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BROOKS THEOREM FOR DART GRAPHS

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Brooks Theorem for Dart Graphs

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Abstract

The well known Brooks theorem says that each graph G of maximum degree $k \geq 3$ is k-colorable unless $G = K_{k+1}$. We generalize this theorem by allowing higher degree vertices with prescribed types of neighborhood.

1 Introduction

A *k-coloring* of a graph is a mapping from the set of vertices to $\{1, \dots, k\}$ such that any two adjacent vertices have different colors. The decision problem whether a given graph G has a k-coloring is a classical NP-complete problem for every fixed $k \geq 3$ (see [3, 4]).

By Brooks' Theorem [1], every graph with maximum vertex degree at most $k \geq 3$ and without a component isomorphic to K_{k+1} (a complete graph on $k + 1$ vertices) has a k-coloring. Furthermore, as follows from [2, 6, 7, 8, 9], there exists a linear time algorithm that finds a k-coloring for such a graph.

Kochol, Lozin, and Randerath [6, Theorem 4.3] proved that if $\mathcal{D}$ is a class of graphs in which the neighborhood of each 4-degree vertex induces a graph isomorphic to a disjoint union of an isolated vertex and a path of length 2, then every graph from $\mathcal{D}$ is either 3-colorable or has a component isomorphic to K_4. Furthermore, there exists a linear time algorithm that finds either a 3-coloring or a component isomorphic to K_4 for each graph from $\mathcal{D}$. This generalizes the Brooks theorem for the case $k = 3$.

The aim of this paper is to generalize the Brooks theorem and the result from [6, Theorem 4.3]. We consider classes of graphs where each vertex of degree at least $k+2$ has

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a strictly prescribed neighborhood, so called "(k,s)-dart graphs", defined in the following section. Our main result, Theorem 1, is that if G is a (k,s)-dart graph, $k \geq \max\{3,s\}$, and $s \geq 2$, then G is $(k+1)$-colorable if and only if it has no component isomorphic to K_{k+2}. Furthermore, if G is $(k+1)$-colorable, then a $(k+1)$-coloring of G can be constructed in a linear time. We also show that if $s > k \geq 3$, then it is an NP-complete problem to decide whether a (k,s)-dart graph is $(k+1)$-colorable (see Theorem 2).

2 Definitions

In this paper we consider simple graphs, i.e., without multiple edges and loops. If G is a graph, then $V(G)$ and $E(G)$ denote the vertex and the edge sets of G, respectively.

Let G be a graph and x,y two vertices of G. Then $G+xy$ denotes the graph constructed from G by adding an edge xy. Since we consider simple graphs, $G+xy = G$ if x,y are adjacent in G. For a vertex v of G, let $d_G(v)$ denote the degree of v in G. Let H,G be two graphs such that H is not a subgraph of G. Then we use to say that G is a H*-free* graph.

A (k,s)-*diamond* is a join of a clique of size $k \geq 1$ and an independent set of size $s \geq 1$. These graphs are also known as split graphs. In a (k,s)-diamond D, vertices that belong to the independent set are called *pick* vertices, and the remaining (i.e. those in the k-clique) are called *central* vertices. Denote by $C(D)$ and $P(D)$ the sets of central vertices and pick vertices of D, respectively. An example of a $(4,2)$-diamond D with $C(D) = \{c_1,\ldots,c_4\}$ and $P(D) = \{p_1,p_2\}$ is in Figure 1.

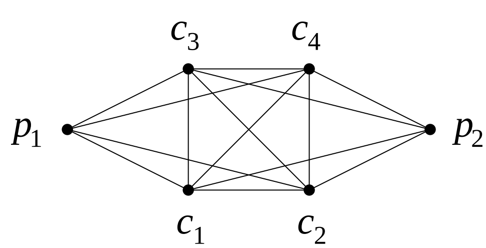

Figure 1: A $(4,2)$-diamond.

