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LINEAR COLOURING OF PLANAR  
GRAPHS WITH PRESCRIBED GIRTH  
AND MAXIMUM DEGREE

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# Linear colouring of planar graphs with prescribed girth and maximum degree

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## Abstract

A linear colouring of a graph is a proper vertex colouring such that the subgraph induced by any two colour classes is a set of vertex disjoint paths. The corresponding linear chromatic number of a graph  $G$ , namely  $\text{lc}(G)$ , is the minimum number of colours in a linear colouring of  $G$ . We prove that for a graph  $G$  with girth  $g \geq 8$  and maximum degree  $\Delta \geq 7$  the inequality on its linear colouring number holds:  $\text{lc}(G) \leq \lceil \frac{\Delta}{2} \rceil + 1$ . This improves the girth assumption of a previously known bound of Raspaud and Wang [9].

## 1 Introduction

All graphs in this paper are undirected, simple and finite. For standard graph definitions the reader is referred to [3]. A *colouring* of the vertices of a graph  $G = (V, E)$  is a mapping  $c : V \rightarrow \mathbb{N}$ ; we call elements of  $\mathbb{N}$  *colours*. A colouring is *proper* if any two adjacent vertices are mapped to different colours. A proper vertex colouring is called *linear* if the subgraph induced by the union of any two colour classes is a set of vertex disjoint paths. The minimum number of colours in a linear colouring of a graph  $G$  is the *linear chromatic number*  $\text{lc}(G)$ .

The notion of a linear colouring was first introduced by Yuster [11] who proved that  $\text{lc}(G) = O(\Delta^{3/2})$  where  $\Delta$  is the maximum degree. He also constructed graphs with  $\text{lc}(G) = \Omega(\Delta^{3/2})$  which showed that his bound is the best possible for general graphs.

In [9] Raspaud and Wang proved that  $\text{lc}(G) = \lceil \frac{\Delta}{2} \rceil + 1$  for a planar graph that satisfies one of the following conditions: (1)  $g \geq 13$  and  $\Delta \geq 3$ ; (2)  $g \geq 11$  and  $\Delta \geq 5$ ; (3)  $g \geq 9$  and  $\Delta \geq 7$ ; (4)  $g \geq 7$  and  $\Delta \geq 13$ . In a wider sense this improves the bound of Yuster to  $\text{lc}(G) = O(\Delta)$  for planar graphs with large enough girth and maximum degree.

To prove our theorem we will use the discharging method which was first used by Wernicke [10]. The discharging method is a technique used to prove statements in structural graph theory, and it is commonly applied in the context of planar graphs. It is most well-known for its central role in the proof of the Four Colour Theorem.

A *linear* colouring is a special case of an acyclic colouring, where it is required that the subgraph induced by the union of any two colour classes is a forest. This colouring has been introduced by Grünbaum [5] and later studied in [1, 2, 7, 8]. It is clear that  $\text{lc}(G) \geq A(G)$ , where  $A(G)$  is the acyclic chromatic number of  $G$ . Acyclic colouring has also been extended to acyclic list colouring by Esperet et al. [4]. A related concept is that of  $k$ -frugal colourings, considered by Hind et al. [6], where a graph is *k-frugal* if it can be properly coloured so that no colour appears  $k$  times in a vertex neighbourhood. Such a colouring is called a  $k$ -frugal colouring. Clearly, a linear colouring is 3-frugal, but the converse is not necessarily true.

We use the following notation. Let  $N(v)$  be the set of neighbours of  $v$  and  $N_2(v)$  the neighbours of  $v$  of degree two. Denote by  $l(f)$  the length of a face  $f$ . A  $n$ -vertex is a vertex of degree  $n$ . A  $\leq n$ -vertex and  $\geq n$ -vertex is a vertex of degree at most  $n$  and at least  $n$  respectively. Let  $c$  be a colouring of a graph. Denote the colours that appear twice in the neighbourhood of a vertex  $v$  by  $c_2(v)$ .

