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ON A CONJECTURE ABOUT WIENER INDEX IN ITERATED LINE GRAPHS OF TREES

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On a conjecture about Wiener index in iterated line graphs of trees

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Abstract

Let G be a graph. Denote by $L^i(G)$ its i-iterated line graph and denote by $W(G)$ its Wiener index. There is a conjecture which claims that there exists no nontrivial tree T and $i \geq 3$, such that $W(L^i(T)) = W(T)$, see [5]. We prove this conjecture for trees which are not homeomorphic to the claw $K_{1,3}$ and the graph of letter H.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. For any two of its vertices, say u and v, by $d(u, v)$ we denote the distance from u to v in G. The *Wiener index* of G, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of G. Wiener index was introduced by Wiener in 1947, see [15]. In the next decades, it was intensively studied by chemists, as it is related to many physical properties of organical molecules, see [9]. Graph theoretists reintroduced this parameter as the distance in 1970 and transmission in 1984, see [6] and [14], respectively. Recently, graph theoretic aspects of Wiener index are intensively studied, see e.g. [7] and [8], or surveys [3] and [4].

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By the definition, if G has a unique vertex, i.e., if $G = K_1$, then $W(G) = 0$. In this case we say that the graph G is *trivial*. We set $W(G) = 0$ also when the set of vertices (and hence also the set of edges) of G is empty.

The line graph of G, $L(G)$, has vertex set identical with the set of edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0 \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In [1], the following theorem was proved.

Theorem 1.1 [1] *If T is a tree on n vertices, then $W(L(T)) = W(T) - \binom{n}{2}$.*

Since $\binom{n}{2} > 0$ if $n \geq 2$, there is no nontrivial tree for which $W(L(T)) = W(T)$. However, there are trees T satisfying $W(L^2(T)) = W(T)$, see for example [2]. In [5] the following conjecture was posed (see also [3]).

Conjecture 1.2 [2] *Let T be a nontrivial tree and $i \geq 3$. Then $W(L^i(T)) \neq W(T)$.*

Denote by P_n a path on n vertices. If $n \geq 2$ then $W(P_n) > W(P_{n-1})$. As $L(P_n) = P_{n-1}$ if $n \geq 2$, while $L(P_1)$ is an empty graph, we have $W(L^i(P_n)) < W(P_n)$ for every $i \geq 1$ if P_n is a nontrivial path. Hence, Conjecture 1.2 is trivially true for paths of length at least 1.

In [11] we prove that for every graph G the function $W(L^i(G))$ is convex in variable i. Hence, the following corollary is a straightforward consequence of this statement.

Corollary 1.3 *Let T be a tree such that $W(L^3(T)) > W(T)$. Then for every $i \geq 3$ we have $W(L^i(T)) > W(T)$.*

Let G be a graph. A *ray* R' in G is a (directed) path, the first vertex of which has degree at least 3, its last vertex has degree 1, and all of its internal vertices (if any exist) have degree 2 in G. Observe that if R' has length t, $t \geq 2$, then the edges of R' correspond to vertices of a ray R in $L(G)$ of length $t - 1$. In [11] we have the following theorem.

Theorem 1.4 [11] *Let T be a tree, all rays of which have length 1, distinct from a path and the claw $K_{1,3}$. Then $W(L^3(T)) > W(T)$.*

Here we extend this statement to trees with arbitrarily long rays. Denote by H a tree on 6 vertices, two of which have degree 3 and four of which have degree 1. (That is, H is a graph which "looks" like the letter H.) The main result of this paper is the following theorem.

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Theorem 1.5 *Let T be a tree, not homeomorphic to a path, claw $K_{1,3}$ and H. Then $W(L^3(T)) > W(T)$.*

Recall that graphs G_1 and G_2 are homeomorphic if and only if the graphs obtained from them by repeatedly removing a vertex of degree 2 (and making its two neighbours adjacent) are isomorphic. Combining Corollary 1.3 and Theorem 1.5 we obtain the following corollary, which proves Conjecture 1.2 for the trees T satisfying the assumption in Theorem 1.5.

Corollary 1.6 *Let T be a tree, not homeomorphic to a path, claw $K_{1,3}$ and H. Then $W(L^i(T)) > W(T)$ for every $i \geq 3$.*

We remark that trees homeomorphic to the claw $K_{1,3}$ and the graph H are considered in forthcoming papers, see [12, 13].

For a tree T, denote $D(T) = W(L^3(T)) - W(T)$. We prove $D(T) > 0$ by induction on the length of the longest ray in T. By Theorem 1.4, $D(T) > 0$ if the longest ray has length 1. Now we describe the induction step:

We suppose that $D(T) > 0$ for all trees, rays of which have length at most $l+1$, and we like to extend this statement to trees with rays of length at most $l+2$. Let a' be the last vertex of a ray of length $l+1$ in T, $l \geq 0$. Since we extend only one ray in turn, namely the ray terminating at a', we assume that all rays of T have lengths at most $l+2$. Add to T one new vertex b' and the edge $a'b'$, and denote the resulting tree by T^*. Denote by a the edge of T containing a' and denote by b the edge $a'b'$. Then ab is an edge of $L(T^*)$ and the degree of b is 1 in $L(T^*)$. Moreover, a is an endvertex of a ray of length l in $L(T)$ and b is an endvertex of a ray of length $l+1$ in $L(T^*)$. By the assumption, all rays of $L(T)$ have lengths at most $l+1$. Define

$$\Delta T = D(T^*) - D(T).$$

In the next section we present an exact formula for ΔT. In section 3 we prove $\Delta T \geq 0$ and this will establish Theorem 1.5 (for more detailed explanation see the proof of Theorem 1.5 below).

Now we introduce notation used throughout the paper. For any set of vertices S and a single vertex z, by $S \setminus \{z\}$ we denote the set $S - \{z\}$. Since we work repeatedly with line graphs of trees, to simplify the notation define $LG = L(G)$ for arbitrary graph G. If z is a vertex, its degree is denoted by d_z. If there are more graphs containing the vertex z, then d_z denotes the degree of z in LT. Analogously, by $d(z, w)$ we denote the distance from z to w, and this distance is preferably considered in LT. A path starting at u and terminating at v is denoted by $u - v$.

2 Preliminaries

Analogously as vertex of $L(G)$ corresponds to an edge of G, vertex of $L^2(G)$ corresponds to a path of length two in G. For $x \in V(L^2(G))$ we denote by $B_2(x)$ the

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corresponding path in G. For two subgraphs S_1 and S_2 of G, by $d(S_1, S_2)$ we denote the shortest distance in G between a vertex of S_1 and a vertex of S_2. If S_1 and S_2 share s edges, then we set $d(S_1, S_2) = -s$.

Let x and y be two vertices of $L^2(G)$, such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Then $d_{L^2(G)}(x, y) = d(B_2(x), B_2(y)) + 2$, see [10, 11].

Let $u, v \in V(G)$, $u \neq v$. Denote by $\beta_i(u, v)$ the number of pairs $x, y \in V(L^2(G))$, with u being the center of $B_2(x)$ and v being the center of $B_2(y)$, such that $d(B_2(x), B_2(y)) = d(u, v) - 2 + i$. Since $d(u, v) - 2 \leq d(B_2(x), B_2(y)) \leq d(u, v)$, we have $\beta_i(u, v) = 0$ for all $i \notin \{0, 1, 2\}$. In [11] we have the following statement.

