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Abstract

In [2] the authors proved that if a sequence of graphs of surjective upper semi-continuous set-valued functions $f_n : X \to 2^X$ converges to the graph of a continuous single-valued function $f : X \to X$, then the sequence of corresponding inverse limits obtained from f_n converges to the inverse limit obtained from f. In this paper a more general result is presented in which surjectivity of f_n is not required. Also, the result is generalized to the case of inverse sequences with non-constant sequences of bonding maps. Finally, these new theorems are applied to inverse limits with tent maps. Among other applications, it is shown that the inverse limits appearing in the Ingram conjecture (with a point added) form an arc.

Key words: Continua; Limits; Inverse limits; Upper semi-continuous set-valued functions; Paths; Arcs
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1 Introduction

It was proved in [2] that if a sequence of graphs of surjective upper semi-continuous (abbreviated as u.s.c.) set-valued functions $f_n : X \to 2^X$ converges to the graph of a continuous single-valued function $f : X \to X$, then the sequence of corresponding inverse limits $\{\varprojlim\{X, f_n\}_{k=1}^{\infty}\}_{n=1}^{\infty}$ converges to the inverse limit $\varprojlim\{X, f\}_{k=1}^{\infty}$.

In this paper we generalize the abovementioned result in two directions (see Section 2 for definitions and notation).

A In Section 3 we give a result about the convergence of any sequence of inverse limits of inverse sequences with constant sequences of bonding maps, which are not necessarily surjective u.s.c. set-valued functions f_n. More precisely, we prove that if X is a compact metric space, and for each positive integer n, $f_n : X \to 2^X$ is a u.s.c. set-valued function, and $f : X \to X$ a continuous single-valued function, such that

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f),$$

then

$$\lim_{n\to\infty} \varprojlim\{X, f_n\}_{k=1}^{\infty} = \varprojlim\{X, f\}_{k=1}^{\infty}$$

and

$$\pi_1(\varprojlim\{X, f\}_{k=1}^{\infty}) \subseteq \liminf \pi_1(\varprojlim\{X, f_n\}_{k=1}^{\infty})$$

are equivalent, where $\pi_1 : \prod X \to X$ is the projection map onto the first factor.

B In Section 5 we generalize the result from [2] to the case where inverse sequences with non-constant sequences of bonding maps are considered, and prove the following. For a compact metric space X, and for any nonempty closed subsets X_n^m and X_n of X, such that for each n,

$$\lim_{m\to\infty} X_n^m = X_n,$$

and for any surjective u.s.c. set-valued functions $f_n^m : X_{n+1}^m \to 2^{X_n^m}$, and continuous single-valued functions $f_n : X_{n+1} \to X_n$, such that

$$\lim_{m\to\infty} \Gamma(f_n^m) = \Gamma(f_n)$$

we prove that

$$\lim_{m\to\infty} \varprojlim\{X_n^m, f_n^m\}_{n=1}^{\infty} = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}.$$

It is a well known fact that for any continuum X, the hyperspace 2^X is path-connected (see [6, p. 113, Theorem 14.9]). In this paper we construct paths from an inverse limit to another inverse limit in $2^{\prod X}$ that go only through inverse limits; see Section 3.

We also add an additional section (Section 4) of applications and examples where the convergence of graphs of tent maps and Knaster continua are considered.

In 1992, W.T. Ingram stated the following conjecture for tent maps $T_0 = f_{(\frac{1}{2}, b_0)}$ and $T_1 = f_{(\frac{1}{2}, b_1)}$, where $b_0, b_1 > \frac{1}{2}$ (see Definition 4.1):

$\varprojlim\{[0,1], T_0\}_{n=1}^{\infty}$ and $\varprojlim\{[0,1], T_1\}_{n=1}^{\infty}$ are homeomorphic if and only if $T_0 = T_1$.

This conjecture has received a large amount of attention in the last seventeen years. It was finally proved in 2009 in [3], where the reader may find an extensive list of references. At the end of Section 4 we show that the inverse limits that appear in the Ingram conjecture (with a point in $2^{\prod [0,1]}$ added) form an arc.

2 Definitions and Notation

Our definitions and notation mostly follow Nadler [13] and Ingram and Mahavier [7].

A *continuum* is a nonempty, compact and connected metric space.

Let $W = \left\{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x \leqslant 1\right\}$. Any continuum homeomorphic to $\mathrm{Cl}(W)$ is called a *sin$\frac{1}{x}$-continuum.*

A continuum S is said to be *indecomposable* provided that whenever A and B are subcontinua of S such that $S = A \cup B$, then $A \subseteq B$ or $B \subseteq A$.

A *harmonic fan* is the continuum, defined as the union $\left(\bigcup_{n=1}^{\infty} K_n\right) \cup K$, where for each n, K_n is the segment in the plane from $(0,0)$ to $(1, \frac{1}{n})$, and K is the segment from $(0,0)$ to $(1,0)$.

Let (X_n, d_n) be a sequence of metric spaces, where all metrics are bounded by 1. Then

$$D(x,y) = \sup_{n \in \{1,2,3,\ldots\}} \left\{\frac{d_n(x_n, y_n)}{n}\right\},$$

where $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$, will be used for the metric on the product space $\prod_{n=1}^{\infty} X_n$. It is well known that the metric D induces the product topology (see [5, p. 190, Theorem 7.2], [12, p. 123, Theorem 9.5]). We will use $\prod X$ for $\prod_{k=1}^{\infty} X_k$, if $X_k = X$ for each positive integer k.

Let (X, d) be a metric space, and $\{a_n\}_{n=1}^{\infty}$ a sequence of points in X. A point $a \in X$ is called an *accumulation point* of the sequence $\{a_n\}_{n=1}^{\infty}$, if for each $\varepsilon > 0$, $d(a_n, a) < \varepsilon$ holds true for infinitely many positive integers n.

If (X, d) is a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X. Let for each $\varepsilon > 0$ and each $A \in 2^X$

$$N_d(\varepsilon, A) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

The set 2^X will be always equipped with the *Hausdorff metric* H_d, which is defined by

$$H_d(H, K) = \inf\{\varepsilon > 0 \mid H \subseteq N_d(\varepsilon, K), K \subseteq N_d(\varepsilon, H)\},$$

for $H, K \in 2^X$. Then $(2^X, H_d)$ is a metric space, called the *hyperspace* of the space (X, d). For more details see [6,13].

If (X, d) is a compact metric space, and $\{A_n\}_{n=1}^{\infty}$ a sequence of subsets of X, $\liminf A_n$, $\limsup A_n$ and $\lim_{n\to\infty} A_n$ are defined as follows:

(a) $\liminf A_n = \{x \in X :$ for each open set U, such that $x \in U, U \cap A_n \neq \emptyset$ for all but finitely many $n\}$,
(b) $\limsup A_n = \{x \in X :$ for each open set U, such that $x \in U, U \cap A_n \neq \emptyset$ for infinitely many $n\}$,
(c) $\lim_{n\to\infty} A_n = A$ if and only if $\limsup A_n = A = \liminf A_n$.

The following are well known results. The proof of Theorem 2.1 can be found in [13, p. 57, Theorem 4.11], the proof of Theorem 2.2 follows from the definitions of $\limsup A_n$ and of the Hausdorff metric.

Theorem 2.1 *Let (X, d) be a compact metric space, and $\{A_n\}_{n=1}^{\infty}$ a sequence of nonempty compact subsets of X. Then $\lim_{n\to\infty} A_n = A$ in the sense of (c) above if and only if the sequence $\{A_n\}_{n=1}^{\infty}$ converges to A in $(2^X, H_d)$.*

When writing $\lim_{n\to\infty} A_n = A$ in 2^X, we shall always mean that $\lim_{n\to\infty} A_n = A$ in $(2^X, H_d)$.

