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STRUCTURE OF FIBONACCI CUBES: A SURVEY

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Abstract

The Fibonacci cube Γ_n is the subgraph of the n-cube induced by the binary strings that contain no two consecutive 1s. These graphs are applicable as interconnection networks and in theoretical chemistry, and lead to the Fibonacci dimension of a graph. In this paper, a survey on Fibonacci cubes is given with an emphasis on their structure, including representations, recursive construction, hamiltonicity, degree sequence and other enumeration results. Their median nature that leads to a fast recognition algorithm is discussed. The Fibonacci dimension of a graph, studies of graph invariants on Fibonacci cubes, and related classes of graphs are also presented. Along the way some new short proofs are given.

Key words: Fibonacci cube; Fibonacci number; Cartesian product of graphs; median graph; degree sequence; cube polynomial

AMS Subj. Class.: 05-02, 05C75, 05C45, 05C07, 05C12.

1 Introduction

Let $B = \{0,1\}$ and for $n \geq 1$ set

$$\mathcal{B}_n = \{b_1 b_2 \ldots b_n \mid b_i \in B, 1 \leq i \leq n\}\,.$$

The *n-cube* Q_n is the graph defined on the vertex set $\mathcal{B}_n$, vertices $b_1 b_2 \ldots b_n$ and $b_1' b_2' \ldots b_n'$ being adjacent if $b_i \neq b_i'$ holds for exactly one $i \in \{1, \ldots, n\}$. *Hypercubes*, as the n-cubes are also called, form one of the central classes in graph theory. On one hand they are important from the theoretical point of view, on the other hand they form a model for numerous applications.

Clearly, $|V(Q_n)| = 2^n$. To obtain additional graphs (or networks) with similar properties as hypercubes, but on vertex sets whose order is not a power of two, Hsu [14] (see also [16]) introduced Fibonacci cubes as follows. For $n \geq 1$ let

$$\mathcal{F}_n = \{b_1 b_2 \ldots b_n \in \mathcal{B}_n \mid b_i \cdot b_{i+1} = 0, 1 \leq i \leq n-1\}\,.$$

The set $\mathcal{F}_n$ thus contains all binary strings of length n that contain no two consecutive 1s. The *Fibonacci cube* Γ_n, $n \geq 1$, has $\mathcal{F}_n$ as the vertex set, two vertices being adjacent if they differ in exactly one coordinate. In other words, Γ_n is the graph obtained from Q_n by removing all vertices that contain at least two consecutive 1s. Note that $\Gamma_1 = K_2$ and Γ_2 is the path on three vertices. The Fibonacci cube Γ_5 is shown in Fig. 1, for a drawing of Γ_{10} see [3]. For convenience we also set $\Gamma_0 = K_1$.

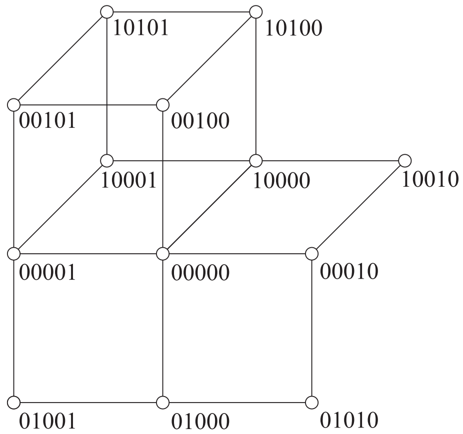

Figure 1: 5-dimensional Fibonacci cube Γ_5

Fibonacci cubes were introduced as a model for interconnection networks. (Many additional alternatives to hypercubes were proposed, let us just mention hierarchical hypercube networks [47] and twisted cubes [29].) In the seminal papers on Fibonacci cubes [6, 14] it was demonstrated that they can emulate many hypercube algorithms as well as they can emulate other topologies, as for instance meshes. However, the subsequent intensive investigations were in great part influenced by the appealing structural properties of Fibonacci cubes.