As we already mentioned, Raspaud and Wang proved that a planar graph with girth  $g \geq 9$  and maximum degree  $\Delta \geq 7$  allows a linear colouring with at most  $\lceil \frac{\Delta}{2} \rceil + 1$  colours. We improved the restriction on girth and so obtained the following:

**Theorem 1.1.** *Let  $G$  be a planar graph with maximum degree  $\Delta \geq 7$  and girth  $g \geq 8$ . Then  $G$  has a linear colouring with at most  $\lceil \frac{\Delta}{2} \rceil + 1$  colours.*

## 2 Proof of the theorem

In the proof, we will use the discharging method. To obtain a contradiction we will suppose that we have a minimal counterexample  $G$  to the theorem regarding the number of vertices. First we will prove that some planar configurations are reducible, i.e. cannot occur in the minimal counterexample. Then we assign the initial charge to every vertex and every face with negative total sum. Next we redistribute the charge in such a way that the final charge is nonnegative for every vertex and face. This will be a contradiction that establishes the theorem.

### 2.1 Reducible configurations

In the proof we use the following reducible configurations:

- (Q1): a 1-vertex;
- (Q2): a 2-vertex adjacent to another 2-vertex and a vertex of degree less than seven;
- (Q3): a 2-vertex adjacent to a 3-vertex and a vertex of degree less than five;
- (Q4): a 2-vertex adjacent to a  $\leq 5$ -vertex and a 3-vertex which is also adjacent to a 2-vertex and a  $\leq 5$ -vertex (see Figure 3);
- (Q5): a 7-vertex adjacent to seven 2-vertices where 5 of them are again adjacent to another 2-vertex and the remaining two 2-vertices have another neighbour of degree  $\leq 3$  (see Figure 4);

(Q6): a 7-vertex adjacent to five 2-vertices where each of them is again adjacent to another 2-vertex and the remaining two neighbours are of degree  $\leq 3$  and both have 2-vertices as the remaining neighbours (see Figure 5).

The reducibility of each of the above configurations is proven in a separate lemma. The lemmas provide the proof only for the case when the degrees are maximal as to the  $\leq$  conditions in the configuration. As it will be clear from the proofs, the reasoning for proving the reducibility for lower degrees is even simpler. It will be implicitly assumed that such configurations are not present in the possible minimal counterexample of Theorem 1.1

**Lemma 2.1.** (Q1) *i.e. a 1-vertex is reducible.*

*Proof.* A 1-vertex  $v$  adjacent to  $x$  is clearly reducible since removing it yields a smaller graph that can be coloured by minimality of  $G$ . This colouring can be extended since colouring  $v$  with a colour that does not appear twice in the neighbourhood of  $x$  or on  $x$  is obviously proper. The number of forbidden colours for  $v$  is then at most  $c_2(x) + 1 = \lfloor \frac{\Delta-1}{2} \rfloor + 1 = \lceil \frac{\Delta}{2} \rceil$ , and so we have at least one available colour for  $v$ . □

**Lemma 2.2.** (Q2) *i.e. a 2-vertex adjacent to another 2-vertex and a vertex of degree at most six is reducible.*

*Proof.* Let  $v$  be a 2-vertex adjacent to a 2-vertex  $x$  and to  $y$  a vertex of degree  $< 7$ . After removing  $v$  we are left with a smaller graph which has a linear colouring  $c$  by minimality of  $G$ . We will now extend this colouring to  $v$  by considering two cases:

- **Case 1:**  $c(x) = c(y)$ . Colouring  $v$  with a colour distinct from  $c(x)$  that does not appear twice in  $N(y)$  or on the other neighbour of  $x$  clearly gives a linear colouring of  $G$ . Thus the number of forbidden colours for  $v$  is at most  $|c_2(y)| + 1 + 1 = \lfloor \frac{d-1}{2} \rfloor + 1 + 1 \leq \lfloor \frac{6-1}{2} \rfloor + 2 = 4$ . But since  $\Delta \geq 7$  we have at least 5 colours, we can choose such a colour for  $v$ .
- **Case 2:**  $c(x) \neq c(y)$ . Colouring  $v$  with a colour distinct from  $c(x)$  and  $c(y)$  that does not appear twice in  $N(y)$  is again linear. The number of forbidden colours in this case is  $|c_2(x)| + 2 \leq \lfloor \frac{6-1}{2} \rfloor + 2 = 4$  and again with at least 5 colours we can choose a colour for  $v$ .