Theorem 2.2 *Let (X, d) be a compact metric space, and $\{A_n\}_{n=1}^{\infty}$ a sequence of nonempty compact subsets of X. Then* $\limsup A_n$ *is the union of all accumulation points of the sequence $\{A_n\}_{n=1}^{\infty}$ in $(2^X, H_d)$.*

Later we shall also need the following uniform property of $\liminf A_n$:

Lemma 2.3 *Let (X, d) be a compact metric space, and $\{A_n\}_{n=1}^{\infty}$ an arbitrary sequence of nonempty closed subsets of X. Then for each $\varepsilon > 0$ there is a positive integer n_0, such that for each $n \geq n_0$ it holds that for each $a \in A = \liminf A_n$ there is an element $b \in A_n$ such that $d(a, b) < \varepsilon$.*

Proof. Assume that this is not true. Then there are an $\varepsilon > 0$ and a strictly increasing sequence

$$i_1 < i_2 < i_3 < \ldots < i_n < \ldots$$

of positive integers, such that for each n there is an element $a_n \in A$ such that for each $b \in A_{i_n}$ it holds that $d(a_n, b) \geq \varepsilon$. Since A is compact, there is an accumulation point $a \in A$ of the sequence $\{a_n\}_{n=1}^{\infty}$. There are infinitely many indices n, for which $d(a_n, a) < \dfrac{\varepsilon}{2}$. For any such index n, and for any $b \in A_{i_n}$, one gets $d(a, b) \geq \dfrac{\varepsilon}{2}$ (otherwise it would follow that $d(a_n, b) < \varepsilon$, contradicting the choice of the sequence $\{a_n\}_{n=1}^{\infty}$). Therefore $a \notin A$, which is a contradiction. $\square$

A set-valued function $f : X \to 2^Y$, where X and Y are compact metric spaces, is *upper semi-continuous* (abbreviated *u.s.c.*) if for each open set $V \subseteq Y$ the set $\{x \in X \mid f(x) \subseteq V\}$ is an open set in X. A u.s.c. set-valued function $f : X \to 2^Y$ is *surjective*, if for each $y \in Y$ there is an $x \in X$, such that $y \in f(x)$.

The *graph* $\Gamma(f)$ of a u.s.c. set-valued function $f : X \to 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

Ingram and Mahavier gave the following characterization of u.s.c. set-valued functions [7, p. 120, Theorem 2.1]:

Theorem 2.4 *Let X and Y be compact metric spaces and $f : X \to 2^Y$ a set-valued function. Then f is u.s.c. if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

An *inverse sequence* of compact metric spaces X_k with u.s.c. bonding functions f_k is a sequence $\{X_k, f_k\}_{k=1}^{\infty}$, where $f_k : X_{k+1} \to 2^{X_k}$ for each k. But by certain misuse of notation, $\{X_k, f_k\}_{k=1}^{\infty}$ is denoted also by

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \ldots$$

Since in this paper we deal only with such inverse sequences, we call them simply inverse sequences.

The *inverse limit* of an inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$ is defined to be the subspace of the product space $\prod_{k=1}^{\infty} X_k$ of all $x = (x_1, x_2, x_3, \ldots) \in \prod_{k=1}^{\infty} X_k$, such that $x_k \in f_k(x_{k+1})$ for each k. The inverse limit of $\{X_k, f_k\}_{k=1}^{\infty}$ is denoted by $\varprojlim \{X_k, f_k\}_{k=1}^{\infty}$. The notion of the inverse limit of an inverse sequence with u.s.c. bonding functions was introduced by Mahavier in [11] and Ingram and Mahavier in [7].

Also, in this paper we use the following standard notation for inverse limits with constant sequences of bonding maps:

$$K_n = \varprojlim \{X, f_n\}_{k=1}^{\infty}, \ K = \varprojlim \{X, f\}_{k=1}^{\infty}.$$

Any continuous function $f : [0,1] \to X$, such that $f(0) = x$ and $f(1) = y$, is a *path* from x to y in X. The space X is *path-connected* if for each pair $x, y \in X$ of points there is a path from x to y in X.

The authors showed in [2] the following results that will be used in the present paper.

Lemma 2.5 *Let X be a compact metric space, and for each positive integer n, let M_n be a nonempty closed subset of X, and let M be a subset of X, such that*

$$\lim_{n\to\infty} M_n = M$$

in 2^X. For each positive integer n, let $x_n \in M_n$. Then $a \in M$ for each accumulation point a of the sequence $\{x_n\}_{n=1}^{\infty}$.

Theorem 2.6 *Let X be a compact metric space, and for each positive integer n, let $f_n : X \to 2^X$ be a u.s.c. set-valued function, and let $f : X \to 2^X$ be a u.s.c. set-valued function, such that*

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f)$$

in $2^{X\times X}$. Then for each accumulation point S of the sequence $\{K_n\}_{n=1}^{\infty}$ in the hyperspace $2^{\prod X}$,

$$S \subseteq K$$

holds.

Theorem 2.7 *Let X be a compact metric space, and for each positive integer n, let $f_n : X \to 2^X$ be a surjective u.s.c. set-valued function, and let $f : X \to X$ be a continuous single-valued function, such that*

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f)$$

in $2^{X\times X}$. Then

$$\lim_{n\to\infty} K_n = K$$

in $2^{\prod X}$.

Also, the following corollaries will be needed:

Corollary 2.8 *Let X be a compact metric space, and for each positive integer n, let $f_n : X \to 2^X$ be a u.s.c. set-valued function, and let $f : X \to 2^X$ be a u.s.c. set-valued function, such that*

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f)$$

in $2^{X\times X}$. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence such that $x_n \in K_n$ for each n and let s be its arbitrary accumulation point in $\prod X$. Then $s \in K$.

Proof. Obviously $s \in \limsup K_n$. By Theorem 2.2 there is an accumulation point S of the sequence K_n in $(2^{\prod X}, H_D)$, such that $s \in S$. By Theorem 2.6 $S \subseteq K$, hence $s \in K$ follows. □

Corollary 2.9 *Let X be a compact metric space, and for each positive integer n, let $f_n : X \to 2^X$ be a u.s.c. set-valued function, and let $f : X \to 2^X$ be a u.s.c. set-valued function, such that*

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f)$$

in $2^{X\times X}$. Then for each $\varepsilon > 0$ there is a positive integer n_0, such that for each positive integer $n \geq n_0$ and each $x \in K_n$ there is $y \in K$, such that $D(x, y) < \varepsilon$.

Proof. Suppose there is an $\varepsilon > 0$ such that for each n_0 there are $n \geq n_0$ and $x \in K_n$, such that $D(x, y) \geq \varepsilon$ for any $y \in K$. Then there is a strictly increasing sequence

$$i_1 < i_2 < i_3 < \ldots < i_n < \ldots$$

of positive integers, such that for each n there is a $x_n \in K_{i_n}$, such that for each $y \in K$ it holds $D(x_n, y) \geq \varepsilon$. Let s be an accumulation point of the sequence $\{x_n\}_{n=1}^{\infty}$ in $\prod X$. It follows from Corollary 2.8 that $s \in K$, which gives a contradiction with $D(s, y) \geq \varepsilon$ for each $y \in K$ (since this implies $D(s, s) \geq \varepsilon$). □

3 Paths through inverse limits

In this section we prove the main theorem of this paper in the direction **A**. We will use Lemma 3.1, which is formulated in such a way that it will be easily used in proving results in the direction **B** as well; see Section 5.

Lemma 3.1 *Let X be a compact metric space, and for each nonnegative integer n, let $X_n, Y_n \in 2^X$, such that in 2^X*

$$\lim_{n\to\infty} X_n = X_0 \quad \textit{and} \quad \lim_{n\to\infty} Y_n = Y_0.$$

Next, let $f_n : X_n \to 2^{Y_n}$ be a u.s.c. set-valued function for each nonnegative integer n, and let $f : X_0 \to Y_0$ be a single-valued continuous function such that in $2^{X\times X}$

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f).$$

Then for each $\varepsilon > 0$ there are a positive integer n_0 and a $\delta > 0$ such that for each positive integer $n \geq n_0$, for each $x \in X_0$, for each $y \in X_n$, and for each $z \in f_n(y)$, it holds that if $d(x, y) < \delta$, then $d(f(x), z) < \varepsilon$.

Proof. Suppose that there is an $\varepsilon > 0$ such that for each positive integer n_0, and for each $\delta > 0$, there exist $n \geq n_0$, $x \in X_0$, $y \in X_n$, and $z \in f_n(y)$, such that $d(x, y) < \delta$ and $d(f(x), z) \geq \varepsilon$. Using this we construct a strictly increasing sequence $n_1 < n_2 < \ldots$ of positive integers in the following way.

- Let a positive integer n_1 and $x_1 \in X_0$, $y_1 \in X_{n_1}$, $z_1 \in f_{n_1}(y_1)$ be such that $d(x_1, y_1) < \delta_1 = 1$ and $d(f(x_1), z_1) \geq \varepsilon$.
- For $m \geq 2$, let $n_m > n_{m-1}$ be a positive integer and let $x_m \in X_0$, $y_m \in X_{n_m}$, $z_m \in f_{n_m}(y_m)$ be such that $d(x_m, y_m) < \delta_m = \frac{1}{m}$ and $d(f(x_m), z_m) \geq \varepsilon$.

