PLANAR GRAPHS WITH LARGEST INJECTIVE CHROMATIC NUMBERS

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Planar graphs with largest injective chromatic numbers[*]

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Abstract

An injective coloring of graphs is a vertex coloring where two vertices receive distinct colors if they have a common neighbor. In 1977, Wegner [12] posed a conjecture on the chromatic number of squares of graphs which remains unsolved for planar graphs with maximum degrees $\Delta \geq 4$. Obviously, the chromatic number of the square of a graph is at least the injective chromatic number of a graph. We present examples and constructions of planar graphs for every Δ, which have the injective chromatic number equal to the bound conjectured by Wegner.

Keywords: Graph coloring, injective coloring, planar graphs

MSC2000: 05C15

1 Introduction

An *injective coloring* of a graph G is a mapping $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every pair of vertices $u, v \in V(G)$ that have a common neighbor. The least k such that G is injectively colorable is the *injective chromatic number* of G, denoted by $\chi_i(G)$. Note that this type of coloring is not necessarily proper.

The injective coloring of graphs was first introduced by Hahn, Kratochvíl, Širáň and Sotteau [6] in 2002. In 2005, first results on planar graphs were given by Doyon, Hahn and Raspaud [4]. As a corollary of the main theorem they obtained that if G is a planar graph of maximum degree Δ and of girth $g(G) \geq 7$, then the injective chromatic number is at most $\Delta+3$. Moreover, if $g(G) \geq 6$ then $\chi_i(G) \leq \Delta+4$, and if $g(G) \geq 5$ then $\chi_i(G) \leq \Delta+8$. The next result is due to Hahn, Raspaud, and Wang [7]. They investigated graphs without K_4-minor and showed that if G is a K_4-minor free graph of maximum degree Δ, then $\chi_i(G) \leq \lceil \frac{3}{2}\Delta \rceil$. For planar graphs, they proved the following theorem:

Theorem 1 *If G is a planar graph of maximum degree $\Delta \geq 3$, then $\chi_i(G) \leq \Delta^2 - \Delta$.*

In 2006, Lužar, Škrekovski, and Tancer [9] proved several results for planar graphs of higher girth:

Theorem 2 *Let G be a planar graph with maximum degree Δ. Then the following hold:*

(a) *if $g(G) \geq 19$, then $\chi_i(G) \leq \Delta$;*

(b) *if $g(G) \geq 10$, then $\chi_i(G) \leq \Delta + 1$;*

(c) *if $g(G) \geq 7$ and $\Delta \leq 3$, then $\chi_i(G) \leq 5$;*

(d) *if $g(G) \geq 5$ and $\Delta \geq 439$, then $\chi_i(G) \leq \Delta + 4$.*

In 2009, Dimitrov, Lužar, and Škrekovski [3] completed Theorem 2 by proving that the injective chromatic number of planar graphs of girth at least 7 is at most $\Delta + 2$. Some of these results were improved by Cranston, Kim, and Yu. They showed that for $\Delta \geq 4$ all planar graphs with girth at least 13 can be injectively colored with Δ colors. Moreover, if a planar graph has girth at least 9, the injective chromatic number is at most $\Delta + 1$.

When considering graphs of girth less than 5 the injective chromatic number considerably increases. In Fig. 1 a planar graph of girth 4 and injective chromatic number $\frac{3}{2}\Delta$ is presented.
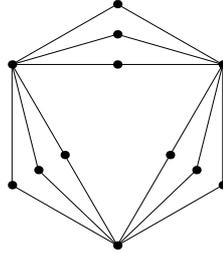

Figure 1: Planar graph with maximum degree Δ and girth 4 that needs $\frac{3}{2}\Delta$ colors for an injective coloring.

Thus, it is natural to ask what is the upper bound for an injective chromatic number of planar graphs. In 2006, Hahn, Raspaud and Wang [7] asked whether it is true that for every planar graph G with maximum degree Δ holds that $\chi_i(G) \leq \left\lceil \frac{3}{2}\Delta \right\rceil$. In this note we present examples of planar graphs with maximum degree $\Delta \geq 4$ and injective chromatic number $\Delta + 5$, for $4 \leq \Delta \leq 7$, and $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, for $\Delta \geq 8$.

2 Planar graphs with largest injective chromatic numbers

The upper bounds of an injective chromatic number presented in this note correspond to orders of maximal planar graphs with diameter two. It is easy to see that the injective chromatic number of planar triangulations with diameter two is equal to their order.