Note that a $(k,1)$-diamond is isomorphic to K_{k+1}; in this case the unique pick vertex does not distinguish from the central vertices but in such a situation this is irrelevant for us.

Definition 1 *A graph G is a (k,s)*-dart *if each vertex of G of degree $\geq k+2$ is a central vertex of some (k,i)-diamond D as an induced subgraph of G with $i \leq s$, for which*

(a) $d_D(x) \geq d_G(x) - 1$ *for each* $x \in V(D)$;

(b) *no two vertices of $C(D)$ have a common neighbor in $G - D$.*

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Every graph of maximum degree $\leq k+1$ is a $(k,1)$-dart graph since in the above definition, we only prescribe the structure on the neighborhood of vertices of higher degree. Also, every (k,s_1)-dart is a (k,s_2)-dart if $s_1 \leq s_2$.

Note that the assumption that x is of degree $\geq k+2$ implies that $i \geq 2$. In a (k,s)-dart graph G, every vertex of degree at least $k+2$ belongs to an induced (k,i)-diamond with $2 \leq i \leq s$. Denote by $\mathcal{D}(G)$ the set of all induced maximal (k,i)-diamonds of G with $i \geq 2$. Observe that we do not require that a diamond of $\mathcal{D}(G)$ must contain a vertex of degree $k+2$ or more, just to satisfy conditions (a) and (b) of Definition 1.

We say that a vertex of a dart G is *central* if it is a central vertex of a diamond of $\mathcal{D}(G)$. Similarly define a *pick* vertex of G. Denote the sets of central vertices and pick vertices by $C(G)$ and $P(G)$, respectively.

Let G be a (k,s)-dart and $D \in \mathcal{D}(G)$. Then, each central vertex $x \in C(D)$ is adjacent to at most one vertex v' from $G-D$. In this case, v' is called *isolated* neighbor of v. The set of all isolated neighbors of the central vertices of D is denoted by $I(D)$. Possibility $I(D)=\emptyset$ is not excluded.

We remark that the following observations for a (k,s)-dart G hold:

(1) A central vertex v of a (k,s)-dart G is not necessarily of degree at least $k+2$. This happens only if v is a central vertex of a $(k,2)$-diamond $D \in \mathcal{D}(G)$ and it has no neighbor in $G-D$. Then, v is of degree $k+1$.

(2) If K_{k+2} is a subgraph of a (k,s)-dart G, then it must be a component of G. Thus a copy of K_{k+2} in G is disjoint from diamonds of $\mathcal{D}(G)$.

(3) No two pick vertices of the same diamond from $\mathcal{D}(G)$ are adjacent.

3 Properties of dart graphs

The following lemma is an easy observation.

Lemma 1 *Let G be a (k,s)-dart graph and $D \in \mathcal{D}(G)$. Let λ be a proper $(k+1)$-coloring of $G-C(D)$ such that all pick vertices $P(D)$ are assigned the same color a. Then, λ can be extended to G unless every central vertex of D has an isolated neighbor and λ assigns the same color $c \neq a$ to all vertices of $I(D)$.*

Proof: Let $L(v) \subset \{1,\ldots,k+1\}$ be the set of available colors for a central vertex $v \in V(G)$ regarding λ. Notice that $k \geq |L(v)| \geq k-1$. And, $|L(v)|=k-1$ if and only if v has an isolated neighbor v' and $\lambda(v') \neq a$. Thus, each central vertex of D has an isolated neighbor and all vertices of $I(D)$ are assigned a same color $c \neq a$, if and only if the unions of all $L(v)$'s is of size $k-1$. Now the proof follows by Hall's theorem.

■

Next lemma assures that diamonds in a dart graph are vertex disjoint:

Lemma 2 *Let G be a (k, s)-dart graph with $k \geq 3$. Then*

(*a*) *$V(D_1) \cap V(D_2) = \emptyset$, for every two distinct diamonds $D_1, D_2 \in \mathcal{D}(G)$.*

(*b*) *$C(G) \cap P(G) = \emptyset$; in particular each pick vertex is of degree k or $k + 1$.*

Proof: We prove (a). Suppose that v is a vertex of two distinct diamonds $D_1, D_2 \in \mathcal{D}(G)$.