□

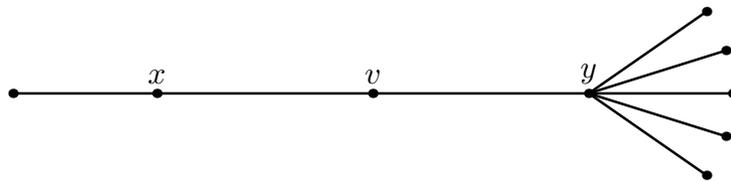


Figure 1: Reducible configuration (Q2).

**Lemma 2.3.** (Q3) *i.e. a 2-vertex adjacent to a 3-vertex and a vertex of degree at most four is reducible.*

*Proof.* Let  $v$  be a 2-vertex adjacent to a 3-vertex  $x$  and  $y$  with degree  $< 5$ . Let  $x_1$  and  $x_2$  be the other neighbours of  $x$ . Again removing  $v$  gives a smaller graph with a linear colouring  $c$  (by minimality of  $G$ ). We distinguish two cases:

- **Case 1:**  $c(x) = c(y)$ . If we colour  $v$  with a colour that is not in  $c_2(y) \cup \{c(x), c(x_1), c(x_2)\}$  the result is a linear colouring of  $G$ . This set contains  $|c_2(y)| + |\{c(x), c(x_1), c(x_2)\}| \leq \lfloor \frac{4-1}{2} \rfloor + 3 = 4$  colours, but as we have at least 5 available colours, we can easily colour  $v$  in the desired way.
- **Case 2:**  $c(x) \neq c(y)$ . Assigning a colour to  $v$  distinct from colours in  $c_2(y)$ ,  $c(x)$ ,  $c(y)$  and  $c(x_1)$  gives a linear colouring. The inequality  $|c_2(y)| + |\{c(x), c(x_1), c(y)\}| \leq \lfloor \frac{4-1}{2} \rfloor + 3 = 4$  assures that we have at least one available colour left for  $v$ .

□

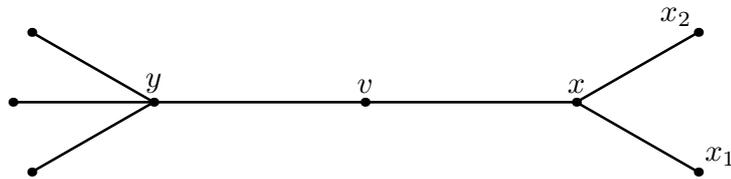


Figure 2: Reducible configuration (Q3).

**Lemma 2.4.** (Q4) *i.e. a 2-vertex adjacent to a 5-vertex and a 3-vertex where its other neighbours are of degree 2 and  $\leq 5$  is reducible.*

*Proof.* Since  $\Delta \geq 7$  the number of colours we use is at least 5. Let  $v$  be a 2-vertex adjacent to a 3-vertex  $x$  and a 5-vertex  $y$ , and let  $x_1$  and  $x_2$  be the other two neighbours of  $x$ . Let  $x_1$  be of degree 2 and  $x_2$  of degree  $\leq 5$ . Let  $x'_1$  be the other neighbour of  $x_1$ . In order to simplify the proof we define the set  $S = N(y) \setminus \{v\} \cup \{x_1, x_2\}$  and  $T = N(x_2) \setminus \{x\} \cup \{x'_1\}$ . We remove  $v$  and colour the remaining graph by minimality of  $G$ . We may assume that  $c(y) = 1$ . We distinguish two cases:

- **Case 1:**  $c(x) = 1$ . Observe that one of the remaining colours appears at most once in the set  $S$  since  $|S| = 6$ . We assign this colour to  $v$  and hereby obtain a linear colouring of  $G$ .
- **Case 2:**  $c(x) \neq 1$ . Say,  $c(x) = 2$ . If the colouring  $c$  cannot be extended to  $v$ , we may assume the following:
  - $c(x_1) = c(x_2) = 3$ ,
  - each of the colours 4 and 5 appear twice in  $N(y) \setminus \{v\}$ .

We uncolour  $x$  and assign the colour 2 to  $v$ . Now we only need to colour  $x$  with 1, 4 or 5. If this is not possible then each of the colours 1, 4 and 5 appear twice in  $T$ , which is impossible since  $|T| = 5$ .

