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TOWARDS THE COMPLETE  
CLASSIFICATION OF TENT MAPS  
INVERSE LIMITS

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# Towards the complete classification of tent maps inverse limits

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## Abstract

We study tent map inverse limits, i.e. inverse limits of inverse sequences of unit segments  $I$  with a tent map being the only bonding function. As the main result we identify an infinite family of curves in  $I^2$  such that if top points of graphs of tent maps belong to the same curve, the corresponding inverse limits are homeomorphic, and if they belong to different curves, the inverse limits are non-homeomorphic. The inverse limits corresponding to certain families of top points are explicitly determined, and certain properties of the inverse limit are proved in the case of  $(0, 1)$  as the top point.

## 1 Introduction

Continua as inverse limits have been studied for a long time. One reason for such intense research in this area is the fact that inverse sequences with very simple spaces and simple bonding maps can give extremely complicated continua as their inverse limits. The inverse limits may be both complicated and useful even in the case, when all the spaces are unit intervals  $[0, 1]$  and all the bonding functions are the same. Such inverse sequences and their inverse limits play an important role in the continuum theory as well as in the theory of topological dynamical systems. They also appear in applications in such diverse areas as economy, mechanics of fluids, physics and more; see [34, 36, 37, 38, 40, 41].

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Even such a simple case when the graph of the bonding function is the union of two segments may be highly non-trivial. Such functions are called tent maps and they are the main object of our study. In Definition 1.1 we introduce the basic notation related to them, which we use later in the paper.

**Definition 1.1.** For any  $a, b \in [0, 1]$ , the tent function  $f_{(a,b)} : [0, 1] \rightarrow [0, 1]$  is defined as the set-valued function with the graph  $\Gamma(f_{(a,b)})$  being the union of the segment (possibly degenerate) from  $(0, 0)$  to  $(a, b)$  and the segment (possibly degenerate) from  $(a, b)$  to  $(1, 0)$ . The point  $(a, b)$  is called the top point of the graph  $\Gamma(f_{(a,b)})$ . The inverse limit obtained from the inverse sequence of closed unit intervals  $[0, 1]$  and the bonding function  $f_{(a,b)}$  is denoted by

$$K_{(a,b)} = \varprojlim \{[0, 1], f_{(a,b)}\}_{n=1}^{\infty}.$$

Note that  $f_{(a,b)}$  is single-valued if and only if  $a \notin \{0, 1\}$  or  $(a, b) \in \{(0, 0), (1, 0)\}$ .

The first and the most famous example of continua  $K_{(a,b)}$  is  $K_{(\frac{1}{2}, 1)}$ , called the Brouwer-Janiszewski-Knaster continuum or sometimes just the Knaster continuum.

The whole family of continua  $K_{(\frac{1}{2}, b)}$ ,  $\frac{1}{2} < b \leq 1$ , has been called Knaster continua and the famous Ingram conjecture, stated in 1992, claimed that all of them are pairwise non-homeomorphic. It generated a large number of articles, such as Barge, Brucks, Diamond [6], Barge, Diamond [8, 9], Barge, Jacklitch, Vago [10], Barge, Martin [11, 12, 13], Block, Jakimovik, Kailhofer, Keesling [14], Block, Keesling, Raines, Štimac [15], Brucks, Bruin [16], Brucks, Diamond [17], Bruin [18, 19, 20], Collet, Eckmann [22], Good, Knight, Raines [26], Good, Raines [27], Kailhofer [32, 33], Raines [47], Raines, Štimac [48, 49], Štimac [50, 51, 52, 53], Swanson, Volkmer [54], and others, in which certain special cases of the conjecture were proved. Finally, the conjecture was proved in 2009 by M. Barge, H. Bruin and S. Štimac [7].

In spite of such great effort and many obtained results the complete classification of all inverse limits  $K_{(a,b)}$  is still an open problem. In this paper we continue the study of inverse limits of tent maps  $f_{(a,b)}$  and their classification.

Note that in some cases the tent maps are not single-valued and therefore the concept of inverse limits of inverse sequences with upper semicontinuous set-valued bonding functions is needed. Such a generalization of the concept of inverse limits was introduced in [31, 39] by W. T. Ingram and W. S.

Mahavier. They gave conditions under which the inverse limit of an inverse sequence of Hausdorff spaces with upper semicontinuous set-valued bonding functions is a Hausdorff continuum, provided some interesting examples of such inverse limits, and discussed their dimension. The concept of these generalized inverse limits has become very popular since their introduction and has been studied by many authors and many papers appeared; for examples see [1, 2, 3, 4, 5, 21, 23, 29, 30, 31, 35, 39, 45, 46, 55].

## 2 Definitions and notation

Our definitions and notation mostly follow [31] and [43].

A *continuum* is a nonempty, compact and connected metric space.

Let  $W = \{(x, \sin \frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$ . Any continuum homeomorphic to  $\text{Cl}(W)$  is called a  *$\sin \frac{1}{x}$ -continuum*.

A *harmonic fan* is any continuum, homeomorphic to the continuum, defined as the union  $\left(\bigcup_{n=1}^{\infty} K_n\right) \cup K$ , where for each  $n$ ,  $K_n$  is the segment in the plane from  $(0, 0)$  to  $(1, \frac{1}{n})$ , and  $K$  is the segment from  $(0, 0)$  to  $(1, 0)$ .

Let  $(X_n, d_n)$  be a sequence of metric spaces, where all metrics are bounded by 1. Then

$$D(x, y) = \sup_{n \in \{1, 2, 3, \dots\}} \left\{ \frac{d_n(x_n, y_n)}{n} \right\},$$

where  $x = (x_1, x_2, x_3, \dots)$ ,  $y = (y_1, y_2, y_3, \dots)$ , will be used for the metric on the product space  $\prod_{n=1}^{\infty} X_n$  (it is well known that the metric  $D$  induces the product topology [24, p. 190], [42, p. 123]).

If  $(X, d)$  is a compact metric space, then  $2^X$  denotes the set of all nonempty closed subsets of  $X$ . Let for each  $\varepsilon > 0$  and each  $A \in 2^X$

$$N_d(\varepsilon, A) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

We will always equip the set  $2^X$  with the *Hausdorff metric*  $H_d$ , which is defined as

$$H_d(H, K) = \inf\{\varepsilon > 0 \mid H \subseteq N_d(\varepsilon, K), K \subseteq N_d(\varepsilon, H)\},$$

for  $H, K \in 2^X$ . Then  $(2^X, H_d)$  is a metric space, called the *hyperspace* of the space  $(X, d)$ . For more details see [28, 43].

Let  $X$  and  $Y$  be compact metric spaces. A single-valued function  $f : X \rightarrow 2^Y$  is also called a set-valued function  $f : X \rightarrow Y$ . A set-valued

function  $f : X \rightarrow Y$  is *upper semi-continuous* (abbreviated *u.s.c.*) if for each open set  $V \subseteq Y$  the set  $\{x \in X \mid f(x) \subseteq V\}$  is an open set in  $X$ .

The *graph*  $\Gamma(f)$  of an u.s.c. set-valued function  $f : X \rightarrow Y$  is the set of all points  $(x, y) \in X \times Y$  such that  $y \in f(x)$ .