Proposition 2.1 *Let G be a connected graph. Then*

$$\begin{aligned} W(L^2(G)) &= \sum_{u \neq v} \left[\binom{d_u}{2}\binom{d_v}{2} d(u, v) + 0\beta_0(u, v) + 1\beta_1(u, v) + 2\beta_2(u, v) \right] \\ &+ \sum_{u} \left[3\binom{d_u}{3} + 6\binom{d_u}{4} \right], \end{aligned}$$

where the first sum runs through all unordered pairs $u, v \in V(G)$ and the second one runs through all $u \in V(G)$.

We apply Proposition 2.1 to line graphs of trees. Let us recall the structure of these graphs. For any tree F, the graph LF consists of cliques in the following sense: Denote by $\mathcal{C}(LF)$ the set of maximal cliques of LF. Then every vertex of LF belongs to at most two cliques from $\mathcal{C}(LF)$; each pair of cliques from $\mathcal{C}(LF)$ intersects in at most one vertex; and the cliques of $\mathcal{C}(LF)$ have a "tree structure", i.e., there are no cliques $C_0, C_1, \ldots, C_{t-1}$, $t \geq 3$, such that C_i and C_{i+1} have nonempty intersection, $0 \leq i \leq t - 1$, the addition being modulo t.

We start with an exact formula for ΔT. For $u \in V(LT) \setminus \{a\}$ define

$$h_{LT}(u) = \left(\binom{d_u}{2} d_a - 1 \right) d(u, a) + (d_u - 1)\left(d_u\, d_a - d_a - \frac{1}{2} d_u \right) - 2 - \phi(u, a), \quad (1)$$

where

$$\phi(u, a) = \begin{cases} (d_a - 1)(d_u - 2) & \text{if } d(u, a) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2 *For a nontrivial tree, the following equality holds:*

$$\Delta T = \sum_{u} h_{LT}(u) + \frac{1}{2} d_a\, (d_a - 1)(2d_a - 1) - 3,$$

where the sum is taken over all vertices $u \in V(LT) \setminus \{a\}$.

Proof Let F be a tree and let u and v be distinct vertices of LF. Consider vertices $x, y \in V(L^2(LF))$ such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Due to the clique structure of LF, there is a unique shortest $u-v$ path in LF. Denote this path by $u = a_0, a_1, \ldots, a_t = v$. If $d(B_2(x), B_2(y)) = d(u, v) - 2$, then we must have $a_1 \in V(B_2(x))$ and $a_{t-1} \in V(B_2(y))$. There are $(d_u - 1)$ ways to choose the other endvertex of $B_2(x)$, and there are $(d_v - 1)$ ways to choose the other endvertex of $B_2(y)$. Hence, $\beta_0(u, v) = (d_u - 1)(d_v - 1)$.

Now we find $\beta_1(u, v)$. We distinguish two cases: $d(u, v) \geq 2$ and $d(u, v) = 1$.

Suppose first $d(u, v) \geq 2$. In this case u and v do not belong to a common clique from $\mathcal{C}(LF)$. If $d(B_2(x), B_2(y)) = d(u, v) - 1$, then either $a_1 \in V(B_2(x))$ or $a_{t-1} \in V(B_2(y))$, but not both. In the first case we obtain $(d_u - 1)\binom{d_v - 1}{2}$ pairs x, y and in the second $\binom{d_u - 1}{2}(d_v - 1)$ pairs x, y. Thus,

$$\beta_1(u, v) = (d_u - 1)\binom{d_v - 1}{2} + \binom{d_u - 1}{2}(d_v - 1).$$

Suppose now that $d(u, v) = 1$. In this case u and v belong to a common clique. All pairs x, y mentioned in the previous case contribute to $\beta_1(u, v)$, but we have to add pairs x, y such that $v \notin V(B_2(x))$, $u \notin V(B_2(y))$ and $d(B_2(x), B_2(y)) = d(u, v) - 1 = 0$. For these pairs the paths $B_2(x)$ and $B_2(y)$ share at least one of their endvertices. Denote by $\alpha_{LF}(u, v)$ the number of these extra pairs. Then

$$\beta_1(u, v) = (d_u - 1)\binom{d_v - 1}{2} + \binom{d_u - 1}{2}(d_v - 1) + \alpha_{LF}(u, v).$$

Since we do not need to evaluate $\alpha_{LF}(u, v)$ in general, we postpone this computation until later.

We have $\binom{d_u}{2}\binom{d_v}{2}$ pairs $x, y \in V(L^2(LF))$, such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Since

$$\begin{aligned}\binom{d_u}{2}\binom{d_v}{2} &= (d_u - 1)(d_v - 1) + (d_u - 1)\binom{d_v - 1}{2} + \binom{d_u - 1}{2}(d_v - 1) \\ &\quad + \binom{d_u - 1}{2}\binom{d_v - 1}{2},\end{aligned}$$

we have $\beta_2(u, v) = \binom{d_u - 1}{2}\binom{d_v - 1}{2} - \alpha_{LF}(u, v)$. By Proposition 2.1

$$\begin{aligned}W(L^2(LF)) &= \sum_{u \neq v} \left[\binom{d_u}{2}\binom{d_v}{2} d(u, v) + (d_u - 1)\binom{d_v - 1}{2}\right. \\ &\quad \left. + \binom{d_u - 1}{2}(d_v - 1) + 2\binom{d_u - 1}{2}\binom{d_v - 1}{2} - \alpha_{LF}(u, v)\right] \\ &\quad + \sum_{u} \left[3\binom{d_u}{3} + 6\binom{d_u}{4}\right]. \qquad (2)\end{aligned}$$

Now we evaluate $W(L^3(T^*)) - W(L^3(T)) = W(L^2(LT^*)) - W(L^2(LT))$; see the notation below Corollary 1.6. The graph LT^* has one more vertex than LT, namely the vertex b of degree 1, and the degree of a increased by 1 to d_a+1 in LT^*. Therefore, all the terms of (2) for pairs u, v which do not contain neither a nor b, cancell out in $W(L^2(LT^*)) - W(L^2(LT))$. But we need to subtract the terms for pairs u, a in LT and to add the terms for pairs u, a in LT^*, $u \in V(LT) \setminus \{a\}$. We can ignore the terms containing b in LT^*, as the degree of b is 1, so that b cannot be a center of $B_2(y)$ for any $y \in V(L^2(LT^*))$. (Observe that all terms of (2) are 0 if one of the vertices has degree 1.) As regards the second sum in (2), we have to subtract the term corresponding to a in LT and add the terms corresponding to a and b in LT^*, the later one being 0 as the degree of b is 1 in LT^*. Denote by $\Delta\alpha(u, v)$ the difference $\alpha_{LT^*}(u, v) - \alpha_{LT}(u, v)$ and denote by ΔWL^2 the difference $W(L^2(LT^*)) - W(L^2(LT))$. By (2) we have

$$\begin{aligned}
\Delta WL^2 &= -\sum_u \Bigg[\binom{d_u}{2}\binom{d_a}{2} d(u,a) + (d_u-1)\binom{d_a-1}{2} + \\
&\quad + \binom{d_u-1}{2}(d_a-1) + 2\binom{d_u-1}{2}\binom{d_a-1}{2} - \alpha_{LT}(u,a)\Bigg] \\
&\quad + \sum_u \Bigg[\binom{d_u}{2}\binom{d_a+1}{2} d(u,a) + (d_u-1)\binom{d_a}{2} + \\
&\quad + \binom{d_u-1}{2} d_a + 2\binom{d_u-1}{2}\binom{d_a}{2} - \alpha_{LT^*}(u,a)\Bigg] \\
&\quad -3\binom{d_a}{3} - 6\binom{d_a}{4} + 3\binom{d_a+1}{3} + 6\binom{d_a+1}{4} \\
&= \sum_u \Bigg[\binom{d_u}{2} d_a\, d(u,a) + (d_u-1)(d_a-1) \\
&\quad + \binom{d_u-1}{2} + 2\binom{d_u-1}{2}(d_a-1) - \Delta\alpha(u,v))\Bigg] \\
&\quad + \frac{1}{4} d_a\,(d_a-1)\Big[-2(d_a-2) - (d_a-2)(d_a-3) \\
&\quad + 2(d_a+1) + (d_a+1)(d_a-2)\Big] \\
&= \sum_u \Bigg[\binom{d_u}{2} d_a\, d(u,a) + (d_u-1)\Big(d_u\, d_a - d_a - \frac{1}{2} d_u\Big) - \Delta\alpha(u,a)\Bigg] \\
&\quad + \frac{1}{2} d_a (d_a-1)(2d_a-1). \qquad (3)
\end{aligned}$$