Assume that $v \in C(D_1) \cap C(D_2)$. If $C(D_1) = C(D_2)$, then by Definition 1(b) we obtain that $P(D_1) = P(D_2)$, whence $D_1 = D_2$. Thus $C(D_1) \neq C(D_2)$.

Suppose first $|C(D_1) \cap C(D_2)| = 1$, i.e., $C(D_1) \cap C(D_2) = \{v\}$. Then by Definition 1, either $k-2$ or $k-1$ vertices of $C(D_2)$ (resp. $C(D_1)$) are pick vertices of D_1 (resp. D_2). But then for $k \geq 4$, we obtain also two adjacent pick vertices of D_1 (resp. D_2), a contradiction to (3). So we may assume that $k = 3$, $C(D_1) = \{u_1, w_1, v\}$, $C(D_2) = \{u_2, w_2, v\}$, and u_1 (resp. u_2) are pick vertices of D_2 (resp. D_1). By (3), w_1 (resp. w_2) is not a pick vertex of D_2 (resp. D_1). Then $w_1 \in I(D_2)$ (resp. $w_2 \in I(D_1)$) is a common neighbor of $v, u_2 \in C(D_2)$ (resp. $v, u_1 \in C(D_1)$), a contradiction with Definition 1(b).

Suppose now $|C(D_1) \cap C(D_2)| \geq 2$. Then each vertex $u \in C(D_1) \setminus C(D_2)$ is a neighbor of at least two vertices from $C(D_2)$, whence by Definition 1(b), $u \in P(D_2)$ and thus $C(D_1) \setminus C(D_2) \subseteq P(D_2)$. Similarly $C(D_2) \setminus C(D_1) \subseteq P(D_1)$. Thus the subgraph of G induced by $C(D_1) \cup C(D_2)$ is a clique, whence $|C(D_1) \cup C(D_2)| = k + 1$, and so $|C(D_1) \cap C(D_2)| = k - 1$. By assumptions, D_1 is a (k, s_1)-diamond, $s \geq s_1 \geq 2$. Thus there exists $x_1 \in P(D_1) \setminus C(D_2)$. By (3), we infer that $x_1 \in I(D_2)$ but then it is a common neighbor of at least two vertices from $C(D_2)$, a contradiction with Definition 1(b).

By the above two paragraphs, we can assume that $C(D_1) \cap C(D_2) = \emptyset$. If $v \in V(D_1) \cap P(D_2)$, then $d_{D_2}(v) + 1 < d_G(v)$, a contradiction with Definition 1(a). Similarly if $v \in V(D_2) \cap P(D_1)$. This proves claim (a). Claim (b) is an easy consequence of (a). ∎

Next lemmas assures that removing small vertices or diamonds in dart graphs we preserve the class of dart graphs.

Lemma 3 *Let G be a (k, s)-dart graph with $k \geq 3$. Then*

(*a*) *if v is a vertex of degree $\leq k$, then $G' = G - v$ is a (k, s)-dart graph,*

(*b*) *if $D \in \mathcal{D}(G)$, then $G' = G - D$ is a (k, s)-dart graph.*

Moreover, in both cases, $\mathcal{D}(G')$ can be determined from $\mathcal{D}(G)$ in a constant time.

Proof: We first show that in both cases G' is a (k, s)-dart graph. Suppose that u' is an arbitrary vertex of degree $\geq k + 2$ in G'. Then, it is also of degree $\geq k + 2$ in G, and hence it belongs to a (k, i)-diamond $D' \in \mathcal{D}(G)$ with $2 \leq i \leq s$. In case (b), diamonds D and D' are disjoint, by Lemma 2, and hence D' is an induced (k, s)-diamond in G'. Consider now case (a). If D' is an induced subgraph of G', then we are done. Otherwise, $v \in V(D')$ is a pick vertex of D'. Since u' is of degree $\geq k + 2$ in G', it follows that $i \geq 3$, and hence $D' - v$ is a $(k, i - 1)$-diamond in G'.