Moreover, recall that the graph does not need to be a triangulation to have an injective chromatic number equal to its order, it is enough that there exists a path of length two between all pairs of its vertices.

The size of largest planar graphs of diameter two has been intensively studied. In 1993, Hell and Seyffarth [8] proved that the order of every planar graph G with diameter two and maximum degree $\Delta \geq 8$, is at most $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$. They also gave a construction of such graphs. Furthermore, they posed the lower bounds for largest possible order n of planar graphs of diameter two and maximum degree $4 \leq \Delta \leq 7$, and presented such graphs. Tishchenko [11] proved that the obtained lower bounds, $n \geq \Delta + 5$, were in fact sharp.

Although these bounds were not proved yet, Wegner [12] posed a notable conjecture in 1977 on coloring squares of planar graphs which uses the same bounds for the required number of colors:

Conjecture 1 (Wegner) *Let G be a planar graph with maximum degree Δ. The chromatic number of square graph G^2 is at most 7, if $\Delta = 3$, at most $\Delta + 5$, if $4 \leq \Delta \leq 7$, and at most $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, otherwise.*

Now we present planar graphs whose injective chromatic number, with exception of cubic graphs, is the bound proposed by Wegner.

Maximum degree $\Delta = 3$. When subcubic graphs are considered, Conjecture 1 states that the square of any planar graph is 7-colorable. This result is announced to be proven by Thomassen, but it is not published yet. For injective colorings it seems that 5 colors suffice. Theorem 1 implies that every subcubic planar graph is injectively 6-colorable. However, until now only the subcubic graphs with injective chromatic number 5 are known. In Fig. 2 a graph presented in [7] is depicted.
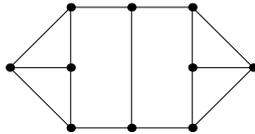

Figure 2: Planar cubic graph with injective chromatic number 5.

Maximum degree $4 \leq \Delta \leq 7$. Now, we present examples of planar graphs with maximum degree $4 \leq \Delta \leq 7$. Note that the graphs depicted in Fig. 3 were already presented in [8] and are the largest planar graphs with diameter 2 and fixed maximum degree Δ. In Fig. 4 we depict new graphs of maximum degree $5 \leq \Delta \leq 9$ with a different construction and injective chromatic number $\Delta + 5$.
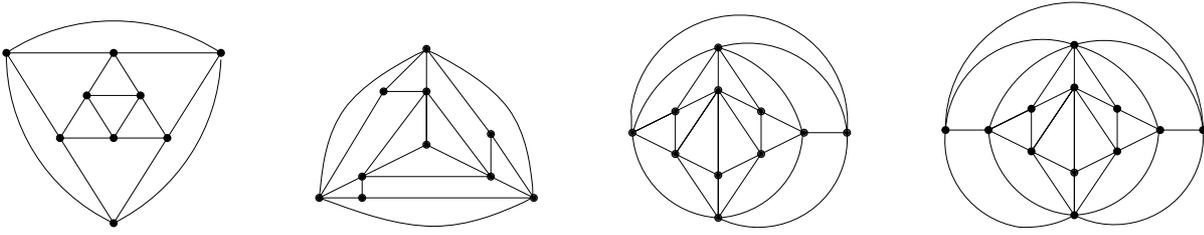
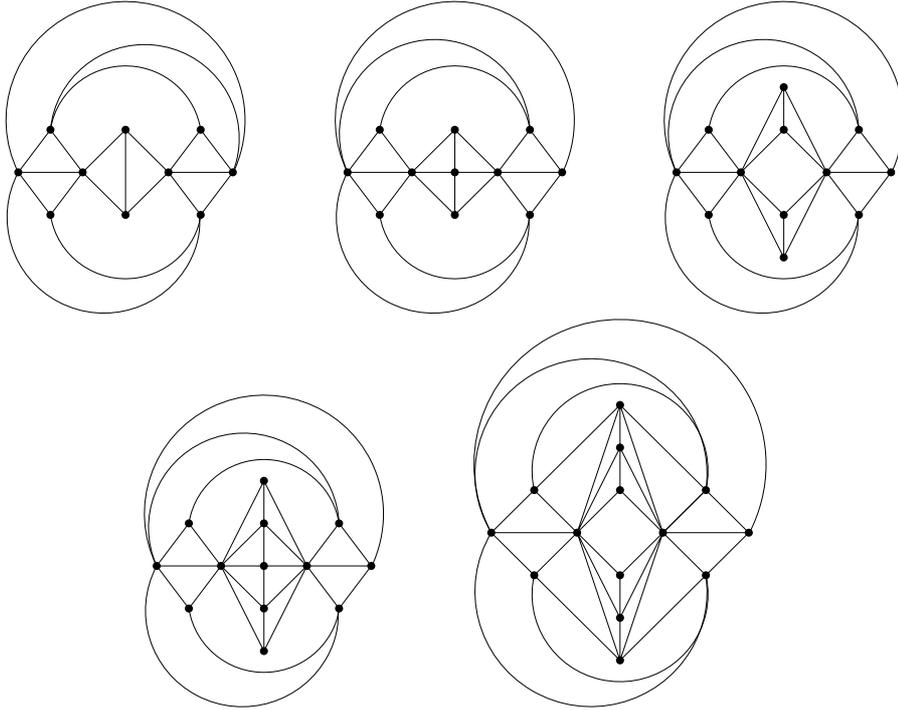Figure 3: Planar graphs with diameter 2 and maximum degree $4 \leq \Delta \leq 7$.