Ingram and Mahavier gave the following characterization of u.s.c. functions [31, p. 120]:

**Theorem 2.1.** Let  $X$  and  $Y$  be compact metric spaces and  $f : X \rightarrow Y$  a set-valued function. Then  $f$  is u.s.c. if and only if its graph  $\Gamma(f)$  is closed in  $X \times Y$ .

In this paper we deal with *inverse sequences*  $\{X_n, f_n\}_{n=1}^{\infty}$ , where  $X_n$  are compact metric spaces and  $f_n : X_{n+1} \rightarrow X_n$  are u.s.c. set-valued functions. We denote  $\{X_n, f_n\}_{n=1}^{\infty}$  also by

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \dots$$

The *inverse limit* of an inverse sequence  $\{X_n, f_n\}_{n=1}^{\infty}$  is defined to be the subspace of the product space  $\prod_{n=1}^{\infty} X_n$  of all  $x = (x_1, x_2, x_3, \dots) \in \prod_{n=1}^{\infty} X_n$ , such that  $x_n \in f_n(x_{n+1})$  for each  $n$ . The inverse limit is denoted by  $\varprojlim \{X_n, f_n\}_{n=1}^{\infty}$ . The notion of the inverse limit of an inverse sequence with u.s.c. bonding functions was introduced by W. S. Mahavier in [39] and W. T. Ingram and W. S. Mahavier in [31].

For any compact metric space  $X$  we use  $\dim(X)$  for the topological (covering) dimension of  $X$  (for the definition see [25, p. 385] or [44, p. 10]).

For the reader's convenience we list the following well-known results that will be used later:

**Theorem 2.2.** [25, p. 393] Let  $X$  and  $Y$  be compact metric spaces such that  $\dim(X) = 0$ . Then

$$\dim(X \times Y) = \dim(Y).$$

**Theorem 2.3.** [44, p. 15] Let  $\{X_n\}_{n=1}^{\infty}$  be a sequence of compact subspaces of a metric space and let  $k$  be a nonnegative integer, such that  $\dim(X_n) \leq k$  for all  $n$ . Then

$$\dim\left(\bigcup_{n=1}^{\infty} X_n\right) \leq k.$$

### 3 Main results

In this section we formulate and prove the main results.

First we introduce basic notation and facts.

Let  $a \in (0, 1)$  and  $b \in (0, 1]$ . Note that  $f_{(a,b)}$  is single-valued. Then

$$f_{(a,b)}(t) = \begin{cases} \frac{b}{a}t & ; \text{ if } t \in [0, a] \\ \frac{b}{a-1}(t-1) & ; \text{ if } t \in [a, 1]. \end{cases}$$

The point  $e = f_{(a,b)}(b) = \frac{b(b-1)}{a-1}$  plays an important role in the study of  $K_{(a,b)}$ . The restrictions of  $f_{(a,b)}$  mapping  $[0, a]$  onto  $[0, b]$ , and  $[a, 1]$  onto  $[e, b]$ , respectively, are bijections. Therefore they have the inverse functions and we denote them by  $L : [0, b] \rightarrow [0, a]$  and  $R : [e, b] \rightarrow [a, 1]$ , respectively. It is easy to see that  $L(t) = \frac{a}{b}t$  and  $R(t) = \frac{a-1}{b}t + 1$ .

For any point  $(x_1, x_2, x_3, \dots) \in K_{(a,b)}$  and any positive integer  $n$ ,  $x_n = f_{(a,b)}(x_{n+1})$ . If  $x_{n+1} < a$  then  $x_{n+1} = L(x_n)$ , if  $x_{n+1} > a$  then  $x_{n+1} = R(x_n)$ , and finally if  $x_{n+1} = a$  then  $x_n = b$  and  $x_{n+1} = L(x_n) = R(x_n)$ . Note that if  $x_{n+1} \geq a$  then  $x_{n+1} \in [a, b]$ , because  $x_{n+1} = f_{(a,b)}(x_{n+2}) \leq b$ . Therefore in that case  $x_n \in [a, e]$ . This fact was the main reason for the introduction of  $e$  and our choice of the restriction of  $f_{(a,b)}$  in the definition of  $R$ .

That means that  $x_{n+1} = R(x_n)$  is possible only for  $x_n \geq e$ , but note that for  $x_{n+1} = L(x_n)$  there are no restrictions.

We continue with the following lemma which will be used in the proof of Theorem 3.2.

**Lemma 3.1.** Let  $X$  be a compact metric space and let  $A$  be a continuum and for each positive integer  $n$ , let  $A_n$  be an arc in  $X$  from  $a_n$  to  $a_{n+1}$  such that

1. for each positive integer  $n$ ,  $A_n \cap A_{n+1} = \{a_{n+1}\}$ ,
2.  $A_i \cap A_j \neq \emptyset$  if and only if  $|i - j| \leq 1$ ,
3. there is a point  $z \in X \setminus \left( \bigcup_{n=1}^{\infty} A_n \right)$  such that  $\lim_{n \rightarrow \infty} A_n = \{z\}$  in  $2^X$ ,
4.  $A = \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \{z\}$ .

Then the subspace  $A$  of  $X$  is an arc.

*Proof.* Let for each positive integer  $n$ ,  $f_n : [\frac{n-1}{n}, \frac{n}{n+1}] \rightarrow A_n$  be a homeomorphism such that  $f_n(\frac{n-1}{n}) = a_n$  and  $f_n(\frac{n}{n+1}) = a_{n+1}$ . Next, let  $f : [0, 1] \rightarrow X$  be a function, defined by  $f(t) = f_n(t)$  if  $t \in [\frac{n-1}{n}, \frac{n}{n+1}]$  for some positive integer  $n$ , and  $f(1) = z$ . Since  $f_n(\frac{n}{n+1}) = a_{n+1} = f_{n+1}(\frac{n}{n+1})$  for each positive integer  $n$ , it follows that  $f$  is continuous on  $[0, 1)$  [24, p. 83]. Let  $\{t_i\}_{i=1}^{\infty}$  be a sequence in  $[0, 1]$  such that  $\lim_{i \rightarrow \infty} t_i = 1$ . Then it follows from 3. that  $\lim_{i \rightarrow \infty} f(t_i) = z$  and therefore  $f$  is continuous at  $t = 1$ . It also follows from 4. that  $f$  is surjective and from 1., 2., and 3. that  $f$  is injective. Therefore  $f$  is a homeomorphism from  $[0, 1]$  onto  $A$ .  $\square$

**Theorem 3.2.** Let  $a, b \in (0, 1)$ ,  $a < b$  and  $b < 1 - a$ . Then  $K_{(a,b)}$  is an arc.

*Proof.* Let  $e = f_{(a,b)}(b) = \frac{b(b-1)}{a-1}$ . It follows from  $a < b < 1$  and  $b < 1 - a$  that  $a < e$ .

Define  $A_1$  to be the set of all points  $(x_1, x_2, x_3, \dots) \in K_{(a,b)}$  such that  $x_{i+1} = L(x_i)$  for each positive integer  $i$ . It follows that  $A_1$  is an arc from

$$a_1 = (0, 0, 0, \dots) \quad \text{to} \quad a_2 = (b, a, L(a), L^2(a), \dots),$$

since one easily proves that  $t \mapsto (t, L(t), L^2(t), \dots)$  is a homeomorphism from  $[0, b]$  onto  $A_1$ .