Obviously $\lim_{m\to\infty} d(x_m, y_m) = 0$. Let (y_0, z_0) be an accumulation point of the sequence (y_k, z_k). For each k, $(y_k, z_k) \in \Gamma(f_{n_k})$, and as $\lim_{k\to\infty} \Gamma(f_{n_k}) = \Gamma(f)$, it follows from Lemma 2.5 that $(y_0, z_0) \in \Gamma(f)$. Therefore $z_0 = f(y_0)$ and hence $d(z_0, f(y_0)) = 0$, and that contradicts the inequality $d(f(y_0), z_0) \geq \varepsilon$, which follows from the fact that for each m, $d(f(x_m), z_m) \geq \varepsilon$ and $d(x_m, y_m) < \delta_m = \frac{1}{m}$. $\square$

In the following theorem we generalize Theorem 2.7 in the direction **A**.

Theorem 3.2 *Let X be a compact metric space and for each positive integer n, let $f_n : X \to 2^X$ be a u.s.c. set-valued function, and let $f : X \to X$ be a continuous single-valued function, such that*

$$\lim_{n\to\infty} \Gamma(f_n) = \Gamma(f)$$

in $2^{X\times X}$. Then the following are equivalent:

(1) $\lim_{n\to\infty} K_n = K$ *in* $2^{\prod X}$,
(2) $L \subseteq \liminf L_n$,
(3) $L = \lim_{n\to\infty} L_n$ *in* 2^X,

where L denotes $\pi_1(K)$ and L_n denotes $\pi_1(K_n)$, with $\pi_1 : \prod X \to X$ being the projection map onto the first factor.

Proof. Assume (1).

Let $t \in \limsup L_n$ be an arbitrary element. Then there are a strictly increasing sequence $\{i_n\}_{n=1}^{\infty}$ of positive integers, and a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \in L_{i_n}$, such that $\lim\limits_{n\to\infty} t_n = t$. For each n, choose $x_n \in K_{i_n}$, such that $\pi_1(x_n) = t_n$. Because of compactness of X, there is a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$. Without loss of generality we may assume that $\{x_n\}_{n=1}^{\infty}$ converges to a point $x \in X$. Obviously $x \in \limsup K_n$, and by Theorem 2.1 it follows that $x \in K$. Therefore $\pi_1(x) \in L$. But $\pi_1(x) = \pi_1(\lim\limits_{n\to\infty} x_n) = \lim\limits_{n\to\infty} \pi_1(x_n) = \lim\limits_{n\to\infty} t_n = t$, hence $t \in L$. This way we proved that

$$\limsup L_n \subseteq L.$$

Let $t \in L$ be arbitrary. Choose an $x \in K$, such that $\pi_1(x) = t$. Let U be an arbitrary open neighborhood of t in X. Then $U \times \prod X$ is an open neighborhood of x in $\prod X$. Choose a positive integer n_0, such that $(U \times \prod X) \cap K_n \neq \emptyset$ for all $n \geq n_0$. For any $n \geq n_0$ choose $x_n \in (U \times \prod X) \cap K_n$. Then $\pi_1(x_n) \in U$, hence $t \in \liminf L_n$. This way we proved that

$$L \subseteq \liminf L_n.$$

It follows that $L = \liminf L_n = \limsup L_n$, hence by Theorem 2.1 $L = \lim\limits_{n\to\infty} L_n$. It means that we have proved that (3) follows from (1).

(2) obviously follows from (3), again by Theorem 2.1.

It remains to prove the implication from (2) to (1). Assuming (2) we shall show that

(a) for each $\varepsilon > 0$ there is a n_0 such that for each $n \geq n_0$ it holds that for each $x \in K_n$ there exists $y \in K$ such that $D(x, y) < \varepsilon$, and that
(b) for each $\varepsilon > 0$ there is a n_0 such that for each $n \geq n_0$ it holds that for each $x \in K$ there exists $y \in K_n$ such that $D(x, y) < \varepsilon$.

Note that (a) has been proved as Corollary 2.9.

Now we prove (b).

Let $\varepsilon > 0$. First choose a positive integer m, such that $\frac{\operatorname{diam}(X)}{m} < \varepsilon$. Next by repeated applications of Lemma 3.1 choose a positive integer n_0 and reals

$$0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots < \varepsilon_{m-2} < \varepsilon_{m-1} < \varepsilon_m = \varepsilon, \qquad (*)$$

(starting at the right-hand side, and then moving to the left) in such a way that for any $n \geq n_0$ and for any i, $1 \leq i \leq m$, for any $t, t' \in X$, and for any $s \in f_n(t')$, from $d(t, t') < \varepsilon_{i-1}$ it follows that $d(f(t), s) < \varepsilon_i$.

Using the assumption (2) and Lemma 2.3 we may assume that for each $n \geq n_0$ and each $t \in L$ there is an $s \in L_n$ such that $d(t, s) < \varepsilon$.

Now, let $n \geq n_0$ and

$$x = (x_1, x_2, x_3, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots) \in K$$

be arbitrary. It follows that $(x_m, x_{m+1}, \ldots) \in K$. Therefore $x_m \in L$. It follows that there is $y_m \in L_n$ such that $d(x_m, y_m) < \varepsilon_0$. Take $y' \in K_n$ such that $\pi_1(y') = y_m$.

Now arbitrarily choose $y_{m-1} \in f_n(y_m)$, $y_{m-2} \in f_n(y_{m-1})$, $y_{m-3} \in f_n(y_{m-2}), \ldots$, $y_1 \in f_n(y_2)$ and let

$$y = (y_1, y_2, \ldots, y_{m-2}, y_{m-1}, y_m, y_{m+1}, y_{m+2}, \ldots),$$

where $(y_m, y_{m+1}, y_{m+2}, \ldots) = y'$. Obviously $y \in K_n$.

Next we observe distances between the coordinates of x and y. For all $i \geq m$,

$$\frac{d(x_i, y_i)}{i} < \varepsilon$$

by the choice of m. It follows from the choice of y_m and from $(*)$ that

$$d(f(x_m), y_{m-1}) < \varepsilon_1 \leq \varepsilon.$$

It holds also that $d(f^2(x_m), y_{m-2}) < \varepsilon_2 \leq \varepsilon$ by the choice of y_{m-2} and from $(*)$.

Inductively for any $i \leq m-1$, $d(f^i(x_m), y_{m-i}) < \varepsilon_i \leq \varepsilon$ by the choice of y_{m-i+1}, and $(*)$.

It follows that

$$D(x, y) = \sup_{n \in \{1,2,3,\ldots\}} \left\{ \frac{d(x_n, y_n)}{n} \right\} < \varepsilon. \qquad \square$$

Now Theorem 2.7 (the theorem from [2] that we generalize in the present paper) easily follows: from surjectivity of f_n it follows that $L_n = X$, for all n, hence $L \subseteq \liminf L_n$, and $\lim_{n\to\infty} K_n = K$ in $2^{\prod X}$ follows from Theorem 3.2.

The following results describe the constructions of paths from an inverse limit to another inverse limit in 2^X that go only through inverse limits.

Lemma 3.3 *Let X be a nonempty compact metric space. Let $F : X \times [0,1] \to X$ be a single-valued continuous function, and let for each $t \in [0,1]$, $f_t : X \to X$ be defined by $f_t(x) = F(x,t)$. Then for each sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$ and each $t_0 \in [0,1]$, from $\lim\limits_{n\to\infty} t_n = t_0$ it follows that $\lim\limits_{n\to\infty} \Gamma(f_{t_n}) = \Gamma(f_{t_0})$.*

Proof. For each $t \in [0,1]$, $\Gamma(f_t) = \{(x, f_t(x)) \mid x \in X\} = \{(x, F(x,t)) \mid x \in X\}$. The proof easily follows from the fact that the function $\varphi : X \times [0,1] \to \Gamma(f_t)$ defined by $\varphi(x,t) = (x, F(x,t))$ is continuous and therefore uniformly continuous. □

The following corollary easily follows.

Corollary 3.4 *Let X be a nonempty compact metric space. Let $F : X \times [0,1] \to X$ be a continuous single-valued function, and let for each $t \in [0,1]$, $f_t : X \to X$ be defined by $f_t(x) = F(x,t)$. Then (1), (2), (3) and (4) below are equivalent:*

(1) The function $\varphi : [0,1] \to 2^{\prod X}$, defined by $\varphi(t) = \varprojlim\{X, f_t\}_{n=1}^{\infty}$, is a path from $\varprojlim\{X, f_0\}_{n=1}^{\infty}$ to $\varprojlim\{X, f_1\}_{n=1}^{\infty}$ in $2^{\prod X}$.
(2) For each sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$ and each $t_0 \in [0,1]$, from $\lim\limits_{n\to\infty} t_n = t_0$ it follows that $\lim\limits_{n\to\infty} K_n = K$.
(3) For each sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$ and each $t_0 \in [0,1]$, from $\lim\limits_{n\to\infty} t_n = t_0$ it follows that $L \subseteq \liminf L_n$.
(4) For each sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$ and each $t_0 \in [0,1]$, from $\lim\limits_{n\to\infty} t_n = t_0$ it follows that $L = \lim\limits_{n\to\infty} L_n$.