Now we determine $\Delta\alpha(u, a)$. For $u \in V(LT) \setminus \{a\}$, the distance from u to a in LT is the same as in LT^*. Therefore $\Delta\alpha(u, a) = \alpha_{LT^*}(u, a) - \alpha_{LT}(u, a) = 0 - 0 = 0$ if $d(u, a) \geq 2$. If $d(u, a) = 1$ then in $\alpha_{LT^*}(u, a) - \alpha_{LT}(u, a)$ we count pairs x, y such that $b \in V(B_2(y))$. Denote by C the clique of $\mathcal{C}(LT)$ containing both a and u. The order of C is $d_a + 1$. We distinguish two cases.

- *Both endvertices of* $B_2(x)$ *are in* C: We have $\binom{d_a-1}{2}$ choices for $B_2(x)$ in this case as $a \notin V(B_2(x))$. For each of these choices there are two choices for $B_2(y)$ such that $B_2(x)$ and $B_2(y)$ share an endvertex and $b \in V(B_2(y))$. Hence, there are $2\binom{d_a-1}{2}$ pairs x, y contributing to $\Delta\alpha(u, v)$ in this case.

- *Only one endvertex of* $B_2(x)$ *is in* C: For this vertex we have $d_a - 1$ choices, as $a \notin V(B_2(x))$, and for the other endvertex of $B_2(x)$ we have $d_u - d_a$ choices. In this case, to every x there is a unique y such that $B_2(x)$ and $B_2(y)$ share an endvertex and $b \in V(B_2(y))$. Hence, there are $(d_a - 1)(d_u - d_a)$ pairs x, y contributing to $\Delta\alpha(u, v)$ in this case.

Thus,

$$\Delta\alpha(u, v) = 2\binom{d_a - 1}{2} + (d_a - 1)(d_u - d_a) = (d_a - 1)(d_u - 2) = \phi(u, a). \qquad (4)$$

Now we evaluate $W(T^*) - W(T)$. If F is a tree with n_0 vertices, then $W(LF) = W(F) - \binom{n_0}{2}$, by Theorem 1.1. Denote by n_1 the number of vertices of LF. Since $n_1 = n_0 - 1$, we have $W(F) = W(LF) + \binom{n_1+1}{2}$. Denote by n the number of vertices of LT. Then

$$\begin{aligned} W(T^*) - W(T) &= W(LT^*) + \binom{n+2}{2} - W(LT) - \binom{n+1}{2} \\ &= W(LT^*) - W(LT) + n + 1. \end{aligned}$$

In $W(LT^*) - W(LT)$, all terms for pairs u, v which do not contain b will cancell out. Therefore

$$\begin{aligned} W(T^*) - W(T) &= \sum_u d(u, b) + d(a, b) + n + 1 \\ &= \sum_u \Big(d(u, a) + 1\Big) + 1 + \sum_u 1 + 2 \\ &= \sum_u \Big(d(u, a) + 2\Big) + 3. \end{aligned} \qquad (5)$$

where the sum goes once again through $n - 1$ vertices $u \in V(LT) \setminus \{a\}$.

Since $\Delta T = D(T^*) - D(T) = W(L^3(T^*)) - W(T^*) - W(L^3(T)) + W(T) = \Delta W L^2 - (W(T^*) - W(T))$, combining (3), (4) and (5) we obtain the required result. □

3 Proof of Theorem 1.5

We prove that $\Delta T \geq 0$ for every tree T which is not homeomorphic to a path, claw $K_{1,3}$ or the graph H. Let l, a', b', a, b, T^* and ΔT be as in the discussion following Corollary 1.6. As explained there, we proceed by induction on l.

First we prove $\Delta T \geq 0$ for the case $l = 0$. In this case a' is adjacent to a vertex of degree at least 3 in T, so that in LT we have $d_a \geq 2$.

Let v be an endvertex of a ray R in LT, i.e., $d_v = 1$. By $\overline{v}$ we denote the first vertex of R, i.e., a vertex at shortest distance to v whose degree is at least 3. Due to the clique structure of LT described below Proposition 2.1, we have:

Observation 3.1 *If u and v are distinct vertices of degree 1 in LT, then $\overline{u} \neq \overline{v}$.*

We use Obseravtion 3.1 repeatedly in the following proofs.

Lemma 3.2 *Let T be a tree different from a path, in which all rays have length at most $l+2$, and let $l = 0$. Then $\Delta T \geq 0$.*

Proof We find a lower bound for $\sum_u h_{LT}(u)$. Consider four cases.

- $d_u = 1$: Then $d(u,a) > 1$, so that $h_{LT}(u) = -d(u,a) - 2$ by (1).
- $d_u = 2$: Since $(d_a - 1)(d_u - 2) = 0$, we have $\phi(u,a) = 0$ also in this case. By (1) we have

  $$h_{LT}(u) = (d_a - 1)d(u,a) + d_a - 3 \geq d_a - 1 + d_a - 3 = 2d_a - 4 \geq 0$$

  as $d_a \geq 2$.
- $d_u \geq 3$ *and* $d(u,a) \geq 2$: By (1) we have

  $$\begin{aligned} h_{LT}(u) &= \left(\binom{d_u}{2} d_a - 1\right) d(u,a) + (d_u - 1)\Big(d_u\, d_a - d_a - \frac{1}{2} d_u\Big) - 2 \\ &\geq 5d(u,a) + (d_u - 1)\frac{1}{2}\Big[d_a\,(d_u - 2) + d_u\,(d_a - 1)\Big] - 2 \\ &\geq 5d(u,a) + 5 - 2 \\ &\geq d(u,a) + 11 \end{aligned}$$

  as $d_u \geq 3$, $d_a \geq 2$ and $d(u,a) \geq 2$.
- $d_u \geq 3$ *and* $d(u,a) = 1$: By (1) we have

  $$\begin{aligned} h_{LT}(u) &= \left(\binom{d_u}{2} d_a - 1\right) d(u,a) + (d_u - 1)\Big(d_u\, d_a - d_a - \frac{1}{2} d_u\Big) - 2 \\ &\quad - (d_a - 1)(d_u - 2) \end{aligned}$$

$$\begin{aligned} &\geq 5d(u,a) + {d_u}^2 d_a - \frac{1}{2}{d_u}^2 - 3d_u\, d_a + \frac{3}{2}d_u + 3d_a - \frac{3}{2} - \frac{5}{2} \\ &= 5d(u,a) + \frac{1}{2}\Big[(2d_a-1)\Big(d_u(d_u-3)+3\Big) - 5\Big] \\ &\geq d(u,a) + 6 \end{aligned}$$

as $d_u \geq 3$, $d_a \geq 2$ and $d(u,a) = 1$.