□

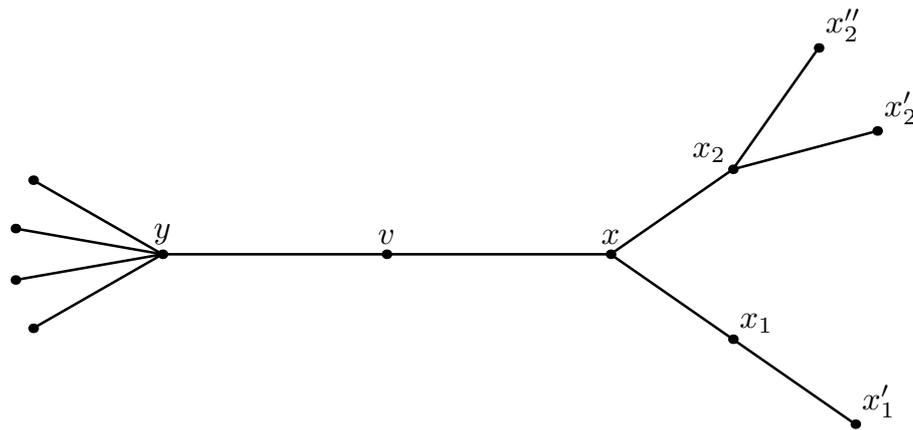


Figure 3: Reducible configuration (Q4).

**Lemma 2.5.** (Q5) *i.e. a 7-vertex adjacent to seven 2-vertices where 5 of them are again adjacent to a 2-vertex and 2 of them are adjacent to 3-vertices is reducible.*

*Proof.* Let  $v, x_1, \dots, x_7, y_1, \dots, y_7, z_1, \dots, z_7$  and  $z'_6, z'_7$  be vertices as in Figure 4. If we remove the star  $S = \{v, x_1, \dots, x_7\}$  the resulting graph has a linear colouring  $c$  by the minimality of  $G$ . Now we want to extend this colouring to the whole graph  $G$ . Let the list of all possible colours be  $\text{col} = \{1, \dots, \lceil \frac{\Delta}{2} \rceil + 1\}$ . Since  $\Delta \geq 7$  we know that  $|\text{col}| \geq 5$ .

Denote the lists of available colours for vertex  $x_i$  by  $L(x_i) = \text{col} \setminus \{c(y_i), c(z_i)\}$  for those  $x_i$  adjacent to 2-vertices and  $L(x_i) = \text{col} \setminus \{c(y_i), c(z_i), c(z'_i)\}$  for those  $x_i$  adjacent to 3-vertices.

Observe that at most two  $x_j$  can be coloured the same. Our problem can be now put in terms of finding a matching in a bipartite graph  $H$  constructed as follows: In partition  $A$  we will have the vertices  $x_1, \dots, x_7$  and in partition  $B$  the vertices  $u_1, \dots, u_5, u'_1, \dots, u'_5$ . Here (in partition  $B$ ) the indices correspond to the 5 colours we have at our disposal. Any vertex  $x_j$  with its list of admissible colours  $L(x_j)$  is connected to the vertices in  $B$  which have indices in  $L(x_j)$ . This means that the vertices in  $A$  are all of degree 6 or 4. If we find a matching in  $H$  which covers  $A$  this will correspond to a proper colouring of  $x_1, \dots, x_7$ , where each colour appears at most twice. The colour of  $x_j$  would simply be the index  $k$  of the vertex  $u_k$  or  $u'_k$  that  $x_j$  is matched by. If we can then also properly colour  $v$  the result will be a linear colouring. Let us show that this is indeed possible.

Hall's theorem states that we can find a matching in a bipartite graph covering the partition  $A$  if for each  $S \subset A$  the inequality  $|S| \geq |N(S)|$  holds. In our case the condition clearly holds for each  $|S| \leq 6$  since each vertex in  $A$  has degree 6 or 4 and only 2 are of degree 4.

The only case where we can have a problem is when  $S = A$  and the lists are of the form:  $L(x_i) = \{1, 2, 3\}$  if  $y_i$  is a 2-vertex and  $L(x_i) \subset \{1, 2, 3\}$  if  $y_i$  is a 3-vertex.