Regarding $\mathcal{D}(G')$ and $\mathcal{D}(G)$, in case (b), Lemma 2 assures that $\mathcal{D}(G)$ consists of D and $\mathcal{D}(G')$. In case (a), $\mathcal{D}(G)$ may change only if v is a pick vertex of some (k,s)-diamond D' of G. So, in this case, $\mathcal{D}(G)$ is either $\mathcal{D}(G')$ or $(\mathcal{D}(G') \setminus \{D' - v)) \cup \{D'\}$.

■

In the next few lemmas, we study properties of a graph G' obtained from G be applying some local changes.

Lemma 4 *Let G be a (k,s)-dart graph with $k \geq 3$ and let a_1, a_2 be two central vertices of a diamond $D \in \mathcal{D}(G)$. Suppose that x_1 and x_2 are the isolated neighbors of a_1 and a_2, respectively. Then, each $(k+1)$-coloring λ^* of $G^* := G - x_1a_1 - x_2a_2 + x_1x_2$ can be modified into a $(k+1)$-coloring of G in a constant time.*

Proof: Clearly $\lambda^*(a_1) \neq \lambda^*(a_2)$ and $\lambda^*(x_1) \neq \lambda^*(x_2)$. By Definition 1, a_1 and x_2 are non-adjacent, and similarly a_2 and x_1 are non-adjacent. Notice that λ^* is not a coloring of G if and only if $\lambda^*(a_1) = \lambda^*(x_1)$ or $\lambda^*(a_2) = \lambda^*(x_2)$. But in that case, we can simply interchange the colors of a_1 and a_2, and obtain a proper $(k+1)$-coloring of G.

■

Lemma 5 *Let G be a K_{k+2}-free (k,s)-dart graph with $k \geq 3$ and $D \in \mathcal{D}(G)$. Let a_1, a_2 be two central vertices of D and let x_1, x_2 be their isolated neighbors, respectively. Then the graph $G' = G - x_1a_1 - x_2a_2 + x_1x_2$ is a K_{k+2}-free graph unless x_1, x_2 are pick vertices of a diamond of $\mathcal{D}(G)$.*

Proof: Suppose that G' contains a copy H of K_{k+2}. Then, x_1, x_2 are vertices of H, thus cannot be adjacent in G and there is a set S of k common neighbors of x_1 and x_2 in G, which induce a clique. Notice that $|S| = k$ and $d_G(x_1), d_G(x_2) \geq k+1$.

Suppose that $d_G(x_1) \geq k+2$. Then, x_1 is a central vertex of some diamond $D' \in \mathcal{D}(G)$, whence by Definition 1(b), $S \subseteq V(D')$ and clearly, $|S \cap C(D')| \geq k-1 \geq 2$. Then x_2 has at least 2 neighbors in $C(D')$, whence x_2 belongs to D', and so it is adjacent with x_1 in G, a contradiction.

Thus, by pervious paragraph, we may assume that $d(x_1) = k+1$, and analogously $d(x_2) = k+1$. Then x_1, x_2 and S belong to a diamond of $D' \in \mathcal{D}(G)$ in which $x_1, x_2 \in P(D')$ and $S = C(D')$.