Hence,

$$h_{LT}(u,a) \geq \begin{cases} -d(u,a) - 2 & \text{if } d_u = 1, \\ 0 & \text{if } d_u = 2, \\ d(u,a) + 6 & \text{if } d_u \geq 3. \end{cases} \tag{6}$$

Since $l = 0$, all rays of T have length at most 2, so that all rays of LT have length at most 1. Hence, if $d_u = 1$ then $d(u,\overline{u}) = 1$ in LT. Thus,

$$h_{LT}(u) + h_{LT}(\overline{u}) \geq -d(u,a) - 2 + d(\overline{u},a) + 6 = -d(\overline{u},a) - 3 + d(\overline{u},a) + 6 \geq 0.$$

Denote by V_1 the set of vertices of degree 1 in $V(LT) \setminus \{a\}$. By Observation 3.1, $\overline{u} \neq \overline{v}$ whenever $u, v \in V_1$, $u \neq v$. Hence, by (6) we have

$$\sum_u h_{LT}(u) \geq \sum_{u \in V_1} \Big(h_{LT}(u) + h_{LT}(\overline{u})\Big) \geq 0.$$

As $d_a \geq 2$, we have $\frac{1}{2}d_a\,(d_a-1)(2d_a-1) \geq 3$, so that

$$\Delta T = \sum_u h_{LT}(u) + \frac{1}{2}d_a\,(d_a-1)(2d_a-1) - 3 \geq 0,$$

by Proposition 2.2. □

Now we prove $\Delta T \geq 0$ for $l \geq 1$, i.e., from now on we consider $l \geq 1$. In this case $\phi(u,a) = 0$ as $d_a = 1$, which simplifies $h_{LT}(u)$, see (1). The problem is that $h_{LT}(u) < 0$ even if $d_u = 2$, so that we need more thight estimations. We prove $\Delta T \geq 0$ by induction on the number of vertices of degree at least 3 in T.

Let G be a graph. A path of length at least one in G is *interior path* if its endvertices have degrees both at least 3, its interior vertices (if any) have degrees 2 in G, and its edges are bridges of G. In the next lemma we show that it suffices to prove $\Delta T \geq 0$ for trees whose interior paths have lengths at most 2, i.e., we reduce the class of trees for which we need to prove $\Delta T \geq 0$.

Lemma 3.3 *Let T^s be obtained from T by subdividing one edge of an interior path of length t, $t \geq 2$, and let $l \geq 1$. Then $\Delta T^s \geq \Delta T$.*

Proof Denote by P' the interior path of T, whose edge was subdivided to obtain T^s. Since P' has length $t \geq 2$, the edges of P' form an interior path P of length $t-1 \geq 1$ in LT. Obviously, LT^s can be obtained from LT by subdividing one edge of P. Denote by e the endvertex of P, which has among the vertices of P the greatest distance from a. Let LT^s be obtained from LT by subdividing that edge of P which is incident to e. Denote the new vertex by w. Observe that for every vertex $u \in V(LT)$, the degree of u in LT is the same as its degree in LT^s.

Since the degree of a is the same in LT^s as in LT, namely 1, by Proposition 2.2 it suffices to show that $\sum_{u \in V(LT^s)\setminus\{a\}} h_{LT^s}(u) \geq \sum_{u \in V(LT)\setminus\{a\}} h_{LT}(u)$. We distinguish three cases.

- *u is a vertex of LT, such that e does not lay on $u-a$ path in LT*: Then $d_{LT^s}(u,a) = d_{LT}(u,a)$, so that $h_{LT^s}(u) = h_{LT}(u)$ and $h_{LT^s}(u) - h_{LT}(u) = 0$, see (1).
- *u is a vertex of LT, such that e lays on $u-a$ path in LT*: Then $d_{LT^s}(u,a) = d_{LT}(u,a)+1$, so that $h_{LT^s}(u) - h_{LT}(u) = \binom{d_u}{2} - 1$ as $d_a = 1$, see (1). Thus, $h_{LT^s}(u) - h_{LT}(u) = -1$ if $d_u = 1$, $h_{LT^s}(u) - h_{LT}(u) = 0$ if $d_u = 2$ and $h_{LT^s}(u) - h_{LT}(u) \geq 2$ if $d_u \geq 3$.
- *$u = w$*: As the degree of w is 2 in LT^s, we have $h_{LT^s}(w) = -2$, by (1).

Every vertex u of degree 1 in LT is an endvertex of a ray starting at vertex $\overline{u}$ of degree at least 3. By Observation 3.1, if u and v are distinct vertices of degree 1 in LT, then $\overline{u} \neq \overline{v}$. Denote by V_e the set of vertices u of LT such that $d_u = 1$ and e lays on $u-a$ path. Observe that $e \neq \overline{u}$ for any $u \in V_e$.

Denote $\Delta h = \sum_{u \in V(LT^s)\setminus\{a\}} h_{LT^s}(u) - \sum_{u \in V(LT)\setminus\{a\}} h_{LT}(u)$. By the analysis above only vertices of $V_e \cup \{w\}$ contribute by negative numbers to Δh. Therefore

$$
\begin{aligned}
\Delta h &\geq \sum_{u \in V_e} \left[\Big(h_{LT^s}(u) - h_{LT}(u) \Big) + \Big(h_{LT^s}(\overline{u}) - h_{LT}(\overline{u}) \Big) \right] \\
&\quad + \Big(h_{LT^s}(e) - h_{LT}(e) \Big) + h_{LT^s}(w) \\
&\geq \sum_{u \in V_e} (-1+2) + 2 - 2 \geq 0.
\end{aligned}
$$

Hence, $\Delta T^s \geq \Delta T$. □

Let F be a tree, such that T is a subgraph of F and the degree of a' is 1 in F. Denote by S_{LF} the set of first edges of rays of F. Then S_{LF} is also a set of vertices of LF. These vertices have degrees at least 3, with the exception when the corresponding edge is incident to vertices of degrees 1 and 3 in F. Let $u \in S_{LF}$. If there is a ray in LF starting at u, then denote by $R_{LF}(u)$ the set of vertices (other than a) of this ray; otherwise set $R_{LF}(u) = \{u\}$. Since $l \geq 1$, there is a ray in LF starting at $\overline{a}$, so that the vertex a is not in $R_{LF}(v)$ for any $v \in S_{LF}$. Observe also that $R_{LF}(u) \cap R_{LF}(v) = \emptyset$ whenever $u, v \in S_{LF}$, $u \neq v$.

Lemma 3.4 *Let F be a tree, rays of which have length at most $l+2$, $l \geq 1$. Moreover, let T be a subgraph of F and let the degree of a' is 1 in F. Let $c \in S_{LF}$ be a vertex of a clique from $\mathcal{C}(LF)$ of order $r \geq 3$. Then*

$$\sum_{u \in R_{LF}(c)} h_{LF}(u) \geq \begin{cases} \left(\binom{r}{2} - 3\right)l + \binom{r-1}{2} & \text{if } c = \overline{a} \\ \left(\binom{r-1}{2} - 1\right)d(c,a) + \binom{r-2}{2} - 2 & \text{if } c \neq \overline{a} \text{ and } |R_{LF}(c)| = 1 \\ \left(\binom{r}{2} - 2\right)d(c,a) - 3l + \binom{r-1}{2} - 5 & \text{if } c \neq \overline{a} \text{ and } |R_{LF}(c)| \geq 2. \end{cases}$$

Proof We distinguish three cases.