Let  $\omega$  be the size of the union of all lists, i.e.  $\omega = |\cup (L(x_i))|$ . Now in other words we only have a problem when  $\omega = 3$ . This means that the lists for those  $x_i$  adjacent to 2-vertices are the same and the lists for those  $x_i$  adjacent to 3-vertices are subsets of this list. In this case we have  $|S| = 7$  and  $N(S) = 6$ . We will deal with this case later.

Let us now suppose that  $\omega \geq 4$  and try to colour  $v$ . If any colour does not appear on  $x_1, \dots, x_7$  we use that colour for  $v$  and the proof is complete. So now we suppose that all colours are used on the vertices  $x_1, \dots, x_7$ . Without loss of generality the colours used on  $x_1, \dots, x_7$  are the multiset  $\{1, 1, 2, 2, 3, 4, 5\}$  and we may assume that the colour 3 is not

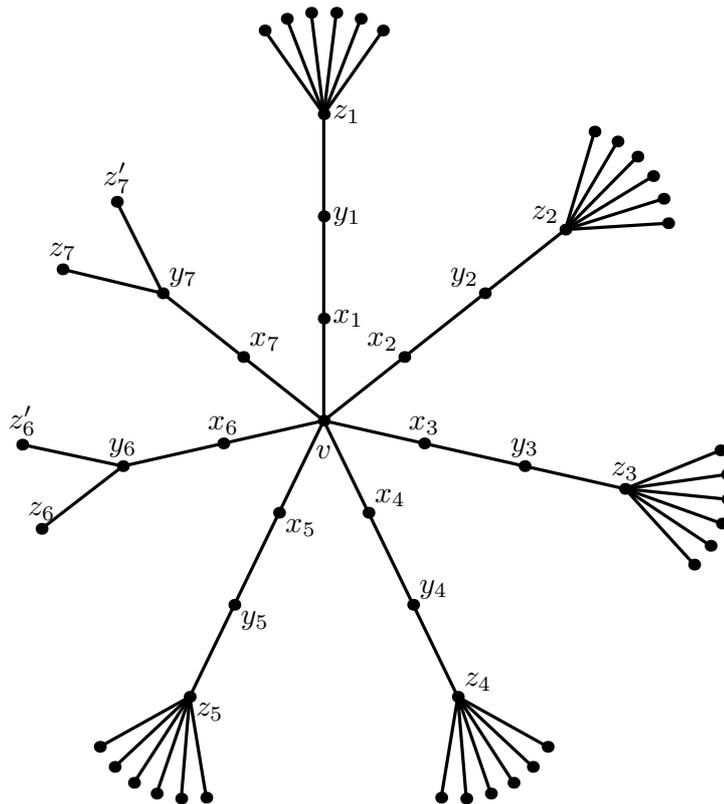


Figure 4: Reducible configuration (Q5).

used for  $x_6$  or  $x_7$ . We now recolor the vertex  $x_t$ , which is coloured with colour 3, with either 4 or 5 depending on which of these two colours is not used for its neighbour  $y_t$ . We need not care about  $z_t$  since all we might create is a longer path in the graph induced by the colour classes of colours  $c(x_t)$  and  $c(y_t)$ . Finally colour  $v$  with 3.

So now all that remains is to check the case when  $\omega = 3$ . Again without loss of generality we suppose that all the lists are equal to or contained in  $\{1, 2, 3\}$ . We choose to colour  $x_1$  and  $x_2$  with 1,  $x_3$  and  $x_4$  with 2 and  $x_5$  and  $x_6$  with 3. Then we set the colour of  $x_7$  to be either 4 or 5 distinct from the colour of  $y_7$  and then we use the remaining colour of these two colours for  $v$ . Similarly as before we do not violate the conditions for a linear colouring this way. This concludes the proof that this configuration is reducible.  $\square$

**Lemma 2.6.** (Q6) *i.e. a 7-vertex adjacent to five 2-vertices where each of them is again adjacent to another 2-vertex and the remaining two neighbours are of degree 3 and both have 2-vertices as the remaining two neighbours is reducible.*

*Proof.* The proof of this lemma is similar to the proof of lemma 2.5 and therefore this proof will be somewhat condensed. Let  $v, x_1, \dots, x_7, y_1, \dots, y_7, z_1, \dots, z_7, y'_6, y'_7$  and  $z'_6, z'_7$  be vertices like in the Figure 5. If we remove the star  $S = v, x_1, \dots, x_7$  the resulting graph has a linear colouring  $c$  by the minimality of  $G$ . Now we want to extend this colouring to  $G$ .

