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DIOPHANTINE STEINER TRIPLES

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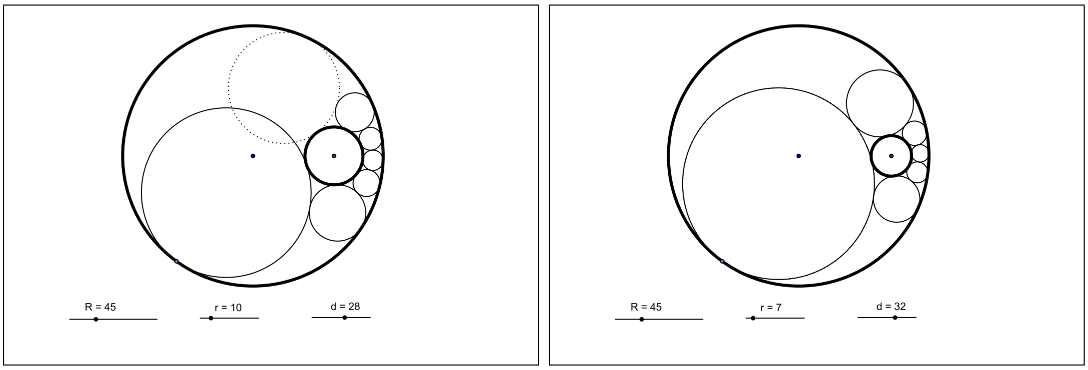
Computer programs for dynamic geometry provide a very effective tool for motivating students. It is my experience that a certain effort in preparing an adequate applet and spending some time on a careful presentation can result in a great change of the atmosphere in the classroom. On the other hand, these programs also open new perspectives in geometry exploration and can provoke new interesting research questions related to the already known topics. The possibility of experimenting and the chance of visual examination of the results adds new dimensions to research and can also be appealing to non-mathematical audiences.

A good example, which can be presented in geometry class as an application of inversion is an applet related to the Steiner Porism. The picture below shows the main idea of the applet.



With three sliders we can dynamically vary the integer values of the radius R of the large circle K, the radius r of the small circle k (the two circles shown bold), and the distance d between the centres. After choosing a point T on K, the applet starts creating the Steiner chain of circles $L_1, L_2, \ldots$ inscribed into the region between K and k. The first circle L_1 touches K at T and also touches k. Next circles $L_2, L_3 \ldots$ touch one another successively and they all touch the two circles K and k. If the n-th circle L_n also happens to touch the first circle L_1, we say that $L_1, \ldots, L_n$ is a *closed Steiner chain* of length n (cf. [3, 6, 7, 10]). The picture on the right shows a closed Steiner chain of lenght 6, while that on the left shows a Stainer chain, that is not closed. Note that a Steiner chain may also close after several loops around the central circle, but we we won't consider this situation here; for us, a closed chain means that it closes in the first round.

The applet provides an experimental method for finding the integer values of R, r and d for which Steiner chain closes.

In this paper, we find all integer triples (R, r, d) giving rise (as above) to closed Steiner chains.

1. Steiner identity and Steiner triples

The famous result, called *Steiner Porism*, states that if the chain is closed for one starting point T, then it is closed for all starting points on K. It is also well known that the existence of a closed Steiner chain of the length $n \geq 3$, is equivalent to the fact that parameters R, r, d and n are connected by the so called Steiner identity (cf. [7]):

$$(R-r)^2 - d^2 = 4Rr\,\tan^2(\tfrac{\pi}{n}). \qquad (1)$$

Let us start our investigation with clarifying some notions. *A triple* (R, r, d) *of real numbers is called an* ***acceptable triple****, if* $0 < r < R$; $d \geq 0$, *and* $R > r + d$.

Acceptable triples therefore define two circles with different radii such that the larger circle contains the smaller one. For every acceptable triple we can then pick a point T on K and start constructing the Steiner chain.

An acceptable triple (R, r, d) *of real numbers is called a* ***Steiner triple****, if the Steiner chain in the configuration described above closes. When a Steiner triple* (R, r, d) *is integer valued, we call it* ***Diophantine Steiner triple****, abbreviated* ***DS triple****. A DS triple is* ***primitive*** *if the greatest common divisor of the members is equal to 1:* $D(R, r, d) = 1$.

As is done in similar circumstances, we will restrict our search of DS triples to primitive DS triples. We will start with the investigation of the possible lengths of the associated Steiner chains.

2. The length n of the Steiner chain

Theorem 1 *Let* (R, r, d) *be a DS triple with the Steiner chain of length* n. *The only possible values for* n *are* $n = 3, n = 4$ *or* $n = 6$.