Here we used K to denote $\varprojlim\{X, f_{t_0}\}_{k=1}^{\infty}$, K_n to denote $\varprojlim\{X, f_{t_{in}}\}_{k=1}^{\infty}$, L to denote $\pi_1(\varprojlim\{X, f_{t_0}\}_{k=1}^{\infty})$, and L_n to denote $\pi_1(\varprojlim\{X, f_{t_{in}}\}_{k=1}^{\infty})$, where $\pi_1 : \prod X \to X$ is the projection map onto the first factor.

Proof. The corollary easily follows from Theorem 3.2 and Lemma 3.3, using the characterization of continuity of functions by the convergence of sequences. □

The following corollary follows directly from Corollary 3.4.

Corollary 3.5 *Let X be a nonempty compact metric space. Let $F : X \times [0,1] \to X$ be a continuous single-valued function, and let for each $t \in [0,1]$,*

$f_t : X \to X$ be defined by $f_t(x) = F(x,t)$. If for each $t \in [0,1]$, f_t is surjective, then the function $\varphi : [0,1] \to 2^{\prod X}$, defined by

$$\varphi(t) = \varprojlim\{X, f_t\}_{n=1}^{\infty},$$

is a path from $\varprojlim\{X, f_0\}_{n=1}^{\infty}$ to $\varprojlim\{X, f_1\}_{n=1}^{\infty}$ in $2^{\prod X}$.

Proof. If f_t is surjective for each $t \in [0,1]$, then (3) in Corollary 3.4 is satisfied. □

4 Applications to tent maps

Inverse limits of inverse sequences of unit intervals $[0,1]$ and tent bonding functions have been studied for a long time. One reason for such intense research in this area is the fact that using such inverse sequences one can obtain the famous examples of continua that have been studied long before they were described as such inverse limits. This way we can describe the arc, the $\sin\frac{1}{x}$-continuum, the Brouwer-Janiszewski-Knaster continuum (see [9], [10, p. 204], [13, p. 22]), Knaster continua that appear in the Ingram conjecture (see [3]), the harmonic fan (see [1, p. 152, Example 3.15].), and more. This is the reason why in this section we provide some examples and applications of Theorem 3.2 in the special case when $X = [0,1]$ and all the bonding functions are tent maps.

Definition 4.1 *For $a, b \in [0,1]$, the tent function $f_{(a,b)} : [0,1] \to 2^{[0,1]}$ is defined as the set-valued function with the graph $\Gamma(f_{(a,b)})$ being the union of the segment (possibly degenerate) from $(0,0)$ to (a,b) and the segment (possibly degenerate) from (a,b) to $(1,0)$. (Note that $f_{(a,b)}$ is single-valued if and only if $a \notin \{0,1\}$ or $(a,b) = (0,0), (1,0)$.) The inverse limit obtained from the inverse sequence of unit intervals $[0,1]$ and bonding functions $f_{(a,b)}$ is denoted by*

$$K_{(a,b)} = \varprojlim\{[0,1], f_{(a,b)}\}_{n=1}^{\infty}.$$

First we provide some additional information about the inverse limits $K_{(a,b)}$.

Theorem 4.2 *If $a, b \in [0,1]$ and $b < a$ or $(a,b) = (0,0)$, then $K_{(a,b)} = \{(0,0,0,\ldots)\}$.*

Proof. If $b = 0$, then clearly $K_{(a,b)} = \{(0,0,0,\ldots)\}$. Next we prove that $K_{(a,b)} = \{(0,0,0,\ldots)\}$ also when $b \neq 0$. Let $(x_1, x_2, x_3, \ldots) \in K_{(a,b)}$, $a, b \in (0,1]$ and $b < a$. If there is a positive integer n such that $x_n > b$, then there is no such $x \in [0,1]$ that $x_n \in f(x)$. Therefore for each positive integer n it holds that

$$x_n \leq b. \tag{$**$}$$

If there is a positive integer n such that $x_n = b$, then $x_{n+1} = a$. Since $a > b$ it contradicts $(**)$. Therefore for each n it holds that $0 \leq x_n < b$. It follows easily that $x_{n+1} = \frac{a}{b} x_n$ for each n. If $x_n \neq 0$ for some n, then $x_{n+k} = (\frac{a}{b})^k x_n$ for each positive integer k. As $x_n \neq 0$ and $a > b$ it follows that $\lim\limits_{k\to\infty} x_{n+k} = \lim\limits_{k\to\infty} (\dfrac{a}{b})^k x_n = \infty$, which is not possible. We have proved that for each n, $x_n = 0$. □

Theorem 4.3 *If $a \in (0,1)$, then $K_{(a,a)}$ is an arc from $(0,0,0,\ldots)$ to $(a,a,a,\ldots)$.*

Proof. Let $(x_1, x_2, x_3, \ldots) \in K_{(a,a)}$. First note that $x_n \leq a$ for each positive integer n. The bonding function $f_{(a,a)} : [0,1] \to [0,a]$ is defined by

$$f_{(a,a)}(t) = \begin{cases} t & ;\ \text{if } t \leq a\,, \\ \frac{a}{a-1}(t-1) & ;\ \text{if } t \geq a\,. \end{cases}$$

Next, for each $a \in (0,1)$, we define the following functions:

(1) $L_a : [0,a] \to [0,a]$, $L_a(t) = t$;
(2) $R_a : [0,a] \to [a,1]$, $R_a(t) = \dfrac{a-1}{a} t + 1$.

For each positive integer n either $x_{n+1} = L_a(x_n)$, or $x_{n+1} = R_a(x_n)$. If there is a positive integer n such that $x_{n+1} \neq x_n$, then $x_{n+1} = R_a(x_n) > a$, which is not possible. Therefore $K_{(a,a)} = \{(t,t,t,\ldots) \mid t \in [0,a]\}$ and hence it is an arc from $(0,0,0,\ldots)$ to $(a,a,a,\ldots)$. □

Theorem 4.4 *If $b \in (0,1)$, then $K_{(0,b)}$ is an arc from $(0,0,0,\ldots)$ to $(\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots)$.*

Proof. Denote by $f : [0,1] \to [0,b]$ the function, defined by $f(x) = b - bx$, the graph of which is the segment from $(0,b)$ to $(1,0)$ and by $g : [0,b] \to [0,1]$ its inverse, $g(x) = 1 - \frac{1}{b}x$. Next denote with $T \subseteq K_{(0,b)}$ the set of all points $(x_1, x_2, x_3, \ldots)$ with infinitely many coordinates different from 0. If there is an integer n such that $x_n = 0$, then $x_k = 0$ for all $k \geq n$. Therefore

$$(x_1, x_2, x_3, \ldots) = (x_1, g(x_1), g^2(x_1), \ldots),$$

with $x_1 \neq 0$. It is easy to see that for each positive integer n

$$g^n(x_1) = 1 - \frac{1}{b} + \frac{1}{b^2} - \frac{1}{b^3} + \ldots + (-1)^{n-1}\frac{1}{b^{n-1}} + (-1)^n\frac{x_1}{b^n} =$$

$$= \frac{1-(-1)^n\frac{1}{b^n}}{1+\frac{1}{b}} + (-1)^n\frac{x_1}{b^n} = \frac{b^{n+1}-(-1)^n b + (-1)^n(1+b)x_1}{b^n(1+b)}.$$

As $x_n \leq b$ for each positive integer n, it follows that

$$g^n(x_1) = \frac{b^{n+1}-(-1)^n b + (-1)^n(1+b)x_1}{b^n(1+b)} \leq b$$

for each positive integer n. This is equivalent to

$$(-1)^n(1+b)x_1 \leq b^{n+2} + (-1)^n b$$

for each positive integer n. If n is even we get $x_1 \leq \frac{b^{n+2}+b}{1+b}$ and by sending n to infinity $x_1 \leq \frac{b}{1+b}$. If n is odd we get $x_1 \geq \frac{-b^{n+2}+b}{1+b}$ and by sending n to infinity $x_1 \geq \frac{b}{1+b}$. Therefore $x_1 = \frac{b}{1+b}$. It follows from $g(\frac{b}{b+1}) = \frac{b}{b+1}$ that $T = \{(\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots)\}$. Next, for each positive integer n, denote with $T_n = \{(x_1, x_2, x_3, \ldots, x_n, 0, 0, 0, \ldots) \in K_{(0,b)} \mid x_1, x_2, x_3, \ldots, x_n \in (0, b]\}$ the set of points with exactly n nonzero coordinates. The following easily follows:

(1) $T_1 = \{(x, 0, 0, 0, 0, 0, 0, \ldots) \mid x \in (0, b]\}$ with $(0, 0, \ldots) \in K_{(0,b)}$ added is an arc

from $(0, 0, 0, 0, 0, 0, 0, \ldots)$ to $(b, 0, 0, 0, 0, 0, 0, \ldots)$.

