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ON THE CONTINUITY OF THE GENERALIZED SPECTRAL RADIUS IN MAX ALGEBRA

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ABSTRACT. Given a bounded set Ψ of $n \times n$ non-negative matrices, let $\rho(\Psi)$ and $\mu(\Psi)$ denote the generalized spectral radius of Ψ and its max version, respectively. We show that

$$\mu(\Psi) = \sup_{t \in (0,\infty)} \left(n^{-1} \rho(\Psi^{(t)}) \right)^{1/t},$$

where $\Psi^{(t)}$ denotes the Hadamard power of Ψ. We apply this result to give a new short proof of a known fact that $\mu(\Psi)$ is continuous on the Hausdorff metric space (β, H) of all nonempty compact collections of $n \times n$ non-negative matrices.

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Key words: Maximum circuit geometric mean; Max algebra; Non-negative matrices; Generalized spectral radius; Joint spectral radius; Continuity; Hausdorff metric; Hadamard powers; Schur powers.

1. Introduction

The algebraic system max algebra and its isomorphic versions provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimisation, mathematical physics, DNA analysis, ... (see e.g. [8], [9], [1], [37], [16], [2], [7], [3], [23], [24], [25], [32]). Max algebra's usefulness arises from a fact that these non-linear problems become linear when decribed in the max algebra language.

Following the notation from ([2], [11], [26], [27], [34], [28], [19]), the max algebra consists of the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab, where $a, b \geq 0$. Let $A = [a_{ij}]$ be a $n \times n$ non-negative matrix, i.e., $a_{ij} \geq 0$ for all $i, j = 1, \dots, n$. We may denote a_{ij} also by $[A]_{ij}$. Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices and $\mathbb{R}_+^{n\times n}$ the set of all $n \times n$ non-negative matrices. The operations between matrices and vectors in the max algebra are defined by analogy with the usual linear algebra. For instance, the product of $A, B \in \mathbb{R}_+^{n\times n}$ in the max algebra is denoted by $A \otimes B$, where $[A \otimes B]_{ij} = \max_{k=1,\dots,n} a_{ik} b_{kj}$. The notation $A_\otimes^2$ means $A \otimes A$, and $A_\otimes^k$

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denotes the k-th max power of A. If $x = [x_i] \in \mathbb{R}^n$ is a non-negative vector, then the notation $A \otimes x$ means $[A \otimes x]_i = \max_{j=1,\ldots,n} a_{ij}x_j$. The usual associative and distributive laws hold in this algebra. Note that the standard products are denoted by AB and Ax.

The weighted directed graph $\mathcal{D}(A)$ associated with A has a vertex set $\{1, 2, \ldots, n\}$ and edges (i, j) from a vertex i to a vertex j with weight a_{ij} if and only if $a_{ij} > 0$. A path of length k is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$. A circuit of length k is a path with $i_{k+1} = i_1$, where $i_1, i_2, \ldots, i_k$ are distinct. Associated with this circuit is the *circuit geometric mean* known as $(a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1})^{1/k}$. The maximum circuit geometric mean in $\mathcal{D}(A)$ is denoted by $\mu(A)$. Note that circuits (i_1, i_1) of length 1 (loops) are included here and that we also consider empty circuits, i.e., circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero.

There are many different descriptions of the maximum circuit geometric mean $\mu(A)$ (see e.g. [14], [15], [10], [22, p. 366], [3, p. 130], [11], [31], [29], [26], [34], [33], [18]). It was proved in [15] that given $A \in \mathbb{R}_+^{n\times n}$

$$\mu(A) = \lim_{t\to\infty} \rho(A^{(t)})^{1/t}, \tag{1}$$

where $A^{(t)} = [a_{ij}^t]$ is a Hadamard (or also Schur) power of A and ρ the spectral radius. Alternative and simplified proofs of (1) can be found in ([10], [22, p. 366], [3, p. 130], [11], [34]). We also have

$$\mu(A) \le \rho(A) \le n\mu(A) \tag{2}$$

(see e.g. [10], [22, p. 366], [26], [27], [34]).