Proof: It follows from (1) that if (R, r, d) is a DS triple with the length of the Steiner chain equal to n, then $y_n = \tan^2(\frac{\pi}{n})$ is a rational number. We will see bellow that y_n is a root of a certain polynomial p_n with integer coefficients. We will get the result investigating the rational roots of p_n.

First, recall that $e^{inx} = (e^{ix})^n$ gives us the formulas for $\sin nx$, $\cos nx$, from where we can obtain the formula for $\tan nx$:

$$\tan nx = \frac{\sin nx}{\cos nx} = \tan x \cdot \frac{\binom{n}{1} - \binom{n}{3}\tan^2 x + \dots}{1 - \binom{n}{2}\tan^2 x + \dots}$$

Considering this equality for $x = \frac{\pi}{n}$, we see that $y_n = \tan^2(\frac{\pi}{n})$ is a root of a polynomial

$$p_n(y) = \binom{n}{1} - \binom{n}{3}y + \binom{n}{5}y^2 - \binom{n}{7}y^3 + \dots.$$

Let us now investigate two different cases: when n is odd and when n is even. We have:

$$\begin{aligned} p_{2k+1}(y) &= (2k+1) - \binom{2k+1}{3}y + \dots + (-1)^k y^k \\ p_{2k}(y) &= (2k) - \binom{2k}{3}y + \dots + (-1)^{k-1}\, 2k\, y^{k-1} \end{aligned}$$

We shall first consider the odd case. For odd $n \geq 5$ we have $0 < y_n = \tan^2(\frac{\pi}{n}) < \tan^2(\frac{\pi}{4}) = 1$, and therefore y_n is not an integer. But y_n is a root of the polynomial $p_n = p_{2k+1} = (-1)^k y^k + \ldots + n$ with integer coefficients. The only possible rational roots of p_n are the divisors of n and therefore integers. Hence, $y_n = \tan^2(\frac{\pi}{n})$ is not rational and therefore for odd values $n, n \geq 5$, there is no DS triples with corresponding Steiner chain of length n.

Let us now consider the even case and prove a similar conclusion for even $n \geq 8$. To prove this we will need the following two Lemmas.

Lemma 1 *For $k \geq 4$ we have:* $\tan^2(\frac{\pi}{2k}) < \frac{1}{k}$.

Proof: Since the function $f(x) = \tan^2(x)$ is convex on the interval $[0, \frac{\pi}{8}]$, its graph on this interval lies under the segment with the vertices $(0, f(0))$ and $(\frac{\pi}{8}, f(\frac{\pi}{8}))$. Using the fact $f(\frac{\pi}{8}) = 3 - 2\sqrt{2}$, this observation can be written as $f(x) \leq \frac{8(3-2\sqrt{2})}{\pi} x$. Applying this inequality for $x = \frac{\pi}{2k};\ k \geq 4$, we obtain

$$\tan^2(\tfrac{\pi}{2k}) \leq \tfrac{4(3-2\sqrt{2})}{k} < \tfrac{1}{k}.$$

□

Lemma 2 *For every even number $n = 2k$ and for all odd numbers $m, 1 \leq m \leq 2k-1$ the binomial coefficient $\binom{2k}{m}$ is even.*

Proof: Since $(1+x)^{2k} = (1+2x+x^2)^k$, we have $(1+x)^{2k} \equiv (1+x^2)^k (\mathrm{mod}\ 2)$. Therefore the coefficients by the odd powers of x on the left hand side are even. □

Using these Lemmas we will prove that, for $k \geq 4$, $y_{2k} = \tan^2(\frac{\pi}{2k})$ is not a rational number. We know that y_{2k} is a root of the polynomial

$$p_{2k} = (-1)^{k-1}\, 2\, k y^{k-1} + \ldots - \binom{2k}{3} y + 2k.$$

Since all the coefficients are even (Lemma 2), we can divide by 2 and see that y_{2k} is a root of a polynomial with integer coefficients of the type $(-1)^{k-1}\, k\, y^{k-1} + \ldots + k$. The positive rational roots of such polynomials are of the form $\frac{p}{q}$, where p and q are divisors of k. The smallest such positive rational number is $\frac{1}{k}$. Lemma 1 states that $0 < y_{2k} < \frac{1}{k}$, and therefore y_{2k} is not among the rational candidates for the roots. This proves that for $k \geq 4$ the number $y_{2k} = \tan^2(\frac{\pi}{2k})$ is not rational. Therefore there is no DS triples with corresponding Steiner chain of length n for even $n \geq 8$. □