■

Lemma 6 *Let G be a (k,s)-dart graph $D \in \mathcal{D}(G)$. Let a_1, a_2 be two central vertices of D and let x_1, x_2 be their isolated neighbors, respectively. Then the graph $G' = G - x_1a_1 - a_2x_2 + x_1x_2$ is a (k,s)-dart graph unless one of the following conditions occurs:*

(a) *x_1, x_2 are pick vertices of the same diamond of $\mathcal{D}(G)$;*

(b) *there exists a diamond $D' \in \mathcal{D}(G)$ and $i \in \{1,2\}$ such that $x_i \in C(D')$ and x_{3-i} is an isolated neighbor of a central vertex from D', which is distinct from x_i.*

Proof: Suppose that G' is not a (k,s)-dart graph. First notice that each vertex preserve its degree from G except a_1, a_2, which belong to D and it is a diamond in G' as well. If there is some $D' \in \mathcal{D}(G)$ that is not induced diamond of G', then x_1 and x_2 must be pick vertices of D', which is the excluded case (a). Next observe that each diamond of $\mathcal{D}(G)$ satisfies Definition 1(a) in G'. Finally, if Definition 1(b) is not satisfied for some $D' \in \mathcal{D}(G)$ in G', then there are two central vertices u and v with a common neighbor w outside D'. Notice that x_1x_2 is one of the edges uw or vw. Then without loss of generality, we may assume that x_1 is a central vertex in D' and x_2 is an isolated neighbor of a central vertex of D' distinct from x_1.

■

Notice that in the exceptional case (a) of the above lemma, G' may still be a dart graph, when D is a $(k,2)$-diamond with no isolated verices. Then, D becomes a copy of K_{k+2} in G'.

4 An extension of Brooks theorem

For a diamond $D \in \mathcal{D}(G)$, a vertex of $I(D)$ could be a central or pick vertex of another diamond of $\mathcal{D}(G)$. Denote by $I_c(D)$ and $I_p(D)$ the subset of all such vertices of $I(D)$, respectively. By Lemma 2(b), sets $I_c(D)$ and $I_p(D)$ are disjoint. Finally, let $I_s(D)$ be the vertices of $I(D)$ that are neither in $I_c(D)$, nor in $I_p(D)$.

Lemma 7 *Let G be a K_{k+2}-free (k,s)-dart graph with given $\mathcal{D}(G) \neq \emptyset$ and $k \geq \max\{3,s\}$ and $s \geq 2$. Then, in a constant time we can construct a K_{k+2}-free (k,s)-dart graph G^* together with $\mathcal{D}(G^*)$ such that*

(a) *$|E(G^*)| < |E(G)|$;*

(b) *From any $(k+1)$-coloring λ of G^* one can construct a $(k+1)$-coloring of G in a constant time.*

Proof: In the construction of G^* we use a bounded number of vertex/edge additions and deletions. Similarly, we obtain $\mathcal{D}(G^*)$ from $\mathcal{D}(G)$ in a finite number of steps. This will preserve that constructions are completed in a constant time. In the sequel consider the following cases:

Case 1. *There exists $v \in V(G)$ of degree $\leq k$.* Then v is not a central vertex. Thus, by Lemma 3(a), $G^* := G - v$ is a (k,s)-dart graph with $|E(G^*)| < |E(G)|$. By the same lemma, one can construct $\mathcal{D}(G^*)$ from $\mathcal{D}(G)$ in a constant time. Obviously, G^* is a K_{k+2}-free graph. A coloring of G^* can be easily extended to a coloring of G by assigning to v a color that miss in its neighborhood.

Case 2. *There exists $v \in C(D)$, $D \in \mathcal{D}(G)$, having no isolated neighbor.* By Lemma 3(b), $G^* := G - D$ is a (k,s)-dart graph and $\mathcal{D}(G^*)$ can be constructed from $\mathcal{D}(G)$ in a constant time. Obviously, G^* is a K_{k+2}-free graph and $|E(G^*)| < |E(G)|$. Let

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λ^* be a $(k+1)$-coloring of G^*. Since each pick vertex of D has at most one neighbor outside D and since $|P(D)| < k+1$, it follows that there exists a color that we can assign to all pick vertices. Since v has no isolated neighbor, we can apply Lemma 1 to extend λ^* to the central vertices of D.