Let  $A_2$  be the set of all points  $(x_1, x_2, x_3, \dots) \in K_{(a,b)}$  such that  $x_2 = R(x_1)$  and  $x_{i+1} = L(x_i)$  for each positive integer  $i \geq 2$ . It follows that  $A_2$  is an arc from

$$a_2 = (b, a, L(a), L^2(a), \dots) \quad \text{to} \quad a_3 = (e, b, a, L(a), L^2(a), \dots),$$

since  $t \mapsto (t, R(t), L(R(t)), L^2(R(t)), \dots)$  is a homeomorphism from  $[e, b]$  onto  $A_2$ .

Also, for each positive integer  $n \geq 3$  define the set  $A_n$  to be the set of all points  $(x_1, x_2, x_3, \dots) \in K_{(a,b)}$  such that  $x_n = R(x_{n-1})$  and  $x_{i+1} = L(x_i)$  for each positive integer  $i \geq n$ . It follows that  $A_n$  is an arc from from

$$a_n = (f^{n-2}(e), \dots, f^2(e), f(e), e, b, a, L(a), L^2(a), L^3(a), \dots)$$

to

$$a_{n+1} = (f^{n-3}(e), \dots, f^2(e), f(e), e, b, a, L(a), L^2(a), L^3(a), \dots),$$

since  $t \mapsto (f^{n-2}(t), \dots, f^2(t), f(t), t, R(t), L(R(t)), L^2(R(t)), \dots)$  is a homeomorphism from  $[e, b]$  onto  $A_n$ .

Since  $R$  is an expansive map the only point of the form  $(t, R(t), R^2(t), \dots)$  is obtained in the case when  $t = R(t)$ . One easily checks that  $t = \frac{b}{1+b-a}$ . Let  $z = (\frac{b}{1+b-a}, \frac{b}{1+b-a}, \frac{b}{1+b-a}, \dots)$ .

Since  $a < e$  and  $L(b) = a$ , it follows that  $L(t) \leq L(b) = a < e$  for each  $t \in [e, b]$ , and therefore  $R(L(t))$  is not defined. It follows that  $K_{(a,b)} = \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \{z\}$ .

One can easily check that  $A_n \cap A_{n+1} = \{a_{n+1}\}$  for each positive integer  $n$  and that for positive integers  $m$  and  $n$  such that  $|m - n| > 1$  it holds that  $A_m \cap A_n = \emptyset$ .

The function  $f_{(a,b)}$  is a contraction mapping on  $[e, b]$  with the contraction factor  $M = \frac{b}{1-a} < 1$ . Using this we show that  $\lim_{n \rightarrow \infty} A_n = \{z\}$  in  $2^{K_{(a,b)}}$ . Take any  $\varepsilon > 0$ . Choose a positive integer  $k$ , such that  $\frac{1}{k} < \varepsilon$ . Let  $n_0$  be a positive integer such that  $M^n < \frac{\varepsilon}{b-e}$  for each  $n \geq n_0$ . Then for each  $n \geq n_0$  and for each  $t \in [e, b]$ ,

$$\left| f^n(t) - \frac{b}{1+b-a} \right| = \left| f^n(t) - f^n \left( \frac{b}{1+b-a} \right) \right| < M^n \left| t - \frac{b}{1+b-a} \right| < \varepsilon.$$

Next we prove that for each  $n \geq n_0 + k + 1$  and for each  $x \in A_n$  it holds that  $D(x, z) < \varepsilon$ . Take any

$$\begin{aligned} x &= (x_1, x_2, \dots, x_k, \dots) \\ &= \underbrace{(f^{n-2}(t), \dots, f^{n-k-1}(t))}_{k}, \dots, f^2(t), f(t), t, R(t), L(R(t)), L^2(R(t)), \dots) \end{aligned}$$

in  $A_n$ . Take arbitrary positive integer  $m$ . If  $m \geq k$  then

$$\frac{d(x_m, \frac{b}{1+b-a})}{m} \leq \frac{1}{m} \leq \frac{1}{k} < \varepsilon.$$

If  $m < k$  then  $n - m - 1 > n - k - 1 \geq n_0$ , hence

$$\frac{d(x_m, \frac{b}{1+b-a})}{m} = \frac{|f^{n-m-1}(t) - \frac{b}{1+b-a}|}{m} < \varepsilon.$$

It follows that  $D(x, z) < \varepsilon$  for each  $x \in A_n$  and therefore  $H_D(A_n, \{z\}) < \varepsilon$ . Using Lemma 3.1 it follows that  $K_{(a,b)}$  is an arc.  $\square$

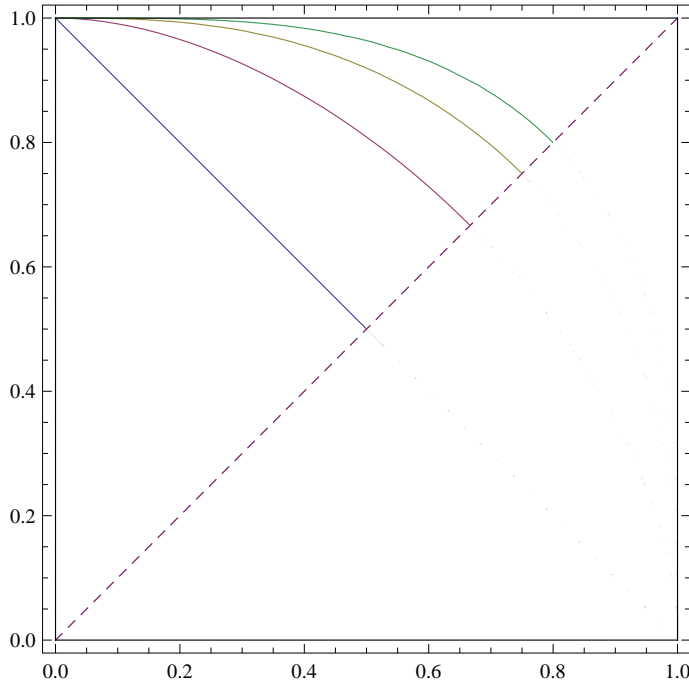
**Definition 3.3.** For any  $t \in [1, \infty)$  let

$$C_t = \{(x, y) \in [0, 1] \times [0, 1] \mid x^{t+1} - x^t = y^{t+1} - y^t, 0 < x < y\}.$$

See Figure 1.

It is easy to see that  $C_1$  is the graph of the function  $f : (0, \frac{1}{2}) \rightarrow (\frac{1}{2}, 1)$ ,  $f(x) = 1 - x$ . One can easily see that for each  $t \in (1, \infty)$ ,  $C_t$  is a subset of  $[0, 1] \times [0, 1]$  containing  $(\frac{1}{2}, y)$  for exactly one  $y \in (\frac{1}{2}, 1)$ .



Figure 1:  $C_t$  for  $t = 1, 2, 3, 4$ .

**Theorem 3.4.** Let  $n$  be a positive integer and  $(a, b), (c, d) \in C_n$ . Then  $K_{(a,b)}$  and  $K_{(c,d)}$  are homeomorphic.

*Proof.* In this proof for any closed interval  $L = [u, v]$  the corresponding open interval  $(u, v)$  will be denoted by  $\overset{\circ}{L}$ .