3. Diophantine equations and solutions

According to Theorem 1 we only have DS triples with Steiner chains of lenghts $n = 3, n = 4$, and $n = 6$. Therefore, the relation (1) simplifies into one of the following three equations:

$$R^2 - 14Rr + r^2 - d^2 = 0 \qquad n = 3 \tag{2}$$
$$R^2 - 6Rr + r^2 - d^2 = 0 \qquad n = 4 \tag{3}$$
$$3R^2 - 10Rr + 3r^2 - 3d^2 = 0 \qquad n = 6 \tag{4}$$

Finding all DS triples means finding general solutions of these three quadratic Diophantine equations. The equations are of the same type as the equations that recently appeared in *Math. Gazette*, namely in papers [2, 4, 8, 9]; and our method of solving these equations is quite similar. Furthermore, it turns out that Diophantine Steiner triples are in fact closely connected to the triples which appear in the above-mentioned papers. These new findings will be presented in another, more geometrically orientated paper.

The table below presents all the primitive solutions (depending on parameters s and t) of the equations above and therefore yields a way to get a list of all primitive DS triples.

In all cases: $s, t \in \mathbb{N}$ and $D(s,t) = 1$. Some additional limitations regarding s and t are specified for each case separately.

		R	r	d	
$n = 3$	(a)	$3t^2 + 4s^2 + 7st$	st	$\lvert 3t^2 - 4s^2 \rvert$	t is odd, s is not divisible by 3
	(b)	$12t^2 + s^2 + 7st$	st	$\lvert 12t^2 - s^2 \rvert$	s is odd, s is not divisible by 3
	(c)	$6t^2 + 2s^2 + 7st$	st	$\lvert 6t^2 - 2s^2 \rvert$	t, s are odd, s is not divisible by 3
$n = 4$		$t^2 + 2s^2 + 3st$	st	$\lvert t^2 - 2s^2 \rvert$	t is odd
$n = 6$	(a1)	$\frac{1}{3}(4s^2 + t^2 + 5st)$	st	$\frac{1}{3} \lvert 4s^2 - t^2 \rvert$	t is odd, one of s, t is congruent 1(mod 3) and the other congruent 2(mod 3)
	(a2)	$4s^2 + t^2 + 5st$	$3st$	$\lvert 4s^2 - t^2 \rvert$	t is odd, either $s \equiv t \not\equiv 0 \pmod 3$ or one of $s, t \equiv 0 \pmod 3$ and $s \not\equiv t \pmod 3$
	(b1)	$\frac{1}{3}(2t^2 + 2s^2 + 5st)$	st	$\frac{2}{3} \lvert t^2 - s^2 \rvert$	t, s are odd, $s \equiv t \not\equiv 0 \pmod 3$
	(b2)	$2t^2 + 2s^2 + 5st$	$3st$	$2 \lvert t^2 - s^2 \rvert$	t, s are odd, $s \not\equiv t \pmod 3$

In the proofs of the results above, the following Lemma will be frequently used:

Lemma 3 *Let p be a prime number and x, y, u positive integers, satisfying the relation:*

$$xy = pu^2.$$

If $D(x, y) = 1$, then one of the factors x, y is of the form ps^2 and the other one is of the form t^2; $s, t \in \mathbb{N}$.

Proof: Since the prime number p divides xy and $D(x, y) = 1$, it must divide one of the factors, for instance $x = px_1$. Therefore we have $yx_1 = u^2$. Write u as the product of prime numbers and the result follows immediately, since $D(x_1, y) = 1$. □

Going back to prooving that the above table presents the primitive solutions to Diophantine Equations (2),(3) and (4), we have to consider the three cases separately. Since the techniques of these three considerations are quite similar, we will present here just the proofs of the cases $n = 3$ and $n = 6$. The easiest case $n = 4$ is omitted.

Proof of the case $n = 3$**:** A straightforward check shows us that R, r, d of the cases (a), (b), (c) above satisfy equation (2). Let us prove that the triple (R, r, d) of the case (a) is primitive. Suppose p is a prime, dividing R, r and d. Note that t and d are odd, so $p \neq 2$. Note also that s and d are not divisible by 3, so $p \neq 3$. Since p divides all three elements of the set $\{R + d, R - d, r\} = \{6t^2 + 7st, 8s^2 + 7st, st\}$, we have $p|6t^2$ and $p|8s^2$. Since $p \neq 2$ and $p \neq 3$, p divides both t and s, which contradicts $D(s, t) = 1$. This contradiction proves that (R, r, d) is primitive. In cases (b) and (c) the proof that (R, r, d) is primitive is similar.