- $c = \overline{a}$: Then $R_{LF}(c)$ has one vertex of degree r, namely c with $d(c,a) = l$, and $l-1$ vertices of degree 2. As the degree of a is 1, by (1) we have

$$\begin{aligned} \sum_{u \in R_{LF}(c)} h_{LF}(u) &= \left(\binom{r}{2} - 1\right)d(c,a) + (r-1)\left(\frac{r-2}{2}\right) - 2 + (l-1)(-2) \\ &= \left(\binom{r}{2} - 3\right)l + \binom{r-1}{2}. \end{aligned}$$

- $c \neq \overline{a}$ *and* $|R_{LF}(c)| = 1$: As the degree of c is $r-1$, by (1) we have

$$\sum_{u \in R_{LF}(c)} h_{LF}(u) = h_{LF}(c) = \left(\binom{r-1}{2} - 1\right)d(c,a) + (r-2)\left(\frac{r-3}{2}\right) - 2.$$

- $c \neq \overline{a}$ *and* $|R_{LF}(c)| \geq 2$: Then $R_{LF}(c)$ has one vertex of degree r, namely c, one vertex of degree 1 at distance at most $d(c,a) + l + 1$ from a and at most l vertices of degree 2 as all rays of LF have length at most $l+1$. By (1) we have

$$\begin{aligned} \sum_{u \in R_{LF}(c)} h_{LF}(u) &\geq \left(\binom{r}{2} - 1\right)d(c,a) + (r-1)\left(\frac{r-2}{2}\right) - 2 \\ &\quad - (d(c,a) + l + 1) - 2 + l(-2) \\ &= \left(\binom{r}{2} - 2\right)d(c,a) - 3l + \binom{r-1}{2} - 5. \end{aligned}$$

□

Before we state the lemmas necessary for the basis of induction, we give the proof of induction step. I.e., we prove that if $\Delta T^h \geq 0$ for every tree T^h homeomorphic to T, rays of which have lengths at most $l+2$, then $\Delta T^{gh} \geq 0$ for all trees T^{gh} homeomorphic to T^g, rays of which have lengths at most $l+2$, where T^g is obtained from T by inserting a star at the end of one ray of T (of course, we cannot attach this star on a').

Let R' be a ray of T which does not terminate at a'. Remove R' from T and replace it by a path PR' of length i, $1 \leq i \leq 2$. Denote by c' the vertex of degree 1 in PR'. Now attach to c' exactly $j-1$ rays, each of length at most $l+2$, and denote the resulting graph by $T_{i,j}$, $j \geq 3$. In the next two lemmas we prove that $\Delta T_{i,j} \geq 0$.

Lemma 3.5 *Suppose that $\Delta T^h \geq 0$ for all trees homeomorphic to T, rays of which have lengths at most $l+2$, $l \geq 1$. Then $\Delta T_{i,3} \geq 0$, $1 \leq i \leq 2$.*

Proof Since $\Delta T^h \geq 0$ for all trees homeomorphic to T, rays of which have length at most $l+2$, we may assume that the length of R' is exactly $l+2$, $l \geq 1$. Then the edges of R' form a ray R in LT of length $l+1$. Denote by e the first vertex of R. By (1) we have

$$\sum_{u \in R(e) \setminus \{e\}} h_{LT}(u) = -2l - (d(e,a) + l + 1) - 2 = -d(e,a) - 3l - 3$$

as $R(e)$ has l vertices of degree 2 and one vertex of degree 1 at distance $d(e,a)+l+1$ from a. We distinguish two cases.

- $i = 1$: Then PR' has length 1 and the unique edge of PR' corresponds to the vertex e in $LT_{1,3}$. In $LT_{1,3}$ the degree of e is $d_e + 3 - 2 = d_e + 1$ as e is in two cliques from $\mathcal{C}(LT_{1,3})$, one of them has order d_e and the other one has order 3. Denote by c any one of the other two vertices of this clique of order 3. Since $d(c,a) \geq l+2$, we have $d(c,a) - 3l - 4 \geq -2l - 2$. Hence, by Lemma 3.4

$$\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq \begin{cases} -2 & \text{if } |R_{LT_{1,3}}(c)| = 1 \\ -2l-2 & \text{if } |R_{LT_{1,3}}(c)| \geq 2. \end{cases}$$

As $-2l-2 \leq -2$, we have $\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq -2l - 2$.

Denote

$$\Delta h = \sum_{u \in V(LT_{1,3}) \setminus \{a\}} h_{LT_{1,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).$$

In Δh all terms cancell out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing e, the vertex e itself, and the vertices of $R(e) \setminus \{e\}$. By (1) we have

$$\begin{aligned}
\Delta h &\geq 2(-2l-2) + \left(\binom{d_e+1}{2} - 1\right) d(e,a) + d_e \left(\frac{d_e+1}{2} - 1\right) - 2 \\
&\quad - \left(\binom{d_e}{2} - 1\right) d(e,a) - (d_e - 1)\left(\frac{d_e}{2} - 1\right) + 2 + (d(e,a) + 3l + 3) \\
&\geq (d_e+1)d(e,a) + (d_e - 1) - l - 1 \\
&\geq 4d(e,a) - l + 1 \geq 0
\end{aligned}$$

as $d_e \geq 3$ and $d(e,a) \geq l+1$. By Proposition 2.2, $\Delta T_{1,3} - \Delta T = \Delta h \geq 0$, so that $\Delta T_{1,3} \geq \Delta T \geq 0$.

- $i = 2$: Then PR' has length 2. One edge of PR' corresponds to e, while the other corresponds to a vertex of degree 3, say f, in $LT_{2,3}$. Observe that the degree of e is d_e in $LT_{2,3}$ and the degree of f is 3 in $LT_{2,3}$. Analogously as in the previous case, denote by c any one of the two vertices of the triangle containing f, $c \neq f$. Since $d(c,a) = d(e,a) + 2 \geq l+3$, we have $d(c,a) - 3l - 4 \geq -2l - 1$. Hence, by Lemma 3.4

$$\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq \begin{cases} -2 & \text{if } |R_{LT_{2,3}}(c)| = 1 \\ -2l-1 & \text{if } |R_{LT_{2,3}}(c)| \geq 2. \end{cases}$$

As $l \geq 1$ we have $-2l - 1 \leq -2$, so that $\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq -2l - 1$. Denote

$$\Delta h = \sum_{u \in V(LT_{2,3}) \setminus \{a\}} h_{LT_{2,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).$$

In Δh all terms cancell out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing f, the vertex f itself, and the vertices of $R(e) \setminus \{e\}$. By (1) we have

$$\begin{aligned}
\Delta h &\geq 2(-2l-1) + (2d(f,a) - 1) + (d(e,a) + 3l + 3) \\
&\geq 3d(e,a) - l + 2 \geq 0
\end{aligned}$$

as $d(f,a) = d(e,a) + 1$ and $d(e,a) \geq l+1$. By Proposition 2.2, $\Delta T_{2,3} - \Delta T = \Delta h \geq 0$, so that $\Delta T_{2,3} \geq \Delta T \geq 0$.

In both cases we have $\Delta T_{i,3} \geq 0$, which completes the proof. □

Now we extend the previous lemma to trees $T_{i,j}$ with higher j.