Figure 4: Planar graphs with diameter 2 and maximum degree $5 \leq \Delta \leq 9$.

Maximum degree $\Delta \geq 8$. Finally, in Fig. 5 we give constructions for planar graphs with diameter 2 and maximum degree $\Delta \geq 8$ which need $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$ colors for an injective coloring. The constructions of planar graphs with arbitrary maximum degrees and chromatic number of the square graph equal to their order were known (see [12, 8]). In fact, we modify those graphs by inserting the paths $P_a = a_1a_2\cdots a_k$, $P_b = b_1b_2\cdots b_k$, and $P_c = c_1c_2\cdots c_{k+1}$ (if Δ is odd, we introduce also the edge $c_{k+1}c_{k+2}$). These additional edges are providing the paths of length two between the vertices u, v, w and the vertices a_i, b_i, and c_i. Note that all the edges of the paths P_a, P_b, and P_c are not really needed to obtain a graph with the desired properties. In fact, it is enough that each vertex of these paths is incident with one edge of the path, so roughly every second is redundant. However, for a sake of having simple construction we prefer to keep all of them. The left graph depicted in Fig. 5 has maximum degree $\Delta = 2k + 2$ and the right one has $\Delta = 2k + 3$, for $k \geq 3$. It is easy to see that there is a path of length two between all vertices, thus every vertex in the graph should receive different color for proper injective
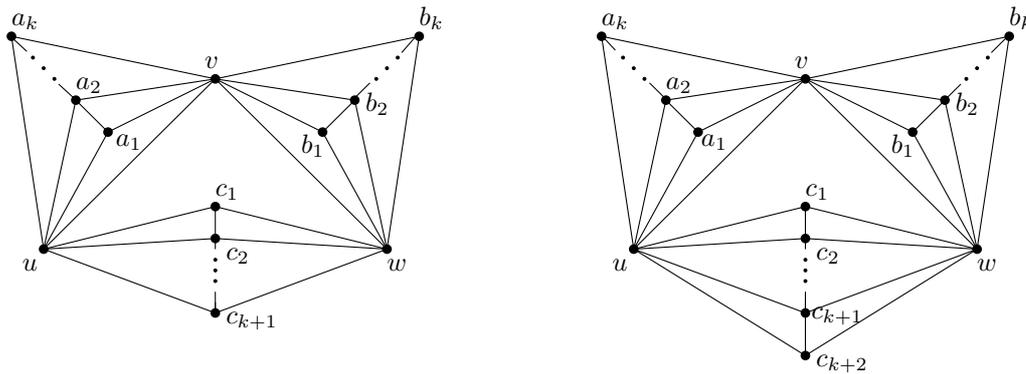
Figure 5: Constructions of diameter 2 planar graphs with maximum degree at least 8 (even on the left and odd on the right) with injective chromatic number $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$.

coloring.

Thus, following Wegner's conjecture, we pose a new variation for the maximum injective chromatic number of planar graphs:

Conjecture 2 *Let G be a planar graph with maximum degree Δ. Then,*

(a) $\chi_i(G) \leq 5$, *if $\Delta = 3$;*

(b) $\chi_i(G) \leq \Delta + 5$, *if $4 \leq \Delta \leq 7$;*

(c) $\chi_i(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, *if $\Delta \geq 8$.*

Note that the injective chromatic number is at most equal to the chromatic number of the square of a graph, therefore proving Wegner's conjecture would imply the proof of Conjecture 2.

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