The presented triples are therefore primitive and satisfy equation (2). Let us prove that all the primitive solutions of (2) are listed above.

First we write (2) in the form

$$(R - 7r + d)(R - 7r - d) = 48r^2 \tag{5}$$

and prove that $D((R - 7r + d), (R - 7r - d))$ is equal either to 2 or to 4. Let $p \notin \{2, 3\}$ be a prime, dividing both numbers. Then p divides the sum and the difference of those numbers and therefore p divides d and $R - 7r$. It follows from (5) that p divides r as well, and therefore also R, contradicting $D(R, r, d) = 1$. In the case of $p = 3$ the proof is similar. Therefore we have:

$$D((R - 7r + d), (R - 7r - d)) = 2^m.$$

Let us check that the only possible values of m are 1 and 2. The possibility $m \geq 3$ would, together with (5) imply that r, d, $R - 7d$ and R are even, again contradicting the fact that the triple is primitive. In the case $m = 0$, one of the factors in the left hand side of equation (5) would be divisible by 16 and the other one would be odd. Thus, the difference is odd. While substracting the two numbers one would get $2d$, which is a contradiction. So, the only possibilities left are $m = 1, 2$.

In case $m = 2$ we have $(R - 7r + d) = 4x_1$, $(R - 7r - d) = 4x_2$ for some $x_1, y_1 \in \mathbb{Z}$ and $D(x_1, y_1) = 1$. Relation (5) can now be rewritten as $x_1 y_1 = 3r^2$. It follows from Lemma 3 that one of the factors is equal to $3t^2$ and the other to s^2. Note that $D(s, t) = 1$ and $r = st$. We get two possibilities, both of them leading to the solution (c).

In case $m = 1$, one of the numbers $(R - 7r + d), (R - 7r - d)$ is divisible by 2 and the other (because of (5)) divisible by 8. We have two possibilities: $(R - 7r + d) = 8x_1$ and $(R - 7r - d) = 2y_1$ or $(R - 7r + d) = 2x_1$ and $(R - 7r - d) = 8y_1$. In both cases we have $D(x_1, y_1) = 1$ and the relation (5) can be rewritten as $x_1 y_1 = 3r^2$. Applying Lemma 3 again and examining all four possibilities, we conclude that they all lead to solutions (a) and (b).

Remarks in the table about t or s being odd or not divisible by 3 are necessary to ensure that the triple is indeed primitive. $\square$

Note that all three relations of the case $n = 3$ could be covered by one single relation

$$R = k(6t^2 + 2s^2 + 7st), \qquad r = kst, \qquad d = k|6t^2 - 2s^2|, \tag{6}$$

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where $k = 1$ if both s, t are odd, and $k = \frac{1}{2}$ otherwise. In the case when s is even, replace s with $2s$ in (6) to obtain case (a). Similarly we obtain case (b) when t is even, and case (c) when both s, t are odd.

Proof of the case n=6: The relation (4), namely: $10Rr = 3(R^2 + r^2 - d^2)$ implies that one of two radii R and r is divisible by 3.

First consider the case $R = 3R_1$. The relation (4) can be rewritten in the form:

$$(9R_1 - 5r + 3d)(9R_1 - 5r - 3d) = 16r^2. \tag{7}$$

Again we consider $D(9R_1 - 5r + 3d, 9R_1 - 5r - 3d)$. With arguments similar to those in the proof of the case $n = 3$ we see that

$$D(9R_1 - 5r + 3d, 9R_1 - 5r - 3d) = 2^m$$

where $m = 1$ or $m = 2$.

The case $m = 2$ gives $9R_1 - 5r + 3d = 4x_1,\ 9R_1 - 5r - 3d = 4y_1,\ D(x_1, y_1) = 1$ and $x_1y_1 = r^2$, which together with Lemma 3 leads to solution (b1).

In the case $m = 1$ one of the factors on the left hand side of (7) is of the form $2x_1$, and the other of the form $8y_1$ where $D(x_1, y_1) = 1$. The relation (7) becomes $x_1y_1 = r^2$ with the solution $x_1 = t^2,\ y_1 = s^2,\ D(s, t) = 1$. This leads to the solution (a1).

If one of the numbers s, t were divisible by 3, since R is integer valued it would follow that the other number would also be divisible by 3, contradicting $D(s, t) = 1$. The requirement about one or two parameters being odd is there to ensure that not all the members of the triple are even. The requirement about congruences (mod 3) ensure that R is integer valued. Since s and t are not divisible by 3, $t^2 \equiv s^2 \equiv 1 (\text{mod } 3)$, so d is certainly integer valued. Now consider R. In solution (a1) we have $4s^2 + t^2 + 5st \equiv 5 + 5st (\text{mod } 3)$. The condition $s \not\equiv t (\text{mod } 3)$ ensures that $5 + 5st \equiv 0 (\text{mod } 3)$, implying that R is integer valued. We consider solution (b1) in a similar way.