Let  $e = f_{(a,b)}(b) = \frac{b(b-1)}{a-1}$  and  $f = f_{(c,d)}(d) = \frac{d(d-1)}{c-1}$ .

Next define  $L_1 : [0, b] \rightarrow [0, a]$  and  $R_1 : [e, b] \rightarrow [a, b]$  with

$$L_1(t) = \frac{a}{b}t, \quad R_1(t) = \frac{a-1}{b}t + 1,$$

and  $L_2 : [0, d] \rightarrow [0, c]$  and  $R_2 : [f, d] \rightarrow [c, d]$  with

$$L_2(t) = \frac{c}{d}t, \quad R_2(t) = \frac{c-1}{d}t + 1.$$

From  $(a, b), (c, d) \in C_n$  it follows that  $(\frac{b}{a})^{n-1}e = (\frac{b}{a})^{n-1}\frac{b(b-1)}{a-1} = a$ ,  $(\frac{b}{a})^ne = (\frac{b}{a})^n\frac{b(b-1)}{a-1} = b$ ,  $(\frac{d}{c})^{n-1}f = (\frac{d}{c})^{n-1}\frac{d(d-1)}{c-1} = c$ ,  $(\frac{d}{c})^nf = (\frac{d}{c})^n\frac{d(d-1)}{c-1} = d$ .  
Moreover

$$0 < e < f_{(a,b)}(e) < f_{(a,b)}^2(e) < \cdots < f_{(a,b)}^{n-1}(e) < f_{(a,b)}^n(e),$$

and  $f_{(a,b)}^k(e) = (\frac{b}{a})^ke$  for each  $k = 0, 1, \dots, n$ . Similarly

$$0 < f < f_{(c,d)}(f) < f_{(c,d)}^2(f) < \cdots < f_{(c,d)}^{n-1}(f) < f_{(c,d)}^n(f),$$

and  $f_{(c,d)}^k(f) = \left(\frac{d}{c}\right)^k f$  for each  $k = 0, 1, \dots, n$ .

Let  $I_0 = [0, e]$  and

$$I_k = \left[ f_{(a,b)}^{k-1}(e), f_{(a,b)}^k(e) \right] = \left[ \left(\frac{b}{a}\right)^{k-1} e, \left(\frac{b}{a}\right)^k e \right]$$

for each  $k = 1, \dots, n$ .

Since  $R_1(e) = b$  and  $R_1(b) = a$ , it follows that  $R_1([e, b]) = I_n$  and therefore  $R_1(I_k) \subseteq I_n$  for each  $k = 1, \dots, n$ . Recall that  $R_1$  is not defined on  $I_0 \setminus \{e\}$ .

Since  $L_1\left(\left(\frac{b}{a}\right)^k e\right) = \left(\frac{b}{a}\right)^{k-1} e$  it follows that  $L_1(I_k) = I_{k-1}$  for each  $k = 1, \dots, n$ . Also  $L_1(I_0) \subseteq I_0$ .

For any  $x \in [0, b]$  let  $S_1(x) = \{I_k \mid k = 0, 1, 2, \dots, n, x \in I_k\}$ . Obviously,  $S_1(x)$  is a singleton, except for  $x = \left(\frac{b}{a}\right)^k e$ ,  $k = 0, 1, \dots, n-1$ , when  $S_1(x) = \{I_k, I_{k+1}\}$ . This notation will simplify the description of the dynamics of the mapping  $f_{(a,b)}$  that will be the crucial part of the proof.

Analogously, we define  $J_0 = [0, f]$ ,

$$J_k = \left[ f_{(c,d)}^{k-1}(f), f_{(c,d)}^k(f) \right] = \left[ \left(\frac{d}{c}\right)^{k-1} f, \left(\frac{d}{c}\right)^k f \right]$$

for each  $k = 1, \dots, n$  and prove that  $R_2([f, d]) = J_n$ ,  $L_2(J_k) = J_{k-1}$  for each  $k = 1, 2, \dots, n$ , and  $L_2(J_0) \subseteq J_0$ .

Also for any  $x \in [0, d]$  we define  $S_2(x) = \{J_k \mid k = 0, 1, 2, \dots, n, x \in J_k\}$  which again turns out to be a singleton, except for  $x = \left(\frac{d}{c}\right)^k f$ ,  $k = 0, 1, \dots, n-1$ , when  $S_2(x) = \{J_k, J_{k+1}\}$ .

Now define the continuous piecewise linear increasing function  $\varphi : [0, b] \rightarrow [0, d]$ , which maps each interval  $I_k$  affinely onto  $J_k$ , for  $k = 0, 1, 2, \dots, n$ . Explicitly  $\varphi$  is given by

$$\varphi(t) = \begin{cases} \frac{f}{e} t & ; \text{ if } t \in I_0 \\ \frac{\frac{d}{c}f - f}{\frac{b}{a}e - e} (t - \frac{b}{a}e) + \frac{d}{c}f & ; \text{ if } t \in I_1 \\ \frac{\left(\frac{d}{c}\right)^2 f - \frac{d}{c}f}{\left(\frac{b}{a}\right)^2 e - \frac{b}{a}e} (t - \left(\frac{b}{a}\right)^2 e) + \left(\frac{d}{c}\right)^2 f & ; \text{ if } t \in I_2 \\ \frac{\left(\frac{d}{c}\right)^3 f - \left(\frac{d}{c}\right)^2 f}{\left(\frac{b}{a}\right)^3 e - \left(\frac{b}{a}\right)^2 e} (t - \left(\frac{b}{a}\right)^3 e) + \left(\frac{d}{c}\right)^3 f & ; \text{ if } t \in I_3 \\ \vdots & \\ \frac{\left(\frac{d}{c}\right)^n f - \left(\frac{d}{c}\right)^{n-1} f}{\left(\frac{b}{a}\right)^n e - \left(\frac{b}{a}\right)^{n-1} e} (t - \left(\frac{b}{a}\right)^n e) + \left(\frac{d}{c}\right)^n f & ; \text{ if } t \in I_n. \end{cases}$$

We will prove that the function  $\Phi : K_{(a,b)} \rightarrow K_{(c,d)}$  defined by

$$(x_1, x_2, x_3, \dots) \mapsto (y_1, y_2, y_3, \dots),$$

where

$$(3.1) \quad y_1 = \varphi(x_1),$$

and

$$(3.2) \quad y_{k+1} = \begin{cases} L_2(y_k) & ; \text{ if } x_{k+1} = L_1(x_k) \\ R_2(y_k) & ; \text{ if } x_{k+1} = R_1(x_k) \end{cases}$$

for each positive integer  $k$ , is well-defined and that it is a homeomorphism.

The well-definedness of  $\Phi$  follows from the following inductive argument.

Let  $x = (x_1, x_2, x_3, \dots) \in K_{(a,b)}$  be arbitrary. By induction on  $m$  we prove that for each  $m$  and for each  $j = 1, 2, \dots, m$ ,  $y_j$  is uniquely determined by (3.1) and (3.2), and that

$$x_j = 0 \iff y_j = 0, \quad x_j = b \iff y_j = d,$$

as well as

$$I_k \in S_1(x_j) \iff J_k \in S_2(y_j),$$

for any  $k = 0, 1, \dots, n$ .

The claim is obviously true for  $m = 1$ , by (3.1) and the definition of  $\varphi$ . Assume that the claim is true for a positive integer  $m$ . Now we distinguish several cases.

