-layer of $Y \Box P_n$. Then*

$$R(G) = \mathrm{pe}(Y \Box P_n;\, X_1)\,.$$

Moreover,

(i) $Y = R(G_{e+}) \Box R(G_{e-})$ *and*
(ii) $X_1 = X = R(G_{e++}) \Box R(G_{e+-}) \Box R(G_{e-+}) \Box R(G_{e--})$.

An *internode* in a tree T is a maximal path of length at least two such that each internal vertex (i.e. not an endvertex) is of degree two.

For a tree T let *augmented tree of* T denote a graph obtained from T such that in every internode π of T the edge between two non-adjacent vertices of π may be inserted. The situation is depicted in Fig. 2. We will also use $aug(T)$ to denote the set of all augmented trees of a given tree T in the sequel.

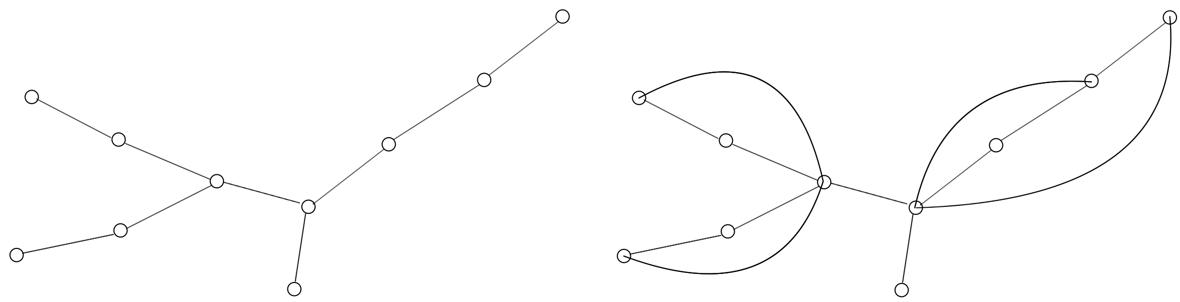

Figure 2: A tree with one of its augmented trees.

A hexagon h of a benzenoid graph G is called a *pendant hexagon* if a common path of the peripheries of h and G is a path of length five. Let h be a pendant hexagon of G and P a common path of the peripheries of h and G. Let $G-h$ denote the resultant subgraph of G by removing the internal vertices and edges of P.

Theorem 6. *Let G be a catacondensed benzenoid graph and T a graph isomorphic to the inner dual of G. Then $X^{\tau}(R(G))$ is isometric to an augmented tree of T.*

Proof. Note that the inner dual of a catacondensed benzenoid graph is isomorphic to a tree with maximal vertex degree three. It is well known that every hexagon of a catacondensed benzenoid graph corresponds to exactly one Θ-class of $R(G)$, e. g. [12].

The proof goes by the induction on the number of hexagons of G. Note first that if G is a single hexagon, the claim readily follows. Suppose than $k > 1$. Let h denote a pendant hexagon of G and let G' denote $G-h$. Let then e denote the edge of h opposite to the edge in the intersection of G' and h. From the decomposition theorem it follows that $R(G) = \mathrm{pe}(Y \Box P_n;\, X_1)$, where Y and X are defined as in Theorem 5. Moreover, $R(G') = \mathrm{pe}(Y \Box P_{n-1};\, X_1)$. Thus, G can be obtained from G' as $pe(G', Y)$. The new Θ-class, denoted E, induced by this peripheral expansion corresponds to the hexagon

$h = h_1$. Moreover, a Θ-class E_i, $i = 2, \dots, n-1$, induced by two consecutive copies Y_i and Y_{i-1} in $Y \Box P_n$ corresponds to the hexagon h_i of the cut C_e. Furthermore, the Θ-class E_n induced by the expansion of X_1 corresponds to the hexagon h_n.

It follows now from the definition of the graph X^τ that E_1 is adjacent in $X^\tau(R(G))$ to every Θ-class of $E_2, \dots, E_n$. Moreover, since Y comprises all other Θ-classes of G, E_1 cannot be adjacent to any other Θ-class in $X^\tau(R(G))$ and the claim now follows. $\square$

Corollary 1. *$X^\tau(\Gamma_d)$ is isomorphic to a path on d vertices.*

A *vine* in a graph G is a maximal path such that at least one endpoint is a leaf and each edge (if any) is incident only to vertices of degree one or two. Note that a vine is nonempty, and may be a single leaf, or G itself, if G is a single path. If a vertex of degree at least three is adjacent to the endpoints of at least two vines, it is called a *path centre*. A *vine path* is either a path (vine) that is itself a connected component of the graph, or the path that is induced by a path centre and two of its adjacent vines.

Lemma 2. *[2] Let G be any graph with a vine path P. Then we may assume that an optimal path cover for G contains P.*

The following lemma shows that similar result can be obtained for any augmented tree.