To conclude the proof, we have to consider the case, when instead of R, the radius r is divisible by 3, thus $r = 3r_1$. In this case, instead of (7) we have the relation:

$$(R - 5r_1 + d)(R - 5r_1 - d) = 16r_1^2. \tag{8}$$

Using standard techniques we prove that $D((R - 5r_1 + d), (R - 5r_1 - d)) = 2^m$ for $m = 1$ or $m = 2$. The case $m = 2$ leads us to the solution (b2), while $m = 1$ leads to the solution (a2). □

Note again that the solutions (a1), (a2), (b1), and (b2) could be presented by a single relation:

$$R = k(2t^2 + 2s^2 + 5st), \qquad r = 3kst, \qquad d = 2k|t^2 - s^2|,$$

where $k \in \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1\}$. When one of t, s is even, put $2t$ or $2s$ instead of t or s in the formula and select k to factor out 2.

The next table brings us a list of DS triples (R, r, d) for the values $R < 100$.

$n = 3$	$(14, 1, 1)$	$(15, 1, 4)$	$(20, 1, 11)$	$(33, 2, 13)$	$(52, 3, 23)$	$(63, 2, 47)$	$(72, 5, 13)$
	$(77, 3, 52)$	$(85, 6, 11)$	$(91, 5, 44)$	$(95, 4, 61)$			
$n = 4$	$(6, 1, 1)$	$(15, 2, 7)$	$(20, 3, 7)$	$(28, 3, 17)$	$(35, 6, 1)$	$(42, 5, 23)$	$(45, 4, 31)$
	$(63, 10, 17)$	$(66, 5, 49)$	$(72, 7, 47)$	$(77, 12, 23)$	$(88, 15, 7)$	$(91, 6, 71)$	$(99, 14, 41)$
$n = 6$	$(3, 1, 0)$	$(9, 2, 5)$	$(10, 3, 3)$	$(18, 5, 7)$	$(28, 9, 5)$	$(35, 9, 16)$	$(42, 5, 33)$
	$(45, 7, 32)$	$(45, 14, 11)$	$(52, 9, 35)$	$(55, 18, 7)$	$(60, 11, 39)$	$(63, 20, 13)$	$(77, 15, 48)$
	$(85, 12, 63)$	$(88, 21, 45)$	$(91, 30, 9)$	$(99, 8, 85)$			

4. Some ideas for possible further investigations

- A look at the list of DS triples shows that there exist values R allowing triples in all three categories: $n = 3$, $n = 4$ and $n = 6$. Such values are, for instance

$$R = 63, 77, 91, 130, 143, 187, 195, 209, 221, \ldots$$

  A closer inspection of this sequence could be interesting.

- Similarly, we could ask whether there exist different primitive DS triples inside the same category sharing the same first term R. The answer is positive. For instance, such triples for category $n = 4$ are $(2145, 32, 2047)$ and $(2145, 368, 17)$. Similarly, $(5005, 348, 887)$ and $(5005, 323, 1588)$ belong to category $n = 3$ and $(5005, 1548, 1273)$ and $(5005, 732, 3657)$ belong to category $n = 6$. The description of all such triples could also be of interest.

- We noticed that the mapping

$$(R, r, d) \mapsto \Big(\frac{5R + r + d}{2}, \frac{R + 5r + d}{2}, R + r - d\Big)$$

  maps primitive DS triples of the case $n = 3$ into DS triples of the case $n = 6$. Direct calculation shows that if the left-hand triple satisfies identity (2), then the right-hand triple satisfies (4). Some deeper reasons for this correspondence will be explained in our geometry paper mentioned above.

- We could, in a similar way, investigate Diophantine Poncelet triples, based on Poncelet Porism ([1, 5, 11]) instead of on Steiner Porism. In Poncelet's case $n = 3$, the relation (2) is replaced by the Euler triangle formula

$$R^2 - 2Rr - d^2 = 0,$$

  which has integer solutions

$$R = k(s + t)^2, \quad r = 2kst, \quad d = k \cdot |s^2 - t^2|,$$

  where $t, s \in \mathbb{N}$, $D(s, t) = 1$ and $k = \frac{1}{2}$ when both s and t are odd, and $k = 1$ otherwise.

  For $n \geq 4$ it seems that essentially different methods would be needed.

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