Lemma 3.6 *Suppose that $\Delta T^h \geq 0$ for all trees homeomorphic to T, rays of which have lengths at most $l+2$, $l \geq 1$. Then $\Delta T_{i,j} \geq 0$ if $j \geq 4$, $1 \leq i \leq 2$.*

Proof We use the notation of the proof of Lemma 3.5. Analogously as in the proof of Lemma 3.5, assume that the length of R' is $l+2$, $l \geq 1$. Then again

$$\sum_{u \in R(e) \setminus \{e\}} h_{LT}(u) = -d(e,a) - 3l - 3.$$

Let c be one of the $j-1$ vertices of the clique of order j obtained from the edges incident to c', other than e (in the case $i=1$) or f (in the case $i=2$). By Lemma 3.4 we have

$$\sum_{u \in R_{LT_{i,j}}(c)} h_{LT_{i,j}}(u) \geq \begin{cases} \left(\binom{j-1}{2} - 1\right) d(c,a) + \binom{j-2}{2} - 2 & \text{if } |R_{LT_{i,j}}(c)| = 1 \\ \left(\binom{j}{2} - 2\right) d(c,a) - 3l + \binom{j-1}{2} - 5 & \text{if } |R_{LT_{i,j}}(c)| \geq 2. \end{cases}$$

As $j \geq 4$ and $d(c,a) \geq l+2 \geq 3$, in any case we have $\sum_{u \in R_{LT_{i,j}}(c)} h_{LT_{i,j}}(u) \geq 0$. Now if $i=1$ then $h_{LT_{i,j}}(e) - h_{LT}(e) \geq 0$ as the degree of e is $e_d + j - 2$ in $T_{i,j}$, see (1). On the other hand if $i=2$ then $h_{LT_{i,j}}(e) = h_{LT}(e)$ while $h_{LT_{i,j}}(f) \geq 0$, as the degree of f is $j \geq 4$ in $T_{i,j}$. Hence

$$\Delta h = \sum_{u \in V(LT_{i,j}) \setminus \{a\}} h_{LT_{i,j}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u) \geq (j-1) \cdot 0 + 0 + d(e,a) + 3l + 3 \geq 0.$$

By Proposition 2.2, $\Delta T_{i,j} - \Delta T = \Delta h \geq 0$, so that $\Delta T_{i,j} \geq \Delta T \geq 0$. □

Now we prove $\Delta T \geq 0$ for the basis of induction. To simplify the notation, we omit the index LT from R_{LT} and h_{LT}.

Lemma 3.7 *Let T be a tree homeomorphic to a star $K_{1,k}$, $k \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$. Then $\Delta T \geq 0$.*

Proof Here $|S_{LT}| = k$ and $\cup_{u \in S_{LT}} R(u) = V(LT) \setminus \{a\}$ where $R(u) \cap R(v) = \emptyset$ if $u \neq v$. Thus $\sum_u h(u) = \sum_{c \in S_{LT}} (\sum_{u \in R(c)} h(u))$. We prove that $\sum_{u \in R(c)} h(u) \geq 1$.

Choose $c \in S_{LT}$. By Lemma 3.4 we have

$$\sum_{u \in R(c)} h(u) \geq \begin{cases} \left(\binom{k}{2} - 3\right) l + \binom{k-1}{2} & \text{if } c = \overline{a} \\ \left(\binom{k-1}{2} - 1\right) d(c,a) + \binom{k-2}{2} - 2 & \text{if } c \neq \overline{a} \text{ and } |R(c)| = 1 \\ \left(\binom{k}{2} - 2\right) d(c,a) - 3l + \binom{k-1}{2} - 5 & \text{if } c \neq \overline{a} \text{ and } |R(c)| \geq 2. \end{cases}$$

As $d(c,a) = l+1$ in the last two cases, $k \geq 4$ and $l \geq 1$, in all three cases we have $\sum_{u \in R(c)} h(u) \geq 1$

We have $\sum_u h(u) = \sum_{c \in S_{LT}} (\sum_{u \in R(c)} h(u)) \geq k \cdot 1 \geq 4$. As $d_a = 1$, we have $\Delta T = \sum_u h(u) - 3$ by Proposition 2.2, so that $\Delta T \geq 0$. □

Denote by $H_{i,j}$ a tree having $i+j$ vertices, $i,j \geq 3$. Out of them one vertex has degree i, another one has degree j and the remaining $i+j-2$ vertices have degrees 1. Obviously, the vertices of degrees i and j must be adjacent in $H_{i,j}$ and $H = H_{3,3}$.

Lemma 3.8 *Let T be a tree homeomorphic to $H_{3,j}$, $j \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$. Suppose that the interior path of $H_{3,j}$ has length at most 2 and moreover suppose that the first vertex of a ray terminating at a' in T has degree 3. Then $\Delta T \geq 0$.*

Proof Denote $e = \overline{a}$. Moreover, denote by P' the unique interior path of T. If P' has length 1, then the unique vertex of LP' (denote it by v) has degree $3+j-2 \geq 5$, while if P' has length 2, then one of the vertices of LP' has degree 3 and the other (denote it by v) has degree $j \geq 4$. Since by (1), $h(u) \geq 0$ if $d_u \geq 3$ and $h(u) \geq 5d(u,a)+1$ if $d_u \geq 4$, the vertices of LP' contribute to $\sum_{u \in V(LT)\setminus\{a\}} h(u)$ by at least $5d(v,a)+1 \geq 5l+6$ as $d(v,a) \geq l+1$.

Denote by c any one of the $j-1$ vertices of the clique of order j from $\mathcal{C}(H_{3,j})$, which is not in LP'. By Lemma 3.4 we have

$$\sum_{u \in R(c)} h(u) \geq \begin{cases} \left(\binom{j-1}{2} - 1\right) d(c,a) + \binom{j-2}{2} - 2 & \text{if } |R(c)| = 1 \\ \left(\binom{j}{2} - 2\right) d(c,a) - 3l + \binom{j-1}{2} - 5 & \text{if } |R(c)| \geq 2. \end{cases}$$

As $j \geq 4$ and $d(c,a) \geq l+2 \geq 3$, in any case we have $\sum_{u \in R(c)} h(u) \geq 0$.

Now consider the rays attached to the clique of order 3 from $\mathcal{C}(H_{3,j})$. By Lemma 3.4

$$\sum_{u \in R(e)} h(u) = \left(\binom{3}{2} - 3\right) l + \binom{3-1}{2} = 1.$$

Denote by f that vertex of the clique of order 3 from $\mathcal{C}(H_{3,j})$, which is different from e and which is not in LP'. By Lemma 3.4 we have

$$\sum_{u \in R(f)} h(u) \geq \begin{cases} -2 & \text{if } |R(f)| = 1 \\ d(f,a) - 3l - 4 & \text{if } |R(f)| \geq 2. \end{cases}$$

Since $d(f,a) = l+1$ and $l \geq 1$, in any case we have $\sum_{u \in R(f)} h(u) \geq -2l-3$.

Now summing the inequalities above we obtain

$$\sum_u h(u) \geq (5l+6) + (j-1) \cdot 0 + 1 + (-2l-3) = 3l+4 \geq 3.$$

As $d_a = 1$, we have $\Delta T = \sum_u h(u) - 3$ by Proposition 2.2, so that $\Delta T \geq 0$. □

Denote by $Y_{i,j}$, $1 \leq i,j \leq 2$, a tree having three vertices of degree 3, namely y_1', y_2' and y_3'. All the other vertices of $Y_{i,j}$ have degrees at most 2. There are two interior paths in $Y_{i,j}$, namely $y'_1 - y'_2$ and $y'_2 - y'_3$, and their lengths are i and j, respectively. Moreover, there are five rays in $Y_{i,j}$. Two such rays start at y'_1, one starts at y'_2 and two start at y'_3. Of course, one of these rays has length exactly $l+1$ and it terminates in a'.