Lemma 3. *Let G be a graph isomorphic to $T^+ \in aug(T)$ and P a vine path of T. Then we may assume that an optimal path cover for G contains P.*

Proof. Let $\mathcal{C}$ be an optimal path cover for G. Since T is connected, P is induced by two vines. Let v be the path centre of P, and let P_1 and P_2 be two vines contained in P. Let $P_1 = u_1, u_2, \dots, u_\ell$ and let P' be a path in $\mathcal{C}$ that contains a vertex of P_1. We first show that P' contains all the vertices of P_1. Note that a vertex of P_1 can be adjacent only to v or to another vertex of P_1 in G. Suppose that for every $i \in [\ell]$, u_i is not adjacent to v in P'. The minimality of the optimal path cover now yields that the vertices of P_1 have to make a single path in $\mathcal{C}$.

If for some $i \in [\ell]$ there exist the edge vu_i that belongs to P', then for $j \neq i$, u_j is adjacent only to a vertex of P_1 (or v) in any path of $\mathcal{C}$. From the minimality of the optimal path cover now again follows that all vertices od P_1 have to belong to the same path in $\mathcal{C}$.

Let Q be a path in $\mathcal{C}$ that includes v. We have to study three cases now.

(i) Q includes the vertices of P_1 and P_2. We showed above that Q includes all the vertices of P_1 and P_2. Moreover, since a vertex of P_1 (P_2) can be adjacent only to v or to another vertex of P_1 (P_2), v has to be adjacent to a vertex of P_1 and to a vertex of P_2. Since the degree of v cannot exceeds two in Q, it follows that Q contains only the vertices of $V(P_1) \cup V(P_2) \cup \{v\}$.

If Q is not equal P (or we are done), by exchanging Q and P in $\mathcal{C}$ we clearly again create a minimal path cover of G.

(ii) Q contains the vertices of only one vine, say P_1. As we showed in the first part of the proof, the vertices of P_2 have to make a path in $\mathcal{C}$, say P_2'. Let Q_- be a subpath of Q containing all the vertices of Q except the vertices in $V(P_1) \cup \{v\}$. Then by exchanging Q and P_2' with P and Q_- in $\mathcal{C}$ we clearly again create a minimal path cover of G.

(iii) Q includes neither P_1 nor P_2. The vertex v divides Q into two subpaths, let denote them Q_1 and Q_2. The vertices of P_1 and P_2 induce two paths in $\mathcal{C}$, let denote them P_1'

and P_2', respectively. Exchange now the paths Q, P_1' and P_2' with P, Q_1 and Q_2 in $\mathcal{C}$. The obtained set of paths is clearly the path cover. Moreover, the number of paths in $\mathcal{C}$ remains the same and we obtained again a minimal path cover of G.

□

Lemma 4. *[2] Every (nonempty) tree contains a vine path.*

If H is a subgraph of G, let $G - H$ denote the graph induced on the set of vertices $V(G) \setminus V(H)$.

Lemma 5. *Let T be a tree. Then an optimal path cover of T is also an optimal path cover of any augmented tree of T.*

Proof. Let $T^+ \in \text{aug}(T)$. The proof is with the induction on the number of paths in an optimal path cover of T^+. If T is a path, then $pc(T) = pc(T^+) = 1$ and T is clearly the path in the optimal path cover of $aug(T)$.

Let then $pc(T) > 1$. For the inductive step see that T contains a vine path by Lemma 4. Let denote it P. From Lemma 3 it follows that P is a path in an optimal path cover of T^+. Let P^+ denote the subgraph of T^+ induced by P. It is clear that $T - P$ is a tree with pc($T - P$) = pc(T) - 1. Moreover, $T^+ - P^+$ is an augmented tree of $T - P$ and the inductive hypothesis gives the result. □

Theorem 7. *Let G be a catacondensed benzenoid graph with the inner dual T. Then $fdim(res(G)) = idim(res(G)) + pc(T) - 1$. Moreover, $fdim(res(G))$ can be computed in linear time.*

Proof. By Theorem 2 we have $fdim(res(G)) = idim(res(G)) + pc(X^\tau(res(G))) - 1$. Theorem 6 says that $X^\tau(res(G))$ is isomorphic to some $T^+ \in \text{aug}(T)$ which yields $fdim(res(G)) = idim(res(G)) + pc(T^+) - 1$. From Lemma 5 than it finally follows $fdim(res(G)) = idim(res(G)) + pc(T) - 1$.

For time complexity of the computation note first that $idim(res(G))$ equals the number of hexagons of G. Since in [2] the algorithm to construct optimal path covers for trees in linear time is presented, the proof is complete. □

Corollary 2. *The path cover number of an augmented tree can be computed in linear time.*

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