Let  $\text{col} = \{1, 2, 3, 4, 5\}$ . We now set the lists of available colours for  $x_i$  to be  $L(x_i) = \text{col} \setminus \{c(y_i), c(z_i)\}$  for those  $x_i$  of degree 2 and  $L(x_i) = \text{col} \setminus \{c(y_i), c(z_i), c(y'_i), c(z'_i)\}$  for those  $x_i$  of degree 3.

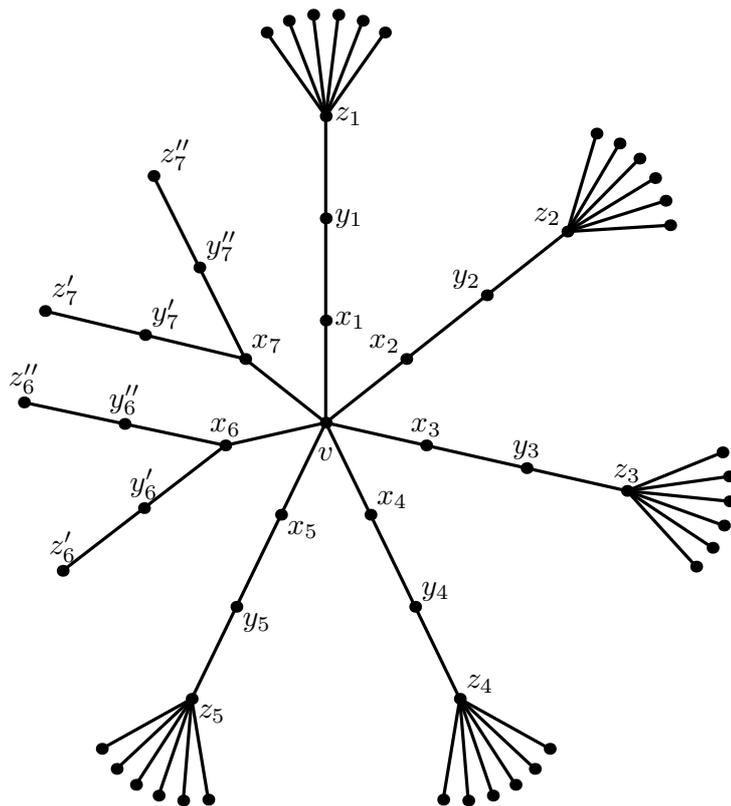


Figure 5: Reducible configuration (Q6).

Again we can only contradict the requirements for a linear colouring within the star  $S$  and if we can extend the colouring to a proper colouring of  $S$  where at most two  $x_j$  are coloured the same the result is a linear colouring.

Similarly as in the previous lemma our problem can be now put in terms of finding a matching in a bipartite graph  $H$  constructed as follows: In partition  $A$  we will have the vertices  $x_1, \dots, x_7$  and in partition  $B$  the vertices  $u_1, \dots, u_5, u'_1, \dots, u'_5$ . In partition  $B$  the indices correspond to the 5 colours we have at our disposal. Any vertex  $x_j$  with its list of admissible colours  $L(x_j)$  is connected to the vertices in  $B$  which have indices in  $L(x_j)$ .

This means that the vertices in  $A$  are all of degree 6 or 2. If we find a matching in  $H$  which covers  $A$  this will correspond to a proper colouring of  $x_1, \dots, x_7$ , where each colour appears at most twice. Again the colour of  $x_j$  would simply be the index  $k$  of the vertex  $u_k$  or  $u'_k$  that  $x_j$  is matched by. If we can then also properly colour  $v$  the result will be a linear colouring. Let us show that this is indeed possible.