Lemma 3.9 *Let T be the tree $Y_{i,j}$, $1 \leq i, j \leq 2$, in which all rays have lengths at most $l+2$, $l \geq 1$. Then $\Delta T \geq 0$.*

Proof Denote by x_1, x_2, x_3, x_4 and x_5 the five vertices of S_{LT} corresponding to the first edges of rays starting at y'_1, y'_1, y'_2, y'_3 and y'_3, respectively. Since the degrees of y'_1, y'_2 and y'_3 are 3 in T, all $x_1, x_2, \ldots, x_5$ are vertices of cliques of order 3 in LT. Let $x_t = \overline{a}$, $1 \leq t \leq 5$. By Lemma 3.4

$$\sum_{u \in R(x_t)} h(u) = 1.$$

For all other x_r, $1 \leq r \leq 5$ and $r \neq t$, by Lemma 3.4 we have

$$\sum_{u \in R(x_r)} h(u) \geq \min\{-2, d(x_r, a) - 3l - 4\}.$$

As $l \geq 1$, this minimum equals $d(x_r, a) - 3l - 4$ if $d(x_r, a) \leq l+4$. If $d(x_r, a) = l+5$ then $\sum_{u \in R(x_r)} h(u) \geq \min\{-2, -2l+1\} \geq -2l$.

Now we consider vertices corresponding to edges of interior paths. If such a path has length 1, then its unique edge corresponds to a vertex, say e, which degree is 4 in LT. By (1) we have

$$h(e) = 5d(e, a) + 1.$$

On the other hand if such a path has length 2, then its edges correspond to two vetices, say e and f, both of degree 3. Suppose that e is closer to a than f. By (1) we have

$$h(e) + h(f) = 2d(e, a) - 1 + 2d(f, a) - 1 = 4d(e, a).$$

In the next, we list contributions to $\sum_u h(u)$ first by vertices of rays starting at $x_1, x_2, \ldots, x_5$ and then by the vertices corresponding to edges of paths $y'_1 - y'_2$ and $y'_2 - y'_3$. By symmetry, there are two cases to consider. First, suppose that $t = 1$, i.e., $\overline{a} = x_1$. We distinguish 4 subcases.

- $i = j = 1$: Then $d(x_2, a) = l+1$, $d(x_3, a) = l+2$ and $d(x_4, a) = d(x_5, a) = l+3$. As $l \geq 1$, we have

$$\sum_u h(u) \geq 1 + (-2l-3) + (-2l-2) + 2(-2l-1) + (5l+6) + (5l+11) \geq 2l+11 \geq 3.$$

- $i = 1$ *and* $j = 2$: Analogously as above we get

$$\sum_u h(u) \geq 1 + (-2l-3) + (-2l-2) + 2(-2l) + (5l+6) + (4l+8) \geq l + 10 \geq 3.$$

- $i = 2$ *and* $j = 1$: We have

$$\sum_u h(u) \geq 1 + (-2l-3) + (-2l-1) + 2(-2l) + (4l+4) + (5l+16) \geq l + 17 \geq 3.$$

- $i = j = 2$: Here $d(x_4, a) = d(x_5, a) = l + 5$. Hence,

$$\sum_u h(u) \geq 1 + (-2l-3) + (-2l-1) + 2(-2l) + (4l+4) + (4l+12) \geq 13 \geq 3.$$

Now suppose that $t = 3$, i.e., $\overline{a} = x_3$. By symmetry, it suffices to consider 3 subcases.

- $i = j = 1$: Then $d(x_1, a) = d(x_2, a) = l+2$ and also $d(x_4, a) = d(x_5, a) = l+2$. As $l \geq 1$, we have

$$\sum_u h(u) \geq 2(-2l-2) + 1 + 2(-2l-2) + (5l+6) + (5l+6) \geq 2l + 5 \geq 3.$$

- $i = 1$ *and* $j = 2$: We have

$$\sum_u h(u) \geq 2(-2l-2) + 1 + 2(-2l-1) + (5l+6) + (4l+4) \geq l + 5 \geq 3.$$

- $i = j = 2$: We have

$$\sum_u h(u) \geq 2(-2l-1) + 1 + 2(-2l-1) + (4l+4) + (4l+4) \geq 5 \geq 3.$$

As $\Delta T = \sum_u h(u) - 3$ by Proposition 2.2, we have $\Delta T \geq 0$. □

Now we prove $\Delta T \geq 0$ for the last graph of the basis of induction. Denote by X_k, $k \geq 4$, a tree having two vertices of degree 3, namely y'_1 and y'_2, and one vertex of degree k, namely y'_3. All other vertices of X_k have degrees at most 2. There are two interior paths in X_k, namely $y'_1 - y'_2$ and $y'_2 - y'_3$, both of lengths at most 2. Moreover, there are $k+2$ rays in X_k. Two such rays start at y'_1, one starts at y'_2 and the remaining $k-1$ start at y'_3.

Lemma 3.10 *Let T be the tree X_k, $k \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$. Suppose that the ray terminating at a' starts at y_1'. Then $\Delta T \geq 0$.*

Proof We use the notation of the proof of Lemma 3.9. Denote by x_1, x_2, x_3, x_4, $\dots x_{k+2}$ the $k+2$ vertices of S_{LT} corresponding to first edges of rays starting at $y_1', y_1', y_2', y_3', \dots, y_3'$, respectively. The vertices x_1, x_2 and x_3 are in cliques of order 3, while $x_4, \dots, x_{k+2}$ are in the clique of order k. Assume that $\overline{a} = x_1$. As shown in the proof of Lemma 3.9, we have $\sum_{u \in R(x_1)} h(u) = 1$. Further, $\sum_{u \in R(x_2)} h(u) \geq -2l-3$ as $d(x_2, a) = l+1$. The vertices corresponding to edges of $y_1' - y_2'$ path contribute to $\sum_u h(u)$ by at least $\min\{5d(e,a)+1, 4d(e,a)\} = 4d(e,a) = 4l+4$ as $d(e,a) = l+1$. Finally, $\sum_{u \in R(x_3)} h(u) \geq \min\{-2, d(x_3,a) - 3l - 4\} \geq -2l-2$ as $d(x_3, a) \geq l+2$.

Since the vertices corresponding to edges of $y_2' - y_3'$ path have degree $k+1$ (in the case when the length of $y_2' - y_3'$ is 1) or 3 and k (in the case when the length of $y_2' - y_3'$ is 2), and since $h(u) \geq 0$ if $d_u \geq 3$ by (1), the contribution of these vertices to $\sum_u h(u)$ is nonnegative.

Finally, consider $\sum_{u \in R(x_i)} h(u)$ when $i \geq 4$. By Lemma 3.4 we have

$$\sum_{u \in R(x_i)} h(u) \geq \begin{cases} \left(\binom{k-1}{2} - 1\right) d(x_i, a) + \binom{k-2}{2} - 2 & \text{if } |R_{LT_{1,3}}(c)| = 1 \\ \left(\binom{k}{2} - 2\right) d(x_i, a) - 3l + \binom{k-1}{2} - 5 & \text{if } |R_{LT_{1,3}}(c)| \geq 2. \end{cases}$$

Since $d(x_i, a) \geq l+3$, $k \geq 4$ and $l \geq 1$, we have $\sum_{u \in R(x_i)} h(u) \geq \min\{7, 11\} = 7$.