$T_2 = \{(x, g(x), 0, 0, 0, 0, 0, \ldots) \mid x \in [f(b), b]\}$ is an arc

from $(b, 0, 0, 0, 0, 0, 0, \ldots)$ to $(f(b), b, 0, 0, 0, 0, 0, \ldots)$.

$T_3 = \{(x, g(x), g^2(x), 0, 0, 0, 0, \ldots) \mid x \in [f(b), f^2(b)]\}$ is an arc

from $(f(b), b, 0, 0, 0, 0, 0, \ldots)$ to $(f^2(b), f(b), b, 0, 0, 0, 0, \ldots)$.

Using induction on n we get that T_n is an arc from

$$(f^{n-2}(b), f^{n-3}(b), f^{n-4}(b), \ldots, f^2(b), f(b), b, 0, 0, 0, \ldots)$$

to

$$(f^{n-1}(b), f^{n-2}(b), f^{n-3}(b), \ldots, f^3(b), f^2(b), f(b), b, 0, 0, \ldots)$$

for each n.

(2) $\lim_{n\to\infty} T_n = T$ in $2^{\prod[0,1]}$,

(3) $K_{(0,b)} = T \cup \{(0, 0, 0, \ldots)\} \cup \left(\bigcup_{n=1}^{\infty} T_n\right)$.

For all positive integers n, m it holds that

$$T_n \cap T_{n+1} = \{(f^{n-1}(b), f^{n-2}(b), f^{n-3}(b), \ldots, f^2(b), f(b), b, 0, 0, 0, \ldots)\},$$

that $T_n \cap T_m = \emptyset$ for $|n-m| \geq 2$, and that $(\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots)\}, (0, 0, 0, \ldots) \notin T_n$. Therefore $K_{(0,b)}$ is an arc from $(0, 0, 0, \ldots)$ to $(\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots)$. $\square$

Theorem 4.5 $K_{(1,1)}$ *is homeomorphic to the harmonic fan.*

Proof. See [1, p. 152, Example 3.15]. $\square$

L.E.J. Brouwer constructed the first indecomposable continuum in [4]. In [8] Z. Janiszewski, inspired by Brouwer's example, described a simplification of Brouwer's continuum, an indecomposable continuum that does not separate the plane. B. Knaster [9] described a similar indecomposable continuum in the plane which is also presented in Kuratowski's book [10, p. 204]. We call this continuum the Brouwer-Janiszewski-Knaster continuum, see Figure 1. It follows from the construction of the Brouwer-Janiszewki-Knaster continuum, that it does not contain a $\sin\frac{1}{x}$−continuum.

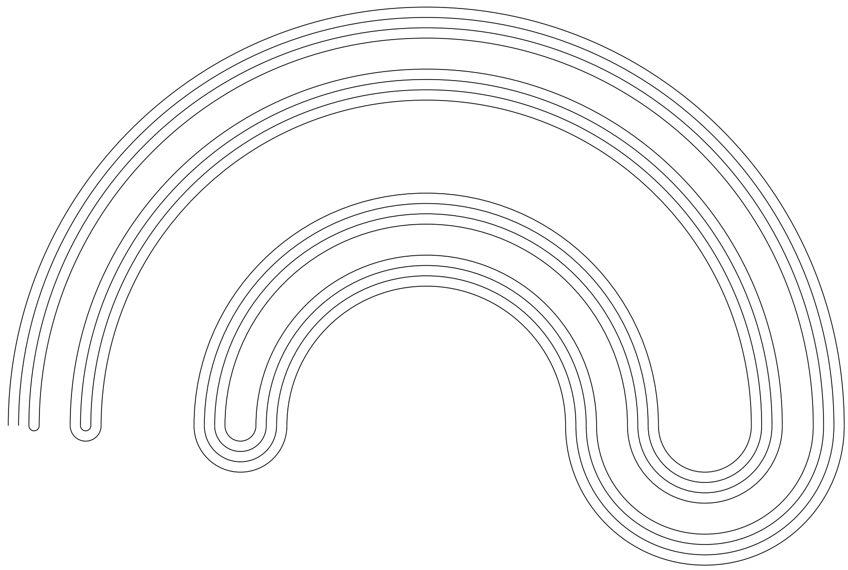

Fig. 1. The Brouwer-Janiszewki-Knaster continuum

Theorem 4.6 *The continuum* $K_{(0,1)}$ *contains a* $\sin\frac{1}{x}$*−continuum.*

Proof. Let

$$L = \{(t, 1-t, t, 1-t, \ldots) \mid t \in [0,1]\},$$

for each positive integer n

$$L_{2n} = \{(\underbrace{t, 1-t, t, 1-t, \ldots, t, 1-t}_{2n}, 0, 0, 0, \ldots) \mid t \in [0,1]\}$$

and

$$L_{2n-1} = \{(\underbrace{t, 1-t, t, 1-t, \ldots, t, 1-t, t}_{2n-1}, 0, 0, 0, \ldots) \mid t \in [0,1]\}.$$

It is easy to see that

(1) $\lim_{n\to\infty} L_n = L$,
(2) for each n, $L_{2n} \cap L_{2n+1} = \{(\underbrace{0, 1, 0, 1, \ldots, 0, 1, 0}_{2n+1}, 0, 0, 0, \ldots)\}$, and
(3) for each n, $L_{2n-1} \cap L_{2n} = \{(\underbrace{1, 0, 1, 0, \ldots, 1, 0, 1, 0}_{2n}, 0, 0, 0, \ldots)\}$.
(4) for each m, n, $L_m \cap L_n = \emptyset$, if $|m-n| \geq 2$.
(5) for each n, $L_m \cap L = \emptyset$.

Therefore

$$L \cup \left(\bigcup_{n=1}^{\infty} L_n \right)$$

is a $\sin \frac{1}{x}$−continuum, contained in $K_{(0,1)}$. □

Theorem 4.7 *The continuum $K_{(0,1)}$ is not homeomorphic to the Brouwer-Janiszewki-Knaster continuum.*

Proof. By Theorem 4.6, $K_{(0,1)}$ contains a $\sin \frac{1}{x}$−continuum. As the Brouwer-Janiszewki-Knaster continuum does not contain a $\sin \frac{1}{x}$-continuum, they are not homeomorphic. □

Theorem 4.8 *If $a \in (0,1)$, then $K_{(a,1)}$ is homeomorphic to the Brouwer-Janiszewki-Knaster continuum.*

Proof. It is a well known fact that $K_{(\frac{1}{2},1)}$ is homeomorphic to the Brouwer-Janiszewki-Knaster continuum; see [13, p. 22]. Let $a \in (0,1)$. We construct a homeomorphism $F_a : K_{(\frac{1}{2},1)} \to K_{(a,1)}$ as follows. The bonding function $f_{(a,1)} : [0,1] \to [0,1]$ is defined by

$$f_{(a,1)}(t) = \begin{cases} \frac{t}{a} & ; \text{ if } t \leq a\,, \\ \frac{t-1}{a-1} & ; \text{ if } t \geq a\,. \end{cases}$$

Next, for each $a \in (0,1)$, define the functions:

(1) $L_a : [0,1] \to [0,a]$, $L_a(t) = at$;
(2) $R_a : [0,1] \to [a,1]$, $R_a(t) = (a-1)t + 1$.