The condition of Hall's theorem clearly holds for each  $|S| \leq 6$  since each vertex in  $A$  has degree 6 or 2 and only 2 are of degree 2. The only case where we can have a problem is when  $S = A$  and the lists are of the form:  $L(x_i) = \{1, 2, 3\}$  if  $x_i$  is a 2-vertex and  $L(x_i) \subset \{1, 2, 3\}$  if  $x_i$  is a 3-vertex. Let  $\omega$  be the size of the union of all lists  $\omega = |\cup(L(x_i))|$ . Let us now suppose that  $\omega \geq 4$  and try to colour  $v$ . If any colour does not appear on  $x_1, \dots, x_7$  we use that colour for  $v$  and the proof is complete. So now we suppose that all colours are used on the vertices  $x_1, \dots, x_7$ . Without loss of generality the colours used on  $x_1, \dots, x_7$  are the multiset  $\{1, 1, 2, 2, 3, 4, 5\}$ . One of the colours 3, 4 and 5 appears on  $x_t$  where  $t \notin \{6, 7\}$ . We recolour  $x_t$  with either 4 or 5 avoiding the colour of  $y_t$  the neighbour of  $x_t$ .

When  $\omega = 3$  we colour the vertices  $x_2, \dots, x_7$  with the colours 1, 2 and 3 without

using any of them twice, which is clearly possible. Then we colour  $x_1$  with either 4 or 5 depending on its neighbour  $y_1$ . At last we use the remaining colour for  $v$  and the proof is complete. □

## 2.2 Initial charge

We assign initial charge to every vertex  $v$  and every face  $f$  of  $G$  as follows:

$$\text{ch}_0(v) = 3\text{deg}(v) - 8 \quad \text{and} \quad \text{ch}_0(f) = l(f) - 8.$$

Observe that by Euler's formula the total charge assigned this way is  $-4$ . By this choice of initial charge all the faces have nonnegative initial charge. Since  $\delta(g) \geq 2$  only 2-vertices have negative charge.

$\text{deg}(v)$	$\text{ch}_0(v)$	$s(v)$
2	-2	
3	1	1/3
4	4	1
5	7	7/5
6	10	5/3
7	13	13/7
$\geq 8$	$\geq 16$	$\geq 2$

Figure 6: This table shows the degree of a vertex, its initial charge and the initial charge divided by the degree.

## 2.3 Discharging rules

We redistribute the initial charge by the following rules:

- $R_1$  : Each vertex  $v$  of degree at least 3 sends  $s(v) = \frac{\text{ch}_0(v)}{\text{deg}(v)}$  to neighbouring vertices of degree 2.
- $R_2$  : A vertex  $v$  of degree at least 3 adjacent to a vertex  $u$  of degree at least 3 sends  $\frac{\text{ch}_0(v)}{\text{deg}(v)(\text{deg}(u)-1)}$  to every vertex of degree 2 adjacent to  $u$ .
- $R_3$  : Let  $v_1v_2v_3v_4v_5$  be a path in  $G$  such that  $\text{deg}(v_1) = \text{deg}(v_2) = \text{deg}(v_4) = 2$  and  $\text{deg}(v_3) = 7$ . Now:
  - if  $\text{deg}(v_5) \geq 4$  then  $v_4$  sends  $1/7$  to  $v_2$ ;
  - if  $\text{deg}(v_5) = 3$  then  $v_4$  sends  $1/21$  to  $v_2$ .
- $R_4$  : If a 7-vertex has positive charge after applying rules  $R_1$ ,  $R_2$  and  $R_3$  it sends it to its neighbours of degree 2 distributed equally.

## 2.4 Final charge

After the discharging rules are applied we will conclude that every face and every vertex has nonnegative charge which will be a contradiction since the total charge is  $-4$ . Because a 1-vertex is reducible we only need to take care of 2-vertices since they are the only ones with negative initial charge.