Case 1.  $S_1(x_{m+1}) = \{I_k\}$ , for some  $k = 0, 1, \dots, n$ .

Subcase 1.1.  $x_{m+1} = R_1(x_m)$ . Now  $x_{m+1} \in I_n$ , i.e.  $k = n$ .  $x_{m+1} \neq a$  therefore  $x_m \neq b$ . It follows that  $x_{m+1} = L_1(x_m)$  does not hold and therefore  $y_{m+1}$  is uniquely determined as  $y_{m+1} = R_2(y_m)$ . Since  $x_m \neq b$  it follows that  $y_m \neq d$ , hence  $y_{m+1} \neq c$ . That means  $S_2(y_{m+1}) = \{J_n\}$ . Note that  $x_{m+1} = b$  implies  $x_{m+1} = R_1(x_m)$ , for  $x_m = e$ , and since  $S_1(e) = \{I_0, I_1\}$ , it follows by the induction assumption that  $S_2(y_m) = \{J_0, J_1\}$ , hence  $y_m = f$ , and therefore  $y_{m+1} = R_2(f) = d$ .

Subcase 1.2.  $x_{m+1} = L_1(x_m)$ ,  $x_m \neq 0$ . In this case  $x_{m+1} \in \overset{\circ}{I}_k$  and  $k < n$ . Therefore  $x_m \in \overset{\circ}{I}_{k+1}$ . It follows that  $y_m \in \overset{\circ}{J}_{k+1}$ , since  $y_m \neq d$  and  $S_2(y_m) = \{J_{k+1}\}$  by the induction assumption. It follows that  $y_{m+1} = L_2(y_m) \in \overset{\circ}{J}_k$  and therefore  $S_2(y_{m+1}) = \{J_k\}$ . Uniqueness of  $y_{m+1}$  is clear, since it is not true that  $x_{m+1} = R_1(x_m)$ .

Subcase 1.3.  $x_{m+1} = L_1(x_m)$ ,  $x_m = 0$ . Note that this is equivalent to  $x_{m+1} = 0$ . By the induction assumption it follows that  $y_m = 0$ , and therefore  $y_{m+1} = L_2(0) = 0$ .  $R_2(0)$  is not defined and the uniqueness is therefore proved. Also,  $S_1(x_{m+1}) = \{I_0\}$  and  $S_2(y_{m+1}) = \{J_0\}$ .

Case 2.  $S_1(x_{m+1}) = \{I_{k-1}, I_k\}$ , for some  $k = 1, \dots, n$ . This is equivalent to  $x_{m+1} = e \left(\frac{b}{a}\right)^{k-1}$ .

Subcase 2.1.  $k < n$ . In this case  $x_{m+1} = L_1(x_m)$ , where  $x_m = e \left(\frac{b}{a}\right)^k$ , and it is not true that  $x_{m+1} = R_1(x_m)$ . Therefore  $S_1(x_m) = \{I_k, I_{k+1}\}$  and by the induction assumption  $S_2(y_m) = \{J_k, J_{k+1}\}$ . It follows that  $y_m = f \left(\frac{d}{c}\right)^k$ . Finally  $y_{m+1}$  is uniquely determined as  $y_{m+1} = L_2(y_m) = f \left(\frac{d}{c}\right)^{k-1}$ , and  $S_2(y_{m+1}) = \{J_{k-1}, J_k\}$ .

Subcase 2.2.  $k = n$ . Now  $x_{m+1} = a$ , and therefore  $x_{m+1} = R_1(x_m) = L_1(x_m)$ , for  $x_m = b$ . By the induction assumption  $y_m = d$ , and it follows that  $y_{m+1} = R_2(d) = L_2(d) = c$ , proving the uniqueness part of the claim. Obviously  $S_2(y_{m+1}) = \{J_{n-1}, J_n\}$ .

The whole inductive proof is completed by the following two observations. First, from  $y_{m+1} = 0$  it follows that  $y_m = 0$  (since  $y_m = f_{(c,d)}(0) = 0$ ), and by the induction assumption it follows that  $x_m = 0$ , and therefore  $x_{m+1} = L_1(0) = 0$  (since  $R_1(0)$  is not defined). Similarly, from  $y_{m+1} = d$  it follows that  $y_m = f$  (since  $y_m = f_{(c,d)}(d) = f$ ), and by the induction assumption it follows that  $x_m = e$  (since from  $S_2(y_m) = \{J_0, J_1\}$  it follows that  $S_1(x_m) = \{I_0, I_1\}$ ), and therefore  $x_{m+1} = R_1(e) = b$  (since  $x_{m+1} = L_1(e)$  would imply  $y_{m+1} = L_2(f) \neq d$ ).

This proves that  $\Phi : K_{(a,b)} \rightarrow K_{(c,d)}$  is a well-defined function. But replacing  $a, b, e, \varphi, x_k$  by  $c, d, f, \varphi^{-1}, y_k$  respectively, one obtains the proof of the well-definedness of the function  $\Psi : K_{(c,d)} \rightarrow K_{(a,b)}$ , which is defined by

$$(y_1, y_2, y_3, \dots) \longmapsto (x_1, x_2, x_3, \dots),$$

where

$$x_1 = \varphi^{-1}(y_1),$$

and

$$x_{k+1} = \begin{cases} L_1(x_k) & ; \text{ if } y_{k+1} = L_2(y_k) \\ R_1(x_k) & ; \text{ if } y_{k+1} = R_2(y_k) \end{cases}$$

for each positive integer  $k$ .

Obviously  $\Psi \circ \Phi = 1$  and  $\Phi \circ \Psi = 1$ . Therefore both  $\Phi$  and  $\Psi$  are bijections.

It remains to be proved that  $\Phi$  and  $\Psi$  are continuous functions.

Let  $x \in K_{(a,b)}$  be an arbitrary point and let  $\{x^i\}_{i=1}^\infty$  be any sequence in  $K_{(a,b)}$  converging to  $x$ . We shall prove that  $\Phi(x^i)$  converges to  $\Phi(x)$ . Coordinatewisely it means that if for each positive integer  $j$ ,  $\lim_{i \rightarrow \infty} x_j^i = x_j$

for  $x^i = (x_1^i, x_2^i, x_3^i, \dots)$  and  $x = (x_1, x_2, x_3, \dots)$ , then

$$\lim_{i \rightarrow \infty} y_j^i = y_j,$$

for each  $j$ , where

$$(y_1^i, y_2^i, y_3^i, \dots) = \Phi(x_1^i, x_2^i, x_3^i, \dots)$$

and

$$(y_1, y_2, y_3, \dots) = \Phi(x_1, x_2, x_3, \dots).$$

For each positive integer  $i$  we fix a sequence  $(N_1^i, N_2^i, N_3^i, \dots)$  of symbols  $L_1, R_1$ , such that for each positive integer  $k$ , it holds that

$$x_{k+1}^i = N_k^i(x_k^i).$$

Then we introduce the sequences  $(O_1^i, O_2^i, O_3^i, \dots)$  of symbols  $L_2, R_2$ , as follows:

$$O_k^i = L_2 \iff N_k^i = L_1, \quad O_k^i = R_2 \iff N_k^i = R_1,$$

for each  $k$  and  $i$ . By the definition of  $\Phi$  it follows that

$$y_{k+1}^i = O_k^i(y_k^i),$$

for each  $k$ .