Case 3. *There exists $D \in \mathcal{D}(G)$, such that $I_c(D) \cup I_s(D) \neq \emptyset$, or some two vertices of $I_p(D)$ do not belong to the same $D' \in \mathcal{D}(G)$.* We can assume that Case 2 does not hold, whence $|I_c(D)| + |I_p(D)| + |I_s(D)| = k$. Let $x_1, x_2 \in I(D)$ be two distinct vertices. And, let $a_i \in C(D)$ be the neighbor of x_i for $i = 1, 2$.

Now, consider the graph $G^* = G - x_1a_1 - x_2a_2 + x_1x_2$. If none of the exceptions of Lemmas 5 or 6 holds, then G^* is a K_{k+2}-free (k, s)-dart graph, and by Lemma 4, we can modify any coloring of G^* to a proper coloring of G in a constant time. Moreover, $|E(G^*)| < |E(G)|$ and $\mathcal{D}(G)$ can be determined in a constant time from $\mathcal{D}(G^*)$.

Assume that for each pair $x_1, x_2 \in I(D)$, some of the exeptions of Lemmas 5 or 6 is satisfied. This implies immediately that $|I_c(D)| \leq 1$ and $|I_s(D)| \leq 1$.

Thus $|I_p(D)| \geq 1$ (because $k \geq 3$). Then $x_1 \in I_s(D) \cup I_c(D)$ and $x_2 \in I_p(D)$ do not satisfy exeptions of Lemmas 5 or 6, whence $I_s(D) \cup I_c(D) = \emptyset$. Thus all vertices of $I(D)$ must be pick vertices of one diamond of $\mathcal{D}(G)$. This contradicts the assumptions of Case 3.

Case 4. *None of Cases 1.–3. occurs.* Thus, by Case 3., for each $D \in \mathcal{D}(G)$, $I_c(D) \cup I_s(D) = \emptyset$, and all vertices of $I_p(D)$ belongs to a same (k, k)-diamond $D' \in \mathcal{D}(G)$. We denote D' by $\varphi(D)$. Furthermore, observe that there exists a perfect matching between $C(D)$ and $P(\varphi(D))$.

Case 4.1. *There exists $D \in \mathcal{D}(G)$, such that $\varphi^2(D) = D$.* Then vertices of D and $\varphi(D)$ induce a component G' of G. Let $G^* := G - G'$. Obviously, G^* is a (k, k)-dart graph, $|E(G^*)| < |E(G)|$ and $\mathcal{D}(G^*) = \mathcal{D}(G) \setminus \{D, \varphi(D)\}$. Moreover, we can construct a $(k+1)$-coloring of G' in a constant time: just color all vertices of $P(D)$ and $P(\varphi(D))$ by the color $k+1$, and assign colors $1, \ldots, k$ to the vertices of $C(D)$ and $C(\varphi(D))$.

Case 4.2. *For each $D \in \mathcal{D}(G)$, $\varphi^2(D) \neq D$.* By the assumptions of lemma, there exists $D \in \mathcal{D}(G) \neq \emptyset$. Let G^* be the graph, we obtain by removing the vertices of $\varphi(D)$ and inserting a perfect matching between $C(D)$ and $P(\varphi^2(D))$. Obviously G^* is a (k, k)-dart graph with less edges than G and $\mathcal{D}(G^*) = \mathcal{D}(G) \setminus \{\varphi(D)\}$. Let λ^* be a $(k+1)$-coloring of G^*. Then λ^* assigns the same color c to all vertices of $P(\varphi^2(D))$. Assign c also to all vertices of $P(\varphi(D))$ and to each of the vertices of $C(\varphi(D))$ an unique color from $\{1, \ldots, k+1\} \setminus \{c\}$. This gives a required coloring of G, completing the proof.

■

Now we are ready to prove the main result.