The final charge of vertices of degree at least 3 stays nonnegative since:

$$\begin{aligned}
 \text{ch}(v) &= \text{ch}_0(v) - \frac{\text{ch}_0(v)}{\text{deg}(v)}n_2(v) - \frac{\text{ch}_0(v)}{\text{deg}(v)(\text{deg}(u) - 1)}n_2(u)(\text{deg}(v) - n_2(v)) \\
 &\geq \text{ch}_0(v) - \frac{\text{ch}_0(v)}{\text{deg}(v)}n_2(v) - \frac{\text{ch}_0(v)}{\text{deg}(v)}(\text{deg}(v) - n_2(v)) \\
 &= \text{ch}_0(v) - \frac{\text{ch}_0(v)}{\text{deg}(v)}\text{deg}(v) \\
 &= 0.
 \end{aligned}$$

Let  $v$  be a 2-vertex and let  $x$  and  $y$  be its two neighbours. If both  $x$  and  $y$  are of degree  $\geq 4$ , then by  $R_1$ ,  $\text{ch}(v) \geq -2 + 2 \cdot 1 = 0$ . Suppose that  $\text{deg}(x) = 3$ . Then  $\text{deg}(y) \geq 5$  by (Q3). If  $\text{deg}(y) \geq 6$  then  $\text{ch}(v) \geq -2 + \frac{1}{3} + \frac{5}{3} = 0$ . So we may assume that  $\text{deg}(y) = 5$ . Now by (Q4) we conclude that  $x$  has either one neighbour of degree  $\geq 6$  or two neighbours of degree  $\geq 3$ . In the first case  $\text{ch}(v) \geq -2 + \frac{1}{3} + \frac{7}{5} + \frac{5}{3 \cdot 2} > 0$  and in the second case  $\text{ch}(v) \geq -2 + \frac{1}{3} + \frac{7}{5} + \frac{1 \cdot 2}{3 \cdot 2} > 0$  by  $R_1$  and  $R_2$ .

Now, we may assume that  $v$  has a neighbour of degree 2, say  $x$ . Then,  $y$  is of degree  $\geq 7$  by the reducibility of (Q2). In case  $\text{deg}(y) \geq 8$ , then  $\text{ch}(v) \geq -2 + 2 = 0$  by  $R_1$ .

So assume that  $\text{deg}(y) = 7$ . After rule  $R_1$  the charge of  $v$  is  $\text{ch}(v) = -2 + \frac{13}{7} = -\frac{1}{7}$ . By (Q5) and (Q6) we know that one of the following cases holds:

- (a) one of the neighbours of  $y$  is a  $\geq 4$ -vertex;
- (b) three neighbours of  $y$  are 3-vertices;
- (c) exactly two neighbours of  $y$  are 3-vertices and at least one of them has another  $\geq 3$ -vertex as a neighbour ;
- (d) one vertex  $z$  at distance 2 from  $y$  is a  $\geq 4$ -vertex;
- (e) three vertices  $u_1, u_2$  and  $u_3$  at distance 2 from  $y$  are 3-vertices.

This means that the final charge of  $v$  in these cases is:

- (a)  $\text{ch}(v) = -\frac{1}{7} + \frac{1}{6} > 0$  by  $R_2$ ;
- (b)  $\text{ch}(v) = -\frac{1}{7} + 3 \cdot \frac{1}{3 \cdot 6} > 0$  by  $R_2$ ;
- (c) in this case  $y$  has at least  $\frac{13}{7 \cdot 2} = 13 - 5\frac{13}{7} - 3\frac{13}{7 \cdot 2}$  charge left after rules  $R_1$  and  $R_2$  while rule  $R_3$  cannot be applied here - this way  $y$  sends at least  $\frac{13}{7 \cdot 2 \cdot 5}$  charge to  $v$  and so  $\text{ch}(v) = -\frac{1}{7} + \frac{13}{7 \cdot 2 \cdot 5} > 0$  by  $R_4$ ;
- (d) the common neighbour of  $z$  and  $y$  sends  $1/7$  charge to  $v$  by  $R_3$  and so  $\text{ch}(v) = -\frac{1}{7} + \frac{1}{7} = 0$ ;
- (e) the common neighbours of  $u_1$  and  $y$ ,  $u_2$  and  $y$  and  $u_3$  and  $y$  send  $1/21$  charge to  $v$  by  $R_3$  and so  $\text{ch}(v) = -\frac{1}{7} + \frac{3}{21} = 0$ .

This concludes our proof that every vertex has nonnegative charge after the discharging step and so a contradiction arises since the sum of the initial charge is  $-4$ . This contradiction establishes the theorem.

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