It is known that $\mu(A)$ is the largest max eigenvalue of A. Moreover, if A is irreducible, then $\mu(A)$ is the unique max eigenvalue and every max eigenvector is positive (see [2, Theorem 2] and [26, Theorem 1]). We also have

$$\mu(A) = \lim_{k\to\infty} \|A_\otimes^k\|^{1/k} \tag{3}$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ (see [11, Lemma 4.1], [26], [27], [34]).

Given an irreducible non-negative matrix A, algorithms for computing $\mu(A)$ and the max eigenvector x were established in [11], [12] and [13]. On the other hand, the infinite-dimensional generalizations of μ can be found in [31], [29] and [33].

Let Σ be a bounded set of $n \times n$ complex matrices. For $m \ge 1$, let

$$\Sigma^m = \{A_1A_2\cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m}. \tag{4}$$

It was shown in [5] that $\rho(\Sigma)$ is equal to the joint spectral radius of Σ, i.e.,

$$\rho(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \|A\|]^{1/m}, \tag{5}$$

where $\|\cdot\|$ is any vector norm on $\mathbb{C}^{n\times n}$. This equality is called the Berger-Wang formula or also the generalized spectral radius theorem. Since then many different type of proofs of (5) were obtained (for references see e.g. [27]). The theory of the generalized spectral radius $\rho(\Sigma)$ has many important applications (see e.g. [5], [36], [4], [20], [35], [30] and the references cited there). In particular, $\rho(\Sigma)$ plays a central role in determining stability in convergence properties of discrete and differential inclusions. In this theory the quantity $\log \rho(\Sigma)$ is known as the maximal Lyapunov exponent (see e.g. [36]).

Let Ψ be a bounded set of $n \times n$ non-negative matrices. For $m \geq 1$, let

$$\Psi_\otimes^m = \{A_1 \otimes A_2 \otimes \cdots \otimes A_m : A_i \in \Psi\}.$$

The max algebra version of the generalized spectral radius $\mu(\Psi)$ of Ψ, defined by

$$\mu(\Psi) = \limsup_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \mu(A)]^{1/m}, \tag{6}$$

has recently received increasing attention (see e.g. [1], [17], [6], [27], [34], [33], [28], [19]). In [27] the max algebra version of the generalized spectral radius theorem was proved, i.e., $\mu(\Psi)$ is equal to the max algebra version of the joint spectral radius $\eta(\Psi)$ of Ψ, which is defined by

$$\eta(\Psi) = \lim_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \|A\|]^{1/m} \tag{7}$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$. The quantity $\log \mu(\Psi)$ measures the worst case cycle time of certain discrete event systems and it is sometimes called the worst case Lyapunov exponent (see e.g. [17], [6], [1] and the references cited there).

A short proof of the max algebra version of the generalized spectral radius theorem was given in [34]. More precisely, it was shown that

$$\mu(\Psi) = \lim_{t\to\infty} \rho(\Psi^{(t)})^{1/t} = \eta(\Psi). \tag{8}$$

Here $\Psi^{(t)}$ denotes the *Hadamard power of* Ψ for $t > 0$, i.e.,

$$\Psi^{(t)} = \{A^{(t)} : A \in \Psi\},$$

which is also a bounded set of $n \times n$ non-negative matrices. Also, $\rho(\Psi^{(t)})^{1/t}$ is decreasing in $t \in (0, \infty)$ and

$$\mu(\Psi) = \inf_{t\in(0,\infty)} \rho(\Psi^{(t)})^{1/t} \tag{9}$$

(see [34, Proposition 2.2]). The basic tool in [34] was the inequality

$$\mu(\Psi) \leq \rho(\Psi) \leq n\, \mu(\Psi) \tag{10}$$

(see [34, Proposition 2.3] and [27, Theorem 3(ii)]), which generalizes (2).