First we show that  $\lim_{i \rightarrow \infty} y_1^i = y_1$ . It follows from the definition of  $\Phi$  that  $y_1 = \varphi(x_1)$  and for each positive integer  $i$ ,  $y_1^i = \varphi(x_1^i)$ . Since  $\varphi$  is continuous,

$$\lim_{i \rightarrow \infty} y_1^i = \lim_{i \rightarrow \infty} \varphi(x_1^i) = \varphi(\lim_{i \rightarrow \infty} x_1^i) = \varphi(x_1) = y_1.$$

Assume that for a positive integer  $j$ , it holds that  $\lim_{i \rightarrow \infty} y_j^i = y_j$ . We show that  $\lim_{i \rightarrow \infty} y_{j+1}^i = y_{j+1}$ .

If  $x_{j+1} < a$ , then there is a positive integer  $i_0$ , such that for all  $i \geq i_0$ ,  $x_{j+1}^i < a$ , and hence  $N_j^i = L_1$ . Therefore for all  $i \geq i_0$ ,  $O_j^i = L_2$ . Obviously  $x_{j+1} = L_1(x_j)$ , and hence  $y_{j+1} = L_2(y_j)$  as well. It follows from the definition of  $\Phi$  and from the continuity of  $L_2$  that

$$(3.3) \quad \lim_{i \rightarrow \infty} y_{j+1}^i = \lim_{i \rightarrow \infty} O_j^i(y_j^i) = \lim_{i \rightarrow \infty} L_2(y_j^i) = L_2(\lim_{i \rightarrow \infty} y_j^i) = L_2(y_j) = y_{j+1}.$$

If  $x_{j+1} > a$ , then there is a positive integer  $i_0$  such that for all  $i \geq i_0$ ,  $x_{j+1}^i > a$ , and hence  $N_j^i = R_1$ . Therefore for all  $i \geq i_0$ ,  $O_j^i = R_2$ . Obviously  $x_{j+1} = R_1(x_j)$ , and hence  $y_{j+1} = R_2(y_j)$  as well. It follows from the definition of  $\Phi$  and from the continuity of  $R_2$  that

$$(3.4) \quad \lim_{i \rightarrow \infty} y_{j+1}^i = \lim_{i \rightarrow \infty} O_j^i(y_j^i) = \lim_{i \rightarrow \infty} R_2(y_j^i) = R_2(\lim_{i \rightarrow \infty} y_j^i) = R_2(y_j) = y_{j+1}.$$

If  $x_{j+1} = a$ , then  $N_j^i = L_1$  may hold true for infinitely or finitely many  $i$ , and also  $N_j^i = R_1$  may hold true for infinitely or finitely many  $i$ . If  $N_j^i = L_1$  is true only for finitely many  $i$ , then (3.4) applies; if  $N_j^i = R_1$  is true only for finitely many  $i$ , then (3.3) applies. In case when both  $N_j^i = L_1$  and  $N_j^i = R_1$  hold true for infinitely many  $i$ , we apply (3.3) and (3.4) respectively for the two subsequences corresponding to the choice of  $L_1$  or  $R_1$  respectively. Clearly in each of these cases we get  $\lim_{i \rightarrow \infty} y_{j+1}^i = y_{j+1}$ .

Therefore  $\Phi$  is continuous. Obviously the proof of continuity of  $\Psi$  can be obtained from the proof above by appropriate replacements.  $\square$

**Corollary 3.5.** If  $(a, b) \in C_m$  and  $(c, d) \in C_n$  for some positive integers  $m, n \geq 2$ ,  $m \neq n$ , then the continua  $K_{(a,b)}$  and  $K_{(c,d)}$  are not homeomorphic.

*Proof.* Let  $t_1, t_2 \in (\frac{1}{2}, 1)$  such that  $(\frac{1}{2}, t_1) \in C_m$ ,  $(\frac{1}{2}, t_2) \in C_n$ . It follows from Theorem 3.4 that  $K_{(a,b)}$  is homeomorphic to  $K_{(\frac{1}{2}, t_1)}$  and  $K_{(c,d)}$  is homeomorphic to  $K_{(\frac{1}{2}, t_2)}$ . Since  $K_{(\frac{1}{2}, t_1)}$  and  $K_{(\frac{1}{2}, t_2)}$  are not homeomorphic, by the positive solution of the Ingram conjecture [7], it follows that  $K_{(a,b)}$  and  $K_{(c,d)}$  are not homeomorphic.  $\square$

We have proved in Theorem 3.4 that  $K_{(a,b)}$  and  $K_{(c,d)}$  are homeomorphic for any  $(a, b), (c, d) \in C_1$ , but we are able to give more precise information about these continua, as shown in the following theorem.

**Theorem 3.6.** If  $(a, b) \in C_1$ , then  $K_{(a,b)}$  is a  $\sin \frac{1}{x}$ -continuum.

*Proof.* It is easy to see that  $(a, b) \in C_1$  if and only if  $1 > b > a$  and  $b = 1 - a$ . Let

$$A_0 = \{(t, 1 - t, t, 1 - t, \dots) \mid t \in [a, b]\},$$

$$A_1 = \{(t, L(t), L^2(t), L^3(t), \dots) \mid t \in [0, b]\},$$

and for each positive integer  $n$ ,

$$A_{2n} = \{\underbrace{(t, 1 - t, t, 1 - t, \dots, t, 1 - t)}_{2n}, L(1 - t), L^2(1 - t), \dots \mid t \in [a, b]\},$$

$$A_{2n+1} = \{\underbrace{(t, 1 - t, t, 1 - t, \dots, t, 1 - t, t)}_{2n+1}, L(t), L^2(t), \dots \mid t \in [a, b]\},$$

where  $L$  has the usual meaning ( $L : [0, b] \rightarrow [0, a]$ ,  $L(t) = \frac{a}{b}t$  for any  $t$ ).

Note that in this case  $e = a$  and  $R(t) = 1 - t$  for each  $t \in [a, b]$  making the above formulas in coherence with what was said about elements of  $K_{(a,b)}$  at the beginning of this section.

Obviously  $K_{(a,b)} = \bigcup_{n=0}^{\infty} A_n$ .

Let  $1 = x_0 > x_1 > x_2 > x_3 > \dots$  be a sequence in  $[0, 1]$  converging to 0. Also let  $T_{2n} = (x_{2n}, -1)$  and  $T_{2n+1} = (x_{2n+1}, 1)$ , for all nonnegative integers  $n$ . Next, let  $B_0 = \{0\} \times [-1, 1]$  be the arc from  $(0, -1)$  to  $(0, 1)$ , and for each nonnegative integer  $n$ ,

$$B_{2n+1} = \left\{ \left( x, \frac{2}{x_{2n+1} - x_{2n}}(x - x_{2n+1}) + 1 \right) \in [0, 1] \times [-1, 1] \mid x \in [x_{2n+1}, x_{2n}] \right\}$$

be the arc from  $T_{2n}$  to  $T_{2n+1}$ ,

$$B_{2n+2} = \left\{ \left( x, \frac{2}{x_{2n+1} - x_{2n+2}}(x - x_{2n+1}) + 1 \right) \in [0, 1] \times [-1, 1] \mid x \in [x_{2n+2}, x_{2n+1}] \right\}$$

be the arc from  $T_{2n+1}$  to  $T_{2n+2}$ .