We proceed as follows. In the rest of this section we present two different representations of Fibonacci cubes while the next section collects definitions and concepts needed. In Section 3 the fundamental decomposition of Fibonacci cubes is given and applied to show that these graphs contain hamiltonian paths. Several additional properties are also given, for instance, short arguments are given for their independence number and the fact that they are prime graphs (w.r.t. the Cartesian product of graphs). Then, in Section 4, several enumeration results are given, in particular the degree sequences. Section 5 points to the median aspect of Fibonacci cubes which culminates with a fast recognition algorithm for this class of graphs. In the subsequent two sections studies of different invariants on Fibonacci cubes and related classes of graphs are described, respectively. The paper is closed with some open problems and ideas for further research.

The *simplex graph* $\kappa(G)$ of a graph G has complete subgraphs of G as vertices, including the empty subgraph, where two vertices are adjacent if the two complete subgraphs differ in a single vertex. In particular, the vertices and the edges of G are vertices of $\kappa(G)$. Simplex graphs were introduced in [1] and turned out to be an important tool in metric graph theory, cf. [19]. Now, consider the complement $\overline{P}_n$ of the path P_n on n vertices. Then complete subgraphs of $\overline{P}_n$ are in a 1-1 correspondence with the sets of pairwise nonconsecutive vertices of $\overline{P}_n$. It is then straightforward to see, cf. [3], that for any $n \geq 1$,

$$\Gamma_n \simeq \kappa(\overline{P}_n)\,.$$

Fig. 2 shows P_5, its complement (the house graph), and $\kappa(\overline{P}_5)$. To emphasize an isomorphism between the latter graph and the one from Figure 1, the induced 3-cube of Γ_5 is drawn bold.

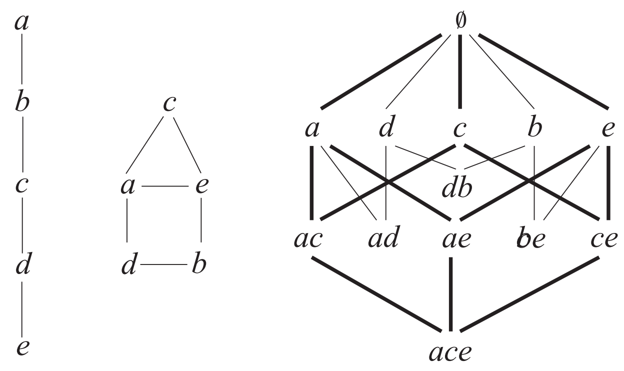

Figure 2: P_5, $\overline{P}_5$, and Γ_5 as the simplex graph of $\overline{P}_5$

A less direct representation of Fibonacci cubes appeared in theoretical chemistry. First recall that a *perfect matching* of a graph G is a set of independent edges of G that meet every vertex of G. Perfect matchings (in hexagonal graphs) play an important role in theoretical chemistry because they reflect the stability of the corresponding (benzenoid) molecule. The *resonance graph* or the *Z-transformation graph* of a hexagonal graph H has perfect matchings as vertices, two vertices being adjacent if they differ on exactly one 6-cycle and on this cycle their symmetric difference is the whole cycle; see [51] and references therein, as well as [53] for a generalization of this concept to all plane bipartite graphs. A *fibonaccene* is a hexagonal chain in which no three hexagons are linearly attached. These concepts lead to the following representation of Fibonacci cubes:

Theorem 1.1 ([27]) *Let G be a fibonaccene with n hexagons. Then the resonance graph of G is isomorphic to Γ_n.*

Theorem 1.1 has been generalized by Zhang, Ou, and Yao [52] who characterized plane bipartite graphs whose resonance graphs are Fibonacci cubes.

2 Preliminaries

To avoid ambiguity with initial conditions, we first define *Fibonacci numbers*: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$. The binary strings from $\mathcal{F}_n$ are called *Fibonacci* *strings*. The name derives from the appealing Zeckendorfs theorem which asserts that any positive integer can be uniquely written as the sum of nonconsecutive Fibonacci numbers. Hence Fibonacci strings are representations of integers in this number system.