Let $\mathcal{K}$ denote the collection of all compact nonempty sets Σ of $n \times n$ complex matrices. The space $\mathcal{K}$ becomes a complete metric space if it is endowed with the usual Hausdorff metric defined by

$$H(\Sigma, \Gamma) = \max \left\{ \max_{A \in \Sigma} \mathrm{dist}(A, \Gamma), \max_{B \in \Gamma} \mathrm{dist}(B, \Sigma) \right\},$$

where $\mathrm{dist}(A, \Gamma) = \inf_{B \in \Gamma} \|A - B\|$. Note that the choice of a vector norm $\| \cdot \|$ on $\mathbb{C}^{n \times n}$ is irrelevant, since they are all equivalent (see e.g. [21, p. 272]). The following result is well known ([4], [20], [36, Lemma 3.5], [35]).

Theorem 1.1. *The generalized spectral radius $\rho(\Sigma)$ is continuous on $(\mathcal{K}, H)$.*

This result was applied to wavelets in [20] (in the case $\Sigma = \{A, B\}$). In [36] and [35] some additional results were proved.

Let (β, H) denote the closed metric subspace of $(\mathcal{K}, H)$ of all nonempty compact subsets of $n \times n$ non-negative matrices. The central result of [28] was the following max algebra version of Theorem 1.1.

Theorem 1.2. *The max version of the generalized spectral radius $\mu(\Psi)$ is continuous on (β, H).*

The main goal of this paper is to give a short proof of Theorem 1.2 by using Theorem 1.1.

2. The main results

The following observation is the key to our proof.

Theorem 2.1. *Let Ψ be a bounded set of $n \times n$ non-negative matrices. Then*

$$\mu(\Psi) = \sup_{t \in (0,\infty)} \left(n^{-1} \rho(\Psi^{(t)}) \right)^{1/t}. \tag{11}$$

Proof. Let $t > 0$. It is easy to see that $\mu(\Psi^{(t)}) = \mu(\Psi)^t$ (see e.g. the proof of [34, Theorem 2.4]). By (10) we have

$$n^{-1} \rho(\Psi^{(t)}) \le \mu(\Psi^{(t)}) = \mu(\Psi)^t.$$

Therefore it follows that

$$\sup_{t \in (0,\infty)} \left(n^{-1} \rho(\Psi^{(t)}) \right)^{1/t} \le \mu(\Psi).$$

On the other hand, we have by (8)

$$\mu(\Psi) = \lim_{t \to \infty} \rho(\Psi^{(t)})^{1/t} = \lim_{t \to \infty} \left(n^{-1} \rho(\Psi^{(t)}) \right)^{1/t} \le \sup_{t \in (0,\infty)} \left(n^{-1} \rho(\Psi^{(t)}) \right)^{1/t}.$$

This completes the proof. □

Corollary 2.2. *If $A \in \mathbb{R}_+^{n \times n}$, then*

$$\mu(A) = \sup_{t \in (0,\infty)} \left(n^{-1} \rho(A^{(t)}) \right)^{1/t}.$$

Remark 2.3. In (11) (and (9)) it suffices to take the supremum (infimum) over all $t \in \mathbb{N}$.

Let us recall that a function f from a metric space (X, d) into $\mathbb{R}$ is lower semi-continuous if and only if the sets $\{x \in X : f(x) > \alpha\}$ are open in (X, d) for all $\alpha \in \mathbb{R}$. It is well known that the supremum of a family of lower semi-continuous functions is lower semi-continuous. A function f is upper semi-continuous if and only if $-f$ is lower semi-continuous. Thus the infimum of a family of upper semi-continuous functions is upper semi-continuous. A function f is continuous if and only if it is both lower semi-continuous and upper semi-continuous.

Now, in view of (9), (11) and Theorem 1.1 we only need the following two results for the proof of Theorem 1.2.