Obviously  $\bigcup_{n=0}^{\infty} B_n$  is a  $\sin \frac{1}{x}$ -continuum.

We define for each nonnegative integer  $n$ ,  $\varphi_n : A_n \rightarrow B_n$  as follows.

$$\varphi_0(t, 1-t, t, 1-t, \dots) = \left( 0, \frac{2}{b-a}(t-a) - 1 \right), \quad t \in [a, b],$$

$$\varphi_1(t, L(t), L^2(t), L^3(t), \dots) = \left( \frac{x_1}{b}t + \left(1 - \frac{1}{b}t\right)x_0, \frac{2}{b}t - 1 \right), \quad t \in [0, b],$$

and for each positive integer  $n$ ,

$$\begin{aligned} & \varphi_{2n}(t, \underbrace{1-t, t, 1-t, \dots, t, 1-t}_{2n}, L(1-t), L^2(1-t), \dots) = \\ & = \left( \frac{x_{2n} - x_{2n-1}}{a-b}(t-b) + x_{2n-1}, \frac{-2}{a-b}(t-b) + 1 \right), \quad t \in [a, b], \end{aligned}$$

$$\begin{aligned} & \varphi_{2n+1}(t, \underbrace{1-t, t, 1-t, \dots, t, 1-t, t}_{2n+1}, L(t), L^2(t), \dots) = \\ & = \left( \frac{x_{2n+1} - x_{2n}}{b-a}(t-a) + x_{2n}, \frac{2}{b-a}(t-a) - 1 \right), \quad t \in [a, b]. \end{aligned}$$

Next, let  $\varphi : \bigcup_{n=0}^{\infty} A_n \rightarrow \bigcup_{n=0}^{\infty} B_n$  be a function, defined by

$$\varphi(x) = \begin{cases} \varphi_0(x) & ; \text{ if } x \in A_0 \\ \varphi_1(x) & ; \text{ if } x \in A_1 \\ \varphi_2(x) & ; \text{ if } x \in A_2 \\ \vdots & \\ \varphi_n(x) & ; \text{ if } x \in A_n \\ \vdots & \end{cases}$$

It is easy to see that  $\varphi$  is a bijection and that  $\varphi$  is continuous on  $\bigcup_{n=1}^{\infty} A_n$ . To see that  $\varphi$  is continuous on  $A_0$ , take arbitrary  $z_0 = (t_0, 1-t_0, t_0, 1-t_0, \dots) \in A_0$  and let  $\{z_n\}_{n=1}^{\infty}$  be a sequence of points in  $\bigcup_{n=0}^{\infty} A_n$  such that  $\lim_{n \rightarrow \infty} z_n = z_0$ . We will show that  $\lim_{n \rightarrow \infty} \varphi(z_n) = \varphi(z_0)$ .

- (a) Suppose there is a positive integer  $n_0$  such that  $z_n \in A_0$  for all  $n \geq n_0$ . Then for each  $n \geq n_0$  fix the unique  $t_n \in [a, b]$ , such that

$$z_n = (t_n, 1-t_n, t_n, 1-t_n, \dots).$$

From  $\lim_{n \rightarrow \infty} z_n = z_0$  it follows that  $\lim_{n \rightarrow \infty} t_n = t_0$ . Obviously, for each positive integer  $n \geq n_0$ ,  $\varphi(z_n) = \varphi_0(z_n) \in B_0$  and hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi(z_n) &= \lim_{n \rightarrow \infty} \varphi_0(t_n, 1-t_n, t_n, 1-t_n, \dots) \\ &= \lim_{n \rightarrow \infty} \left(0, \frac{2}{b-a}(t_n - a) - 1\right) \\ &= \left(0, \frac{2}{b-a}(t_0 - a) - 1\right) \\ &= \varphi_0(t_0, 1-t_0, t_0, 1-t_0, \dots) \\ &= \varphi(z_0). \end{aligned}$$

- (b) Suppose there is a positive integer  $n_0$ , such that for each  $n \geq n_0$ ,  $z_n \notin A_0$ .

If there is  $m_0 \geq n_0$ , such that for each  $n \geq m_0$ ,  $z_n \in A_{2k(n)}$  for some positive integer  $k(n)$ , then

$$z_n = \underbrace{(t_n, 1-t_n, t_n, 1-t_n, \dots, t_n, 1-t_n)}_{2k(n)}, L(1-t_n), L^2(1-t_n), \dots$$

for some  $t_n \in [a, b]$ . Obviously  $\lim_{n \rightarrow \infty} t_n = t_0$  and  $\lim_{n \rightarrow \infty} x_{k(n)} = 0$ . Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi(z_n) &= \lim_{n \rightarrow \infty} \varphi_{2k(n)}(z_n) \\ &= \lim_{n \rightarrow \infty} \left( \frac{x_{2k(n)} - x_{2k(n)-1}}{a-b}(t_n - b) + x_{2k(n)-1}, \frac{-2}{a-b}(t_n - b) + 1 \right) \\ &= \left(0, \frac{-2}{a-b}(t_0 - b) + 1\right) \\ &= \varphi(z_0), \end{aligned}$$



If there is  $m_0 \geq n_0$ , such that for each  $n \geq m_0$ ,  $z_n \in A_{2k(n)+1}$  for some positive integer  $k(n)$ , then

$$z_n = \underbrace{(t_n, 1 - t_n, t_n, 1 - t_n, \dots, t_n, 1 - t_n, t_n)}_{2k(n)+1}, L(t_n), L^2(t_n), \dots$$

for some  $t_n \in [a, b]$ , and therefore by the same reasoning as above

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi(z_n) &= \lim_{n \rightarrow \infty} \varphi_{2k(n)+1}(z_n) \\ &= \lim_{n \rightarrow \infty} \left( \frac{x_{2k(n)+1} - x_{2k(n)}}{b - a} (t_n - a) + x_{2k(n)}, \frac{2}{b - a} (t_n - a) - 1 \right) \\ &= \left( 0, \frac{2}{b - a} (t_0 - a) - 1 \right) \\ &= \varphi(z_0), \end{aligned}$$

If  $z_n \in A_{2k(n)}$  for infinitely many  $n$  and  $z_n \in A_{2k(n)+1}$  for infinitely many  $n$ , then we apply calculations from the previous subcases respectively for the two subsequences corresponding to the choice of  $A_{2k(n)}$  or  $A_{2k(n)+1}$ . Clearly in each of these cases we get  $\lim_{n \rightarrow \infty} \varphi(z_n) = \varphi(z_0)$ .

- (c) If both  $z_n \in A_0$  and  $z_n \notin A_0$  hold true for infinitely many  $n$ , we apply (a) and (b) respectively for the two subsequences. Clearly in each of these cases we get  $\lim_{n \rightarrow \infty} \varphi(z_n) = \varphi(z_0)$ .

We have shown that  $\varphi$  is a continuous bijection from the compact space  $\bigcup_{n=0}^{\infty} A_n$  onto the metric space  $\bigcup_{n=0}^{\infty} B_n$  and therefore  $\varphi$  is a homeomorphism. □

## 4 $K_{(0,1)}$

$K_{(0,1)}$  turns out to be a very complicated continuum. In this section we give a detailed description of  $K_{(0,1)}$ , which helps us to recognize some of its subcontinua as certain familiar continua. The continuum has already been studied in [5, 21].