Summing these inequalities we obtain

$$\sum_u h(u) \geq 1 + (-2l-3) + (4l+4) + (-2l-2) + 0 + (k-1)7 = 7k - 7 \geq 3.$$

As $d_a = 1$, by Proposition 2.2 we have $\Delta T = \sum_u h(u) - 3 \geq 0$. □

Now we summarize the proof of Theorem 1.5

Proof of Theorem 1.5 Let T be a tree, not homeomorphic to a path, claw $K_{1,3}$ and H. We prove that $D(T) = W(L^3(T)) - W(T) > 0$. Denote by $l+2$, $l \geq -1$, the length of a longest ray in T. If $l = -1$ then $D(T) = W(L^3(T)) - W(T) > 0$ by Theorem 1.4.

Hence, suppose that $l \geq 0$ and assume that the statement is true for all trees, not homeomorphic to a path, claw $K_{1,3}$ and H, rays of which have lengths at most $l+1$. Let $R_1', R_2', \dots, R_t'$ be rays of T having length $l+2$. Further, denote by c_i' the first vertex of R_j', denote by b_i' its last vertex and denote by a_i' the neighbour of b_i' in T, $1 \leq i \leq t$. Finally, denote by T_i a tree obtained from T by removing the vertices $b_{i+1}', b_{i+2}', \dots, b_t'$ and edges $a_{i+1}'b_{i+1}', a_{i+2}'b_{i+2}', \dots, a_t'b_t'$, $0 \leq i \leq t$. Then

$T_t = T$ and T_0 is a tree, rays of which have length at most $l+1$. By induction we have $D(T_0) = W(L^3(T_0)) - W(T_0) > 0$. Denote $\Delta T_i = D(T_{i+1}) - D(T_i)$, $0 \le i \le t-1$.

Suppose that $l = 0$. All rays of T_i have length at most $l + 2$, and the ray R'_{i+1} terminating at a'_{i+1} has length $l+1$. Moreover, T_{i+1} is obtained from T_i by adding the vertex b'_{i+1} and the edge $a'_{i+1}b'_{i+1}$. Hence $\Delta T_i \ge 0$ by Lemma 3.2, $0 \le i \le t-1$, where the vertex a'_{i+1} and the tree T_i play the role of a and T, respectively. Consequently $\sum_{i=0}^{t-1} \Delta T_i \ge 0$. Since

$$0 \le \sum_{i=0}^{t-1} \Delta T_i = D(T_t) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)],$$

we have $W(L^3(T)) - W(T) \ge W(L^3(T_0)) - W(T_0) > 0$.

Now suppose that $l \ge 1$. Denote by T_i^- a tree obtained from T_i by shortening all interior paths, which length is at least 3, to paths of length 2. Analogously as T_{i+1} is obtained from T_i, the tree T_{i+1}^- is obtained from T_i^- by adding the vertex b'_{i+1} and the edge $a'_{i+1}b'_{i+1}$. We prove that $\Delta T_i^- = D(T_{i+1}^-) - D(T_i^-) \ge 0$ by induction on the number of vertices of degree at least 3. Observe that T_i^-, so as T_i, is a tree, rays of which have length at most $l + 2$ and the ray terminating at a'_{i+1} has length $l + 1$, $0 \le i \le t - 1$.

Denote by V_i^3 the set of vertices of degree at least 3 in T_i^-. We distinguish four cases.

- $|V_i^3| = 1$: Then T_i^- is homeomorphic to $K_{1,k}$. Since T is not homeomorphic to $K_{1,3}$, we have $k \ge 4$. By Lemma 3.7 we have $\Delta T_i^- \ge 0$.

- $|V_i^3| = 2$: If the degree of c'_{i+1} is 3, then $\Delta T_i^- \ge 0$ by Lemma 3.8, as T is not homeomorphic to $H = H_{3,3}$. On the other hand if the degree of c'_{i+1} is $k \ge 4$, then denote by c'' the other vertex of V_i^3. Remove the rays starting at c'' from T_i^-, and denote the resulting graph by T''. Then T'' is a tree, rays of which have length at most $l + 2$, and T'' is homeomorphic to $K_{1,k}$. By Lemma 3.7 we have $\Delta T'' \ge 0$. If the degree of c'' is 3 then $\Delta T_i^- \ge 0$ by Lemma 3.5, while if the degree of c'' is at least 4 then $\Delta T_i^- \ge 0$ by Lemma 3.6.

- $|V_i^3| = 3$: Denote by T^* a graph obtained from T_i^- by removing the edges of all rays. Then T^* is a tree, so that it has at least two vertices of degree 1. (We remark that in this case T^* is a path.) Denote by c'' such a vertex of degree 1 in T^*, $c'' \ne c'_{i+1}$, which degree in T_i^- is the smallest possible. Finally, denote by T'' a tree obtained from T_i^- by removing all rays starting at c''. We distinguish two subcases.

  - *T'' is homeomorphic to H*: If the degree of c'' is 3 in T then $\Delta T_i^- \ge 0$ by Lemma 3.9. Hence, suppose that the degree of c'' is $k \ge 4$. By the choice of c'', the vertex c'_{i+1} is a leaf of T^*. Hence, T is X_k and c'_{i+1} is the vertex y'_1 in the notation of Lemma 3.10. Therefore $\Delta T_i^- \ge 0$ by Lemma 3.10.

- *T'' is homeomorphic to $H_{i,j}$, $i \le j$ and $j \ge 4$:* Since T'' is not homeomorphic to H, we have $\Delta T'' \ge 0$ by the previous case (the case $|V_i^3| = 2$). If the degree of c'' is 3 then $\Delta T_i^- \ge 0$ by Lemma 3.5, while if the degree of c'' is at least 4 then $\Delta T_i^- \ge 0$ by Lemma 3.6.

Thus, we proved $\Delta T_i^- \ge 0$ for every tree T_i^-, rays of which have length at most $l+2$ and $|V_i^3| = 3$.

- $|V_i^3| \ge 4$: Analogously as in the previous case, denote by T'' a tree obtained from T_i^- by removing all rays starting at a pendant vertex c'' of T^*, $c'' \ne c'_{i+1}$. By induction we assume that $\Delta T'' \ge 0$. If the degree of c'' is 3 then $\Delta T_i^- \ge 0$ by Lemma 3.5, while if the degree of c'' is at least 4 then $\Delta T_i^- \ge 0$ by Lemma 3.6.

Hence, in any case we have $\Delta T_i^- \ge 0$. If $T_i^- = T_i$ then we have also $\Delta T_i \ge 0$. Otherwise form a sequence $T_i^- = F_0, F_1, \ldots, F_r = T_i$ such that F_{j+1} is obtained from F_j by subdividing one edge of one interior path, $0 \le j \le r-1$. By Lemma 3.3 we have $\Delta F_{j+1} - \Delta F_j \ge 0$. Hence, $\sum_{j=0}^{r-1}(\Delta F_{j+1} - \Delta F_j) \ge 0$. Since

$$0 \le \sum_{j=0}^{r-1}(\Delta F_{j+1} - \Delta F_j) = \Delta T_i - \Delta T_i^-,$$

we have $\Delta T_i \ge \Delta T_i^- \ge 0$.

Thus, we proved that $\Delta T_i \ge 0$ for every $i \in \{0, 1, \ldots, t-1\}$. Hence, $\sum_{i=0}^{t-1} \Delta T_i \ge 0$. Since

$$0 \le \sum_{i=0}^{t-1} \Delta T_i = D(T_t) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)],$$

we have $W(L^3(T)) - W(T) \ge W(L^3(T_0)) - W(T_0) > 0$. □

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