Theorem 1 *Let G be a (k, s)-dart graph with $k \geq \max\{3, s\}$ and $s \geq 2$. Then G is $(k+1)$-colorable if and only if it has no component isomorphic to K_{k+2}. Furthermore, if G is $(k+1)$-colorable, then a $(k+1)$-coloring of G can be constructed in a linear time.*

Proof: The necessity of the first part of the theorem is trivial. To see the sufficiency, observe that a (k, s)-dart graph is K_{k+2}-free if and only if it has no component isomorphic to K_{k+2}. The same is true if G is a graph with vertex degree at most $k + 1$. Therefore, the sufficiency follows from Lemma 7 and Brooks' Theorem [1].

We can check whether a dart graph G is K_{k+2}-free in linear time. Analogously, we can find the set $\mathcal{D}(G)$ in linear time. Consequently, by means of Lemma 7 we can create in linear time a K_{k+2}-free graph G' without vertices of degree more than $k + 1$ such that any $(k + 1)$-coloring of G' can be transformed into a $(k + 1)$-coloring of G in linear time. By [7] (see also [9, 6]), a $(k + 1)$-coloring of G' can be found in linear time, which proves the statement.

■

5 NP-Completeness

In this section we show that Theorem 1 cannot be extended for (k, s)-dart graphs where $s > k \geq 2$ unless P = NP.

We need some more notation. Take n vertex disjoint copies of $(k, k + 1)$-diamonds $D_1, \ldots, D_n$, $k, n \geq 2$. For $i = 1, \ldots, n$, denote by $v_{i,1}, \ldots, v_{i,k}$ and $u_{i,1}, \ldots, u_{i,k+1}$ the central and pick vertices of D_i, respectively. Add nk new edges $v_{i,j}u_{i+1,j}$, $i = 1 \ldots, n$, $j = 1, \ldots, k$ (considering the sum $i + 1$ mod n). Then the resulting graph is called a $(n, k + 1)$-*bracelet* and vertices $u_{1,k+1}, \ldots, u_{n,k+1}$ are called its *connectors*. An example of a $(4, 3)$-bracelet with connectors $u_{1,3}, \ldots, u_{4,3}$ is in Figure 2.

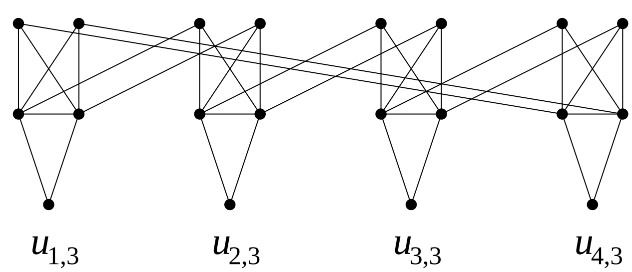

Figure 2: A $(3, 4)$-bracelet.

We study complexity of the following problem.

DART-(k, s)-$(k + 1)$-COL
Instance: A (k, s)-dart graph G.
Question: Is G $k + 1$-colorable?

Theorem 2 *The problem* DART-(k, s)-$(k + 1)$-COL, $k \geq 3$, *is*

(a) NP-*complete for* $s > k$,

(b) *solvable in linear time for* $s \leq k$.

Proof: Claim (b) holds true by Theorem 1. We prove (a). Let G be a graph. Replace each vertex v of G of degree ≥ 2 by a $(d_G(v), k+1)$-bracelet H_v. Let H_v be an isolated vertex if $d_G(v) = 1$. Each edge uv of G replace by an edge joining a connector of H_v with a connector of H_u so that each connector is attached to at most one new edge. Denote the resulting graph by G'. Clearly, G' is a $(k, k+1)$-dart graph. By any $(k+1)$-coloring of H_v, $v \in V(G)$, all connectors of H_v must be colored by the same color. Hence G' is $(k+1)$-colorable if and only if G is so. Thus the problem from item (a) can be polynomially reduced to the problem of $(k+1)$-coloring. This problem is NP-complete for every fixed $k \geq 2$ by Garey and Johnson [3, GT4].

■

We have proved item (a) also for $k = 2$. Let us note that item (b) for this case is a consequence of [6, Theorem 4.3].

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