Let  $x = (x_1, x_2, x_3, \dots) \in K_{(0,1)}$ . Suppose there is an integer  $n$  such that  $x_n \in \{0, 1\}$ . If  $x_n = 0$ , then  $x_{n+1} \in \{0, 1\}$ . If  $x_n = 1$ , then  $x_{n+1} = 0$ . In the case where  $x_n = t \in (0, 1)$ , one can easily see that  $x_{n+1} \in \{0, 1 - t\}$ .

Let

$$A = \{(x_1, x_2, x_3, x_4, \dots) \in K_{(0,1)} \mid x_1 = 0\} \subseteq K_{(0,1)}$$

and

$$B = \{(x_1, x_2, x_3, x_4, \dots) \in K_{(0,1)} \mid x_1 = 1\} \subseteq K_{(0,1)}.$$

If  $x = (x_1, x_2, x_3, \dots) \in K_{(0,1)}$ , then exactly one of the following is possible.

1.  $x \in A$ .
2.  $x \in B$ .
3. There are an odd positive integer  $n$  and  $a \in A$  such that

$$x \in A_n(a) = \{(t, \underbrace{1-t, t, 1-t, \dots, t}_n, a) \mid t \in (0, 1)\}.$$

4. There are an even positive integer  $n$  and  $a \in A$  such that

$$x \in A_n(a) = \{(t, \underbrace{1-t, t, 1-t, \dots, t, 1-t}_n, a) \mid t \in (0, 1)\}.$$

5.  $x \in A_\infty = \{(t, 1-t, t, 1-t, \dots) \mid t \in (0, 1)\}$ .

One can easily see that  $\text{Cl}(A_\infty) = \{(t, 1-t, t, 1-t, \dots) \mid t \in [0, 1]\}$  is the arc from  $(0, 1, 0, 1, \dots) \in A$  to  $(1, 0, 1, 0, \dots) \in B$  in  $K_{(0,1)}$ . Here and in the rest of this section by  $\text{Cl}$  we denote the closure operator in the Hilbert cube.

For each  $n$ ,  $\text{Cl}(A_n(a))$  is the arc  $\{(t, \underbrace{1-t, t, 1-t, \dots, t}_n, a) \mid t \in [0, 1]\}$  from

$$\underbrace{(0, 1, 0, 1, \dots, 0, a)}_n \in A \quad \text{to} \quad \underbrace{(1, 0, 1, 0, \dots, 1, a)}_n \in B$$

if  $n$  is odd, and the arc  $\{(t, \underbrace{1-t, t, 1-t, \dots, t, 1-t}_n, a) \mid t \in [0, 1]\}$  from

$$\underbrace{(0, 1, 0, 1, \dots, 1, a)}_n \in A \quad \text{to} \quad \underbrace{(1, 0, 1, 0, \dots, 0, a)}_n \in B$$

if  $n$  is even. We will show that

$$(4.1) \quad K_{(0,1)} = \left( \bigcup_{n=1}^{\infty} \left( \bigcup_{a \in A} \text{Cl}(A_n(a)) \right) \right) \cup \text{Cl}(A_\infty).$$

For each  $a \in A$  there are two possibilities: either  $a = (0, 0, \dots)$  or  $a = (0, 1, \dots)$ . In the first case there is an  $a_0 \in A$  such that  $a = (0, a_0)$  and hence  $a \in \text{Cl}(A_1(a_0))$ . In the second case  $a$  is of the form  $a = (0, 1, 0, \dots)$

and therefore there is an  $a_0 \in A$  such that  $a = (0, 1, a_0)$  and hence  $a \in \text{Cl}(A_2(a_0))$ . That proves

$$A \subseteq \left( \bigcup_{n=1}^{\infty} \left( \bigcup_{a \in A} \text{Cl}(A_n(a)) \right) \right) \cup \text{Cl}(A_{\infty}).$$

For each  $b \in B$  there is  $a_0 \in A$  such that  $b = (1, a_0)$ . Therefore  $b \in \text{Cl}(A_1(a_0))$  and hence

$$B \subseteq \left( \bigcup_{n=1}^{\infty} \left( \bigcup_{a \in A} \text{Cl}(A_n(a)) \right) \right) \cup \text{Cl}(A_{\infty}),$$

and 4.1 follows.

Using 4.1 we are able to prove some additional properties of  $K_{(0,1)}$  as follows.

- (a)  $K_{(0,1)}$  contains  $\sin \frac{1}{x}$ -continua. For example, a proof similar to the proof of Theorem 3.6 can be obtained in order to prove that for  $a = (0, 0, 0, \dots) \in A$

$$\left( \bigcup_{n=1}^{\infty} \text{Cl}(A_n(a)) \right) \cup \text{Cl}(A_{\infty})$$

is a  $\sin \frac{1}{x}$ -continuum. See also [5].

- (b)  $K_{(0,1)}$  contains harmonic fans. For example,

$$F_1 = \left( \bigcup_{n=1}^{\infty} \text{Cl}(A_{2n-1}(a)) \right) \cup \text{Cl}(A_{\infty}),$$

where  $a = (0, 1, 0, 1, 0, 1, \dots) \in A$ , and

$$F_2 = \left( \bigcup_{n=1}^{\infty} \text{Cl}(A_{2n}(a)) \right) \cup \text{Cl}(A_{\infty}),$$

where  $a = (0, 1, 0, 1, 0, 1, \dots) \in A$ , are harmonic fans in  $K_{(0,1)}$ .

- (c)  $K_{(0,1)}$  is one-dimensional. It is easy to see that  $A$  is a Cantor set and that for each positive integer  $n$ ,

$$\bigcup_{a \in A} \text{Cl}(A_n(a))$$

is homeomorphic to the product  $A \times [0, 1]$ . Since  $\dim(A) = 0$ , it follows from Theorem 2.2 that

$$\dim \left( \bigcup_{a \in A} \text{Cl}(A_n(a)) \right) = 1.$$

A countable union of one-dimensional compacta is a one-dimensional compactum, see Theorem 2.3, therefore

$$\dim(K_{(0,1)}) = \dim \left( \left( \bigcup_{n=1}^{\infty} \left( \bigcup_{a \in A} \text{Cl}(A_n(a)) \right) \right) \cup \text{Cl}(A_{\infty}) \right) = 1.$$

See also [21].

- (d) It has been proved in [21] that  $K_{(0,1)}$  has trivial shape and is therefore tree-like.

## 5 A question

Unfortunately the techniques that were used in the proof of Theorem 3.4 do not work in general, i.e. using them one cannot prove that  $K_{(a,b)}$  is homeomorphic to  $K_{(c,d)}$  if  $(a, b), (c, d) \in C_t$  for arbitrary  $t \in [1, \infty)$ . Initially we conjectured such a result, but many unsuccessful attempts to prove it provided us with evidence of a very complicated behavior, and we are not so confident anymore. Therefore we just pose the following question.

**Question 5.1.** Is it true that for any  $a, b, c, d \in (0, 1)$ , from  $(a, b), (c, d) \in C_t$ , for some  $t \in [1, \infty)$ , it follows that  $K_{(a,b)}$  and  $K_{(c,d)}$  are homeomorphic?

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