It is easy to see that for each $t \in [0,1]$ it holds that

$f_{(a,1)}(L_a(t)) = t$ and $f_{(a,1)}(R_a(t)) = t$. Define

$$F_a : K_{(\frac{1}{2},1)} \to K_{(a,1)}$$

by

$$F_a(x_1, x_2, x_3, \ldots) = (y_1, y_2, y_3, \ldots),$$

where $y_1 = x_1$ and for each positive integer n,

$$y_{n+1} = \begin{cases} L_a(y_n) & ; \text{ if } x_{n+1} = L_{\frac{1}{2}}(x_n)\,, \\ R_a(y_n) & ; \text{ if } x_{n+1} = R_{\frac{1}{2}}(x_n)\,. \end{cases}$$

By a rather straightforward analysis one can see that the function $F_a : K_{(\frac{1}{2},1)} \to K_{(a,1)}$ is a continuous bijection from a compact space to a metric space, and therefore it is a homeomorphism. □

For each $T = f_{(\frac{1}{2},b)}$, with $\frac{1}{2} < b \leq 1$, we will call $K_{(\frac{1}{2},b)} = \varprojlim\{[0,1], T\}_{n=1}^{\infty}$ a Knaster continuum. Although all Knaster continua are mutually non-homeomorphic (as it follows from the proof of the Ingram conjecture [3]), we show in Theorem 4.18 that together with the point $K_{(\frac{1}{2},\frac{1}{2})}$ they form an arc in $2^{\prod[0,1]}$.

First we provide some theorems concerning the convergence of the sequences $\{K_{(a_n,b_n)}\}_{n=1}^{\infty}$ to some $K_{(a,b)}$ in $2^{\prod[0,1]}$. Regarding Theorem 4.2, from now on we only consider the case when $(a,b) \in \{(x,y) \in [0,1] \times [0,1] \mid y \geq x\}$.

Theorem 4.9 *Let $(a,b) \in \{(x,y) \in [0,1] \times [0,1] \mid y \geq x\}$, $a \notin \{0,1\}$, and let $\{(a_n,b_n)\}_{n=1}^{\infty}$ be a sequence of points in $\{(x,y) \in [0,1] \times [0,1] \mid y \geq x\}$ such that $\lim_{n\to\infty}(a_n,b_n) = (a,b)$. Then $\lim_{n\to\infty} K_{(a_n,b_n)} = K_{(a,b)}$ in $2^{\prod[0,1]}$.*

Proof. First, for each $(c,d) \in \{(x,y) \in [0,1] \times [0,1] \mid y \geq x\}$, $c \notin \{0,1\}$, denote by $L_{(c,d)} = \pi_1(K_{(c,d)})$, where $\pi_1 : \prod[0,1] \to [0,1]$ is the projection map onto the first factor. It is obvious that $L_{(c,d)} = [0,d]$. Therefore $L_{(a,b)} = [0,b]$. Without any loss of generality we may assume that $a_n \notin \{0,1\}$ for each positive integer n, and therefore $L_{(a_n,b_n)} = [0,b_n]$. It follows from $\lim_{n\to\infty}(a_n,b_n) = (a,b)$ that $\lim_{n\to\infty} b_n = b$, and therefore $\lim_{n\to\infty} L_{(a_n,b_n)} = L_{(a,b)}$ in $2^{[0,1]}$. It is easy to see that $\lim_{n\to\infty} \Gamma(f_{(a_n,b_n)}) = \Gamma(f_{(a,b)})$ and hence $\lim_{n\to\infty} K_{(a_n,b_n)} = K_{(a,b)}$, by Theorem 3.2. □

Theorem 4.10 *Let* $\{(a_n, b_n)\}_{n=1}^{\infty}$ *be a sequence of points in* $\{(x, y) \in [0, 1] \times [0, 1] \mid y \geq x\}$ *such that* $\lim_{n\to\infty}(a_n, b_n) = (0, 0)$*. Then* $\lim_{n\to\infty} K_{(a_n,b_n)} = K_{(0,0)}$ *in* $2^{\prod[0,1]}$.

Proof. In this case $L_{(0,0)} = \{0\}$, obviously $L_{(0,0)} \subseteq \liminf L_{(a_n,b_n)}$. It is easy to see that $\lim_{n\to\infty} \Gamma(f_{(a_n,b_n)}) = \Gamma(f_{(0,0)})$ and hence $\lim_{n\to\infty} K_{(a_n,b_n)} = K_{(0,0)}$ by Theorem 3.2. □

Theorem 4.11 *Let* $\{(a_n, b_n)\}_{n=1}^{\infty}$ *be a sequence of points in* $[0, 1] \times [0, 1]$ *such that* $\lim_{n\to\infty}(a_n, b_n) = (0, b)$ *for* $b \notin \{0, 1\}$*. Then* $\lim_{n\to\infty} K_{(0,b_n)} = K_{(0,b)}$ *in* $2^{\prod[0,1]}$.

Proof. It follows from Theorem 4.4 that $K_{(0,b)}$ is an arc from $(0, 0, 0, \ldots)$ to $(\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots)$.

First assume that $a_n = 0$ for each n. Without any loss of generality we assume that $b_n \notin \{0, 1\}$ for each n, and therefore $K_{(a_n,b_n)}$ is an arc from $(0, 0, 0, \ldots)$ to $(\frac{b_n}{1+b_n}, \frac{b_n}{1+b_n}, \frac{b_n}{1+b_n}, \ldots)$ for each n. It follows from Theorem 2.6 that for each accumulation point S of the sequence $\{K_{(0,b_n)}\}_{n=1}^{\infty}$, $S \subseteq K_{(0,b)}$. But it follows from

$$\lim_{n\to\infty} b_n = b$$

that for each such accumulation point S,

$$(0, 0, 0, \ldots), (\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots) \in S,$$

hence $K_{(0,b)} \subseteq S$. Therefore $S = K_{(0,b)}$ and $\lim_{n\to\infty} K_{(0,b_n)} = K_{(0,b)}$.

Next assume that $a_n \neq 0$ for each n. It follows from Theorem 2.6 that for each accumulation point S of the sequence $\{K_{(a_n,b_n)}\}_{n=1}^{\infty}$, $S \subseteq K_{(0,b)}$. But it follows from $\lim_{n\to\infty}(a_n, b_n) = (0, b)$ and $(0, 0, 0, \ldots) \in K_{(a_n,b_n)}$ for each n that for each such accumulation point S,

$$(0, 0, 0, \ldots), (\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots) \in S.$$

Since $K_{(0,b)}$ is an arc from $(0, 0, 0, \ldots)$ to $(\frac{b}{1+b}, \frac{b}{1+b}, \frac{b}{1+b}, \ldots)$, it follows that $S = K_{(0,b)}$ for each accumulation point S, and therefore $\lim_{n\to\infty} K_{(a_n,b_n)} = K_{(0,b)}$. The general case follows easily by considering appropriate subsequences. □

Example 4.12 *Here we show that* $\lim_{n\to\infty} K_{(1-\frac{1}{n},1-\frac{1}{n})} = K_{(1,1)}$ *does not hold.*

By Theorem 4.3 for each positive integer $n \geq 2$*,*

$$K_{(1-\frac{1}{n},1-\frac{1}{n})} = \{(t, t, t, \ldots) \mid t \in [0, 1 - \frac{1}{n}]\}$$

is an arc from $(0,0,0,\ldots)$ *to* $(1-\frac{1}{n},1-\frac{1}{n},1-\frac{1}{n},\ldots)$. *Obviously for each such* n *and each* $x=(t,t,t,\ldots)\in K_{(1-\frac{1}{n},1-\frac{1}{n})}$, *and for* $(0,1,1,1,\ldots)\in K_{(1,1)}$,

$$D(x,(0,1,1,1,\ldots)) = \max\left\{t,\frac{1-t}{2}\right\} \geq$$

$$\min\left\{\max\left\{s,\frac{1-s}{2}\right\} \mid s\in[0,1-\frac{1}{n}]\right\} = \frac{1}{3}$$

and therefore $\lim_{n\to\infty} K_{(1-\frac{1}{n},1-\frac{1}{n})} = K_{(1,1)}$ *does not hold.*

Theorem 4.13 *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence in* $[0,1]$, *such that* $\lim_{n\to\infty} a_n = 1$. *Then* $\lim_{n\to\infty} K_{(a_n,1)} = K_{(1,1)}$ *in* $2^{\prod[0,1]}$.

Proof. In this case $f_{(1,1)}$ is not a single-valued function, hence Theorem 3.2 cannot be applied.

Without any loss of generality we may assume that $a_n \neq 1$ for each n. It follows from Corollary 2.9 that for each $\varepsilon > 0$ there is a n_0 such that for each $n \geq n_0$, for each $x \in K_{(a_n,1)}$ there is a $y \in K_{(1,1)}$ such that $D(x,y) < \varepsilon$. Therefore we only need to show that for each $\varepsilon > 0$ there is a n_0 such that for each $n \geq n_0$, for each $x \in K_{(1,1)}$ there is a $y \in K_{(a_n,1)}$ such that $D(x,y) < \varepsilon$. Let $\varepsilon > 0$ and $x = (x_1,x_2,x_3,\ldots) \in K_{(1,1)}$.