Lemma 2.4. *If $\Psi \in \beta$ then $\Psi^{(t)} \in \beta$ for all $t > 0$.*

Proof. Let $t > 0$, $\Psi \in \beta$, $\varepsilon > 0$ and $\|\cdot\|_\infty$ a vector norm on $\mathbb{R}^{n\times n}$ defined by $\|A\|_\infty = \max_{i,j=1,\ldots,n} |a_{ij}|$. Since $\Psi^{(t)}$ is obviously nonempty and bounded, we only need to show that it is also a closed subset of $\mathbb{R}_+^{n\times n}$. To prove this, let $\{A_n\}_{n\in\mathbb{N}} \subset \Psi$ such that $\|A_n^{(t)} - B\|_\infty \to 0$ as $n \to \infty$ for some $B \in \mathbb{R}_+^{n\times n}$. If $M = \sup_{A\in\Psi} \|A^{(t)}\|_\infty$, then it is easy to see that $\|B\|_\infty < M + 1$. Since $x \mapsto x^{1/t}$ is an uniformly continuous function from the compact interval $[0, M + 1]$ to $[0, (M + 1)^{1/t}]$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|x^{1/t} - y^{1/t}| < \varepsilon$.

Let $C = B^{(1/t)}$ and thus $B = C^{(t)}$. Since there exists $n_0 \in \mathbb{N}$ such that $\|A_n^{(t)} - C^{(t)}\|_\infty < \delta$ for all $n \geq n_0$, we also have $\|A_n - C\|_\infty < \varepsilon$ for these n. Therefore $\|A_n - C\|_\infty \to 0$ as $n \to \infty$. Since Ψ is a closed subset of $\mathbb{R}_+^{n\times n}$, we have $C \in \Psi$ and thus $B \in \Psi^{(t)}$, which completes the proof. □

Lemma 2.5. *Let $t > 0$. The map $\Psi \mapsto \Psi^{(t)}$ is a homeomorphism from (β, H) onto (β, H).*

Proof. It suffices to prove that the map $\Psi \mapsto \Psi^{(t)}$ is continuous on (β, H), since the rest is obvious. To prove this, choose $\Psi \in \beta$ and $0 < \varepsilon < 1$. Let $K(\Psi, \varepsilon)$ denote the open ball in (β, H) with the center Ψ and the radius ε, i.e.,

$$K(\Psi, \varepsilon) = \{\Gamma \in \beta : H(\Psi, \Gamma) < \varepsilon\}.$$

If $M = \sup_{A\in\Psi} \|A\|_\infty$, then

$$\sup_{B\in\Gamma} \|B\|_\infty \leq M + \varepsilon < M + 1$$

for all $\Gamma \in K(\Psi, \varepsilon)$. Similarly as in the proof of Lemma 2.4 there exists $\delta_1 > 0$ such that $|x - y| < \delta_1$ and $x, y \in [0, M + 1]$ imply $|x^t - y^t| < \varepsilon$.

Let $\delta = \min\{\varepsilon, \delta_1\}$, $\Gamma \in K(\Psi, \delta)$, $A \in \Psi$ and $B \in \Gamma$. Then there exist $C \in \Gamma$ and $D \in \Psi$ such that $\|A - C\|_\infty < \delta$ and $\|B - D\|_\infty < \delta$. Since $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in [0, M + 1]$ for all $i, j = 1, \ldots, n$, we have that $\|A^{(t)} - C^{(t)}\|_\infty < \varepsilon$ and $\|B^{(t)} - D^{(t)}\|_\infty < \varepsilon$. This implies $H(\Psi^{(t)}, \Gamma^{(t)}) < \varepsilon$, which completes the proof. □

Having all the preliminaries prepared it is now easy to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.5 and Theorem 1.1 the function $\Psi \mapsto \rho(\Psi^{(t)})$ is continuous on (β, H) for every $t > 0$. Therefore $\mu(\Psi)$ is upper semi-continuous on (β, H) by (9) and it is lower semi-continuous on (β, H) by (11). This completes the proof. □

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