The *Cartesian product* $G \,\square\, H$ of graphs G and H has the vertex set $V(G) \times V(H)$, the edge set $E(G \,\square\, H)$ consists of pairs $(g, h)(g', h')$ where either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. A graph is called *prime* (with respect to the Cartesian product) if it is nontrivial and cannot be represented as the Cartesian product of two nontrivial graphs.

The *distance* $d_G(u, v)$ between vertices u and v of a graph G is the length of a shortest path between u and v in G. If the graph G is clear from the context, we simply write $d(u, v)$. The *diameter* of a connected graph G is the maximum distance between two vertices of G; the *eccentricity* of a vertex u is the maximum distance between u and all the other vertices; the *radius* of G is the minimum eccentricity in G; and the *center* of G is the set of vertices with eccentricity equal to the radius of G.

A (connected) graph G is a *median graph* if every triple u, v, w of its vertices has a unique *median*: a vertex x such that $d(u, x)+d(x, v) = d(u, v)$, $d(v, x)+d(x, w) = d(v, w)$ and $d(u, x) + d(x, w) = d(u, w)$. Hypercubes are median graphs, the median of a triple is obtained by the majority rule in each of their coordinates. A subgraph H of a (connected) graph G is an *isometric subgraph* if $d_H(u, v) = d_G(u, v)$ holds for any $u, v \in V(H)$. It is a *convex subgraph* if for any $u, v \in V(H)$, every u, v-shortest path in G lies in H. Isometric subgraphs of hypercubes are called *partial cubes*. It is well-known that median graphs are partial cubes.

We will need two relations defined on the edge set of a graph: relation Θ and relation τ.

The Djoković-Winkler relation Θ [8, 44] is defined on the edge set of a graph G in the following way. Edges xy and uv of G are in relation Θ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. Relation Θ is reflexive and symmetric, hence its transitive closure Θ^* forms an equivalence relation on $E(G)$. Its equivalence classes are called Θ^*-*classes*. Winkler [44] proved that a connected graph G is a partial cube if and only if G is bipartite and $\Theta^* = \Theta$. Hence for partial cubes we may speak of Θ-classes instead of Θ^*-classes.

Edges e and f of a graph G are said to be in relation τ (see [12, 21]) if $e = f$ or if they form a convex path on three vertices. In other words, $e\tau f$ if $e = uv$, $f = vw$, where $uw \notin E(G)$ and v is the only common neighbor of u and w.

Finally, $\alpha(G)$ and $\gamma(G)$ denote the independence number and the domination number of G, respectively.

3 Recursive structure and applications

Let $n \geq 1$, then $\mathcal{F}_n$ naturally partitions into the sets of strings

$$A_n = \{b_1 b_2 \ldots b_n \in \mathcal{F}_n \mid b_1 = 1\} \quad \text{and} \quad B_n = \{b_1 b_2 \ldots b_n \in \mathcal{F}_n \mid b_1 = 0\}$$

that start with 1 and 0, respectively. Setting $A_0 = \emptyset$ and $B_0 = \{\lambda\}$, where λ is the empty string, A_n and B_n can be for any $n \geq 1$ recursively defined with

$$A_n = \{1\alpha \mid \alpha \in B_{n-1}\} \quad \text{and} \quad B_n = \{0\alpha \mid \alpha \in A_{n-1} \cup B_{n-1}\}\,.$$

This partition of $\mathcal{F}_n$ reveals the intrinsic recursive structure of Γ_n. Since a string of A_n ($n \geq 2$) necessarily starts with 10, the set A_n induces a subgraph of Γ_n isomorphic to Γ_{n-2}. Similarly, B_n induces Γ_{n-1} in Γ_n. Moreover, each vertex 1α of A_n has exactly one neighbor in B_n, the vertex 0α. This recursive structure is illustrated in Fig. 3 and will be called the *fundamental decomposition* of Γ_n. If needed, the fundamental decomposition of Γ_n can be recursively applied to its subgraphs Γ_{n-1} and/or Γ_{n-2}, a typical example will be given in Section 4 in the computation of degree sequences.