The bonding function $f_{(a_n,1)} : [0,1] \to [0,1]$ is defined by

$$f_{(a_n,1)}(t) = \begin{cases} \frac{t}{a_n} & ;\ \text{if } t \leq a_n\ , \\ \frac{t-1}{a_n-1} & ;\ \text{if } t \geq a_n\ . \end{cases}$$

Next define the following functions:

(1) $L_{a_n} : [0,1] \to [0,a_n]$, $L_{a_n}(t) = a_n t$;
(2) $R_{a_n} : [0,1] \to [a_n,1]$, $R_{a_n}(t) = (a_n-1)t+1$.

It is easy to see that for each $t \in [0,1]$ it holds that

$f_{(a_n,1)}(L_{a_n}(t)) = t$ and $f_{(a_n,1)}(R_{a_n}(t)) = t$.

One can easily find a positive integer n_1 such that for each $n \geq n_1$ and for each $x = (t,t,t,t,\ldots) \in K_{(1,1)}$, $t \neq 1$, it holds that $D(x,y) < \varepsilon$, if y is chosen to be

$$y = (t, L_{a_n}(t), L_{a_n}^2(t), L_{a_n}^3(t), \ldots) \in K_{(a_n,1)}.$$

It is also easy to find a positive integer n_2 such that for each $n \geq n_2$ and for $x = (1,1,1,1,\ldots) \in K_{(1,1)}$, it holds that $D(x,y) < \varepsilon$, if y is chosen to be

$$y = (1, R_{a_n}(1), R_{a_n}^2(1), R_{a_n}^3(1), \ldots) \in K_{(a_n,1)}.$$

One can also find a positive integer n_3 such that for each $n \geq n_3$, for each k, and for each $x = (\underbrace{t, t, \ldots, t}_{k}, 1, 1, 1, \ldots) \in K_{(1,1)}$, it holds that $D(x, y) < \varepsilon$, if $y \in K_{(a_n,1)}$ is chosen to be

$$y = (\underbrace{t, L_{a_n}(t), L_{a_n}^2(t), L_{a_n}^3(t), \ldots, L_{a_n}^{k_1-1}(t)}_{k},$$

$$R_{a_n}(L_{a_n}^{k_1-1}(t)), R_{a_n}^2(L_{a_n}^{k_1-1}(t)), R_{a_n}^3(L_{a_n}^{k_1-1}(t)), \ldots).$$

Now let $n_0 = \max\{n_1, n_2, n_3\}$. Then for each $n \geq n_0$, for each $x \in K_{(1,1)}$ there is a $y \in K_{(a_n,1)}$ such that $D(x, y) < \varepsilon$. $\square$

Example 4.14 *Let* $(a_n, b_n) = \left(0, \frac{n}{n+1}\right)$ *for each positive integer* n. *Obviously,* $\lim_{n\to\infty}(a_n, b_n) = (0, 1)$. *We show that in this case* $\lim_{n\to\infty} K_{(a_n,b_n)} = K_{(0,1)}$ *does not hold, by showing that there is a positive integer* n_0 *such that for each* $n \geq n_0$ *and for each* $y = (y_1, y_2, y_3, y_4, y_5, \ldots) \in K_{(a_n,b_n)}$,

$$D(x, y) \geq \frac{1}{7},$$

where $x = \left(\frac{1}{2}, 0, 1, 0, 1, \ldots\right) \in K_{(0,1)}$.

We distinguish the following two cases.

(1) *If* $y_2 = 0$, *then* $y_3 = 0$ *or* $y_3 = 1$. *If* $y_3 = 1$, *then there is not such* y_4 *that* $y_3 \in f_{(a_n,b_n)}(y_4)$. *Therefore* $y_3 = 0$. *In this case* $D(x, y) \geq \frac{1}{3} \geq \frac{1}{7}$.
(2) *If* $y_2 \neq 0$, *then* $y_2 \in \left(0, \frac{1}{3}\right]$ *or* $y_2 \in \left(\frac{1}{3}, 1\right]$. *If* $y_2 \in \left(\frac{1}{3}, 1\right]$, *then* $D(x, y) \geq \frac{1}{7}$. *If* $y_2 \in \left(0, \frac{1}{3}\right]$, *then* $y_1 = f_{(a_n,b_n)}(y_2)$. *In this case*

$$d\left(y_1, \frac{1}{2}\right) = \left|y_1 - \frac{1}{2}\right| = \left|f_{(a_n,b_n)}(y_2) - \frac{1}{2}\right| = \left|-\frac{n}{n+1}y_2 + \frac{n}{n+1} - \frac{1}{2}\right|.$$

As

$$\lim_{n\to\infty}\left|-\frac{n}{n+1}y_2 + \frac{n}{n+1} - \frac{1}{2}\right| \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \geq \frac{1}{7},$$

there is a positive integer n_0 *such that for each* $n \geq n_0$ *and each* $y \in K_{(a_n,b_n)}$ *it holds* $D(x, y) \geq \frac{1}{7}$.

The following theorem can be proved using a similar approach that has been used to prove Theorem 4.13.

Theorem 4.15 *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of points in* $[0, 1]$ *such that* $\lim_{n\to\infty} a_n = 0$. *Then* $\lim_{n\to\infty} K_{(a_n,1)} = K_{(0,1)}$ *in* $2^{\prod[0,1]}$. $\square$

Next we present several results about paths in the hyperspace $2^{\prod[0,1]}$.

Theorem 4.16 *Let $\varphi : [0,1] \to [0,1] \times [0,1]$ be a path. If for each $t \in [0,1]$, $\varphi(t) \in \{(x,y) \in [0,1] \times [0,1] \mid y \geq x\} \setminus \{(0,1),(1,1)\}$, then $\phi : [0,1] \to 2^{\prod[0,1]}$, defined by $\phi(t) = K_{\varphi(t)}$, is a path from $K_{\varphi(0)}$ to $K_{\varphi(1)}$.*

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ and $t_0 \in (0,1)$ such that $\lim_{n\to\infty} t_n = t_0$. As φ is continuous, it follows that $\lim_{n\to\infty} \varphi(t_n) = \varphi(t_0)$. It follows from Theorems 4.9 and 4.10 that $\lim_{n\to\infty} K_{\varphi(t_n)} = K_{\varphi(t_0)}$ and therefore ϕ is continuous. □

In the following theorem we describe a path from $K_{(0,1)}$, the continuum we studied in Theorem 4.6, to $K_{(1,1)}$, which is homeomorphic to the harmonic fan, see Theorem 4.5. The described path goes only through $K_{(t,1)}$, $t \in (0,1)$, where for each $t \in (0,1)$, $K_{(t,1)}$ is homeomorphic to the the Brouwer-Janiszewki-Knaster continuum, see Theorem 4.8.

Theorem 4.17 *Let $\phi : [0,1] \to 2^{\prod[0,1]}$ be defined by $\phi(t) = K_{(t,1)}$. Then ϕ is a path from $K_{(0,1)}$ to $K_{(1,1)}$; in fact, $\phi([0,1])$ is an arc.*

Proof. The function $\varphi : [0,1] \to [0,1] \times [0,1]$, defined by $\varphi(t) = (t,1)$, is a path from $(0,1)$ to $(1,1)$. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ and $t_0 \in [0,1]$ such that $\lim_{n\to\infty} t_n = t_0$. As φ is continuous, it follows that $\lim_{n\to\infty} \varphi(t_n) = \varphi(t_0)$. It follows from Theorems 4.9, 4.15 and 4.13 that $\lim_{n\to\infty} K_{\varphi(t_n)} = K_{\varphi(t_0)}$ and therefore ϕ is continuous. Since there is a point of the form $(1, t, \ldots)$ in $K_{(t,1)}$, but not in $K_{(s,1)}$ for $s \neq t$, it follows that $\phi([0,1])$ is an arc. □

In the following theorem we construct a path from $K_{(\frac{1}{2},\frac{1}{2})}$ to $K_{(\frac{1}{2},1)}$ only through $K_{(\frac{1}{2},t)}$, $t \in (\frac{1}{2},1)$. Here, $K_{(\frac{1}{2},\frac{1}{2})}$ is an arc, see Theorem 4.3, while $K_{(\frac{1}{2},1)}$ is homeomorphic to the Brouwer-Janiszewki-Knaster continuum, see Theorem 4.8. For each $t \in (\frac{1}{2},1)$, $K_{(\frac{1}{2},t)}$ is a Knaster continuum from the Ingram conjecture.

Theorem 4.18 *$\phi : [\frac{1}{2},1] \to 2^{\prod[0,1]}$, defined by $\phi(t) = K_{(\frac{1}{2},t)}$ is a path from $K_{(\frac{1}{2},\frac{1}{2})}$ to $K_{(\frac{1}{2},1)}$, and $\phi([0,1])$ is an arc.*

Proof. By Theorem 4.16, $\phi : [\frac{1}{2},1] \to 2^{\prod[0,1]}$, defined by $\phi(t) = K_{(\frac{1}{2},t)}$ is a path from $K_{(\frac{1}{2},\frac{1}{2})}$ to $K_{(\frac{1}{2},1)}$. Since all these inverse limits are non-homeomorphic, it follows that $\phi([0,1])$ is an arc. □

5 Another generalization of Theorem 2.7

In the following theorem we generalize Theorem 2.7 in the direction **B**. But first we introduce the notation that will be used. For the metric spaces X_n^m and X_n and for u.s.c. set-valued functions $f_n^m : X_{n+1}^m \to 2^{X_n^m}$ and $f_n : X_{n+1} \to X_n$, we will use

$$K_m = \varprojlim \{X_n^m, f_n^m\}_{n=1}^\infty \,,$$
$$K = \varprojlim \{X_n, f_n\}_{n=1}^\infty \,.$$

Theorem 5.1 *Let X be a compact metric space and for all positive integers m, n, let X_n^m and X_n be nonempty closed subsets of X, such that for each n,*

$$\lim_{m\to\infty} X_n^m = X_n.$$

Next, let for all positive integers m, n, $f_n^m : X_{n+1}^m \to 2^{X_n^m}$ be surjective u.s.c. set-valued functions, and let $f_n : X_{n+1} \to X_n$ be continuous single-valued functions, such that

$$\lim_{m\to\infty} \Gamma(f_n^m) = \Gamma(f_n)$$

in $2^{X\times X}$. Then

$$\lim_{m\to\infty} K_m = K$$

in $2^{\prod X}$.

The following diagram may help the reader to get a better idea about the situation:

$$\begin{array}{lcl}
X_1^1 \overset{f_1^1}{\leftarrow} X_2^1 \overset{f_2^1}{\leftarrow} X_3^1 \overset{f_3^1}{\leftarrow} X_4^1 \overset{f_4^1}{\leftarrow} X_5^1 \overset{f_5^1}{\leftarrow} X_6^1 \overset{f_6^1}{\leftarrow} \ldots & & K_1 \\
X_1^2 \overset{f_1^2}{\leftarrow} X_2^2 \overset{f_2^2}{\leftarrow} X_3^2 \overset{f_3^2}{\leftarrow} X_4^2 \overset{f_4^2}{\leftarrow} X_5^2 \overset{f_5^2}{\leftarrow} X_6^2 \overset{f_6^2}{\leftarrow} \ldots & & K_2 \\
X_1^3 \overset{f_1^3}{\leftarrow} X_2^3 \overset{f_2^3}{\leftarrow} X_3^3 \overset{f_3^3}{\leftarrow} X_4^3 \overset{f_4^3}{\leftarrow} X_5^3 \overset{f_5^3}{\leftarrow} X_6^3 \overset{f_6^3}{\leftarrow} \ldots & & K_3 \\
X_1^4 \overset{f_1^4}{\leftarrow} X_2^4 \overset{f_2^4}{\leftarrow} X_3^4 \overset{f_3^4}{\leftarrow} X_4^4 \overset{f_4^4}{\leftarrow} X_5^4 \overset{f_5^4}{\leftarrow} X_6^4 \overset{f_6^4}{\leftarrow} \ldots & & K_4 \\
X_1^5 \overset{f_1^5}{\leftarrow} X_2^5 \overset{f_2^5}{\leftarrow} X_3^5 \overset{f_3^5}{\leftarrow} X_4^5 \overset{f_4^5}{\leftarrow} X_5^5 \overset{f_5^5}{\leftarrow} X_6^5 \overset{f_6^5}{\leftarrow} \ldots & & K_5 \\
X_1^6 \overset{f_1^6}{\leftarrow} X_2^6 \overset{f_2^6}{\leftarrow} X_3^6 \overset{f_3^6}{\leftarrow} X_4^6 \overset{f_4^6}{\leftarrow} X_5^6 \overset{f_5^6}{\leftarrow} X_6^6 \overset{f_6^6}{\leftarrow} \ldots & & K_6 \\
\qquad\qquad\qquad \vdots & & \vdots \\
X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow} X_3 \overset{f_3}{\leftarrow} X_4 \overset{f_4}{\leftarrow} X_5 \overset{f_5}{\leftarrow} X_6 \overset{f_6}{\leftarrow} \ldots & & K
\end{array}$$

Proof. We must prove that for each $\varepsilon > 0$, there exists a positive integer m_0 such that $H_D(K_m, K) < \varepsilon$ holds true for all positive integers $m \geq m_0$. By the definition of H_D that means that for all positive integers $m \geq m_0$ it holds that

(a) for all $x \in K$, there exists $y \in K_m$, such that $D(x,y) < \varepsilon$, and
(b) for all $x \in K_m$, there exists $y \in K$, such that $D(x,y) < \varepsilon$.

In order to prove that for any given $\varepsilon > 0$ we can choose a positive integer m_0, such that (a) holds for any $m \geq m_0$, let us first fix a positive integer n such that $\frac{\operatorname{diam}(X)}{n} < \varepsilon$.

Then choose numbers

$$0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \ldots < \varepsilon_{n-2} < \varepsilon_{n-1} < \varepsilon_n = \varepsilon, \qquad (***)$$

starting at the right-hand side, and then moving to the left in such a way that for any chosen ε_i the next ε_{i-1} is chosen, so that for any $x \in X_i$ and $x' \in X_i^n$, it follows from $d(x,x') < \varepsilon_{i-1}$ that $d(f_{n-i+1}(x), y) < \varepsilon_i$ for $y \in f^n_{n-i+1}(x')$, as guaranteed by Lemma 3.1.

Fix an arbitrary $x \in K$ and an arbitrary $m \geq m_0$. Then x is of the form

$$x = (f_1(f_2(\ldots(f_{n-1}(x_n)\ldots)), \ldots, f_{n-2}(f_{n-1}(x_n)), f_{n-1}(x_n), x_n, x_{n+1}, \ldots)\,.$$

First choose a $y_n \in X_n^m$, such that $d(x_n, y_n) < \varepsilon_1$. It follows from $\lim_{m\to\infty} X_n^m = X_n$ that such a y_n exists. Now arbitrarily choose $y_{n-1} \in f^m_{n-1}(y_n)$, $y_{n-2} \in f^m_{n-2}(y_{n-1})$, $y_{n-3} \in f^m_{n-3}(y_{n-2})$, …, $y_1 \in f^m_1(y_2)$ and let $y \in K_m$ be of the form

$$y = (y_1, y_2, \ldots, y_{n-2}, y_{n-1}, y_n, y_{n+1}, y_{n+2}, \ldots),$$

where y_{n+1}, y_{n+2}, … have been chosen using surjectivity of the bonding functions.

Next we observe distances between the coordinates of x and y. For all $i \geq n$,

$$\frac{d(x_i, y_i)}{i} < \varepsilon$$

by the choice of n. It follows from the choice of y_n and from $(***)$ that

$$d(f_{n-1}(x_n), y_{n-1}) < \varepsilon_2 \leq \varepsilon.$$

It holds that $d(f_{n-2}(f_{n-1}(x_n)), y_{n-2}) < \varepsilon_3$ by the choice of y_{n-2} and from $(***)$.

Inductively for any $i \leq n-1$,

$$d(f_{n-i}(f_{n-i-1}(\ldots(f_{n-1}(x_n)\ldots))), y_{n-i}) < \varepsilon_{i+1} \leq \varepsilon$$

by the choice of y_{n-i+1}, and $(***)$.

It follows that

$$D(x,y) = \sup_{n\in\{1,2,3,\ldots\}} \left\{ \frac{d(x_n, y_n)}{n} \right\} < \varepsilon,$$

proving (a).

In order to prove that for any given $\varepsilon > 0$ we can choose a positive integer m_0, such that (b) holds for any $m \geq m_0$, assume that it is not so. In that case there would exist a strictly increasing sequence i_m of integers and a sequence $x_m \in K_{i_m}$ such that for all m and all $y \in K$, $D(x_m, y) \geq \varepsilon$. Let z be an accumulation point of $\{x_m\}_{m=1}^{\infty}$. By Corollary 2.8 $z \in K$, hence $d(x_m, z) \geq \varepsilon$ for all positive integers m, which contradicts the choice of z. □

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