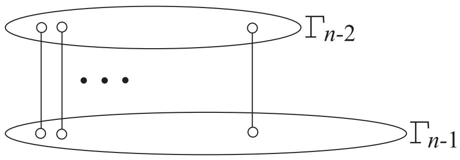

Figure 3: The recursive structure of Γ_n

In the rest of the section we demonstrate how important properties of the cubes can be deduced using the fundamental decomposition.

Note first that since $|V(\Gamma_0)| = 1$ and $|V(\Gamma_1)| = 2$, the fundamental decomposition immediately implies that $|V(\Gamma_n)| = F_{n+2}$. (Another way to see it is to apply Zeckendorf's theorem.)

With the following construction that mimics the fundamental decomposition, Cong, Zheng, and Sharma [6] showed that Fibonacci cubes contain hamiltonian paths. The empty sequence $g_0 = \lambda$ and the sequence $g_1 = 0, 1$ are clearly spanning paths of Γ_0 and Γ_1, respectively. For $n \geq 2$, let

$$g_n = 0\overline{g}_{n-1}, 10\overline{g}_{n-2}\,,$$

where $\overline{g}$ denotes the reverse of the sequence g and αg is the sequence obtained from g by appending a fixed string α in front of each of the terms of g. The first few sequences g_i are thus:

$$\begin{aligned} g_0 &= \lambda \\ g_1 &= 0, 1 \\ g_2 &= 01, 00, 10 \\ g_3 &= 010, 000, 001, 101, 100 \\ g_4 &= 0100, 0101, 0001, 0000, 0010, 1010, 1000, 1001 \end{aligned}$$

It follows from the fundamental decomposition that the sequence g_n contains all the vertices of Γ_n. Moreover, by induction, consecutive terms in $0\overline{g}_{n-1}$ as well as in $10\overline{g}_{n-2}$ differ in one position. Finally, showing (using induction again) that the last term of $0\overline{g}_{n-1}$ and the first term of $10\overline{g}_{n-2}$ also differ in exactly one position, we have:

Proposition 3.1 ([6]) *Γ_n has a hamiltonian path for any $n \geq 0$.*

Proposition 3.1 deserves several remarks.

The last terms of $0\overline{g}_{n-1}$ in the construction of g_n are 00, 001, 0010, 00100, 001001, 0010010, 00100100, ..., which leads to the sequence A033138 of [39].

Since Γ_n is bipartite, it can only have a hamiltonian cycle if it has an even number of vertices, that is, n must be of the form $3k-1$, $k \geq 2$. In [6] it is briefly mentioned that all Fibonacci cubes with an even number of vertices indeed have a hamiltonian cycle, see [50] for details. We also add that Zagaglia Salvi [48] proved that for $n \geq 7$, every edge of Γ_n belongs to cycles of every even length.

Proposition 3.1 in particular implies the following result:

Corollary 3.2 ([34]) *For any $n \geq 0$, $\alpha(\Gamma_n) = \left\lceil \frac{F_{n+2}}{2} \right\rceil$.*

Proof. As Γ_n has a hamiltonian path, $\alpha(\Gamma_n) \leq \lceil |V(\Gamma_n)|/2 \rceil = \lceil F_{n+2}/2 \rceil$. On the other hand, let $X+Y$ be the bipartition of Γ_n, then $\alpha(\Gamma_n) \geq \max\{|X|, |Y|\} \geq \lceil F_{n+2}/2 \rceil$. □

The n-cube Q_n can be equivalently defined as the Cartesian product of n copies of K_2. Hence hypercubes are the simplest multiple Cartesian products of graphs and it is natural to ask whether Fibonacci cubes admit such a representation or they are prime. Zhang, Ou, and Yao [52] proved that the latter is true, we next give a short proof of this fact. For this sake we recall Feder's theorem from [12] asserting that for a connected graph G, the relation $\sigma = (\Theta \cup \tau)^*$ is its product relation. In particular, G is prime if and only if σ has a single equivalence class.

Proposition 3.3 *For any $n \geq 1$, Γ_n is prime with respect to the Cartesian product.*

Proof. Γ_1, Γ_2, and Γ_3 each have a prime number of vertices, hence they are clearly prime with respect to the Cartesian product. Let $n \geq 4$ and consider the fundamental decomposition of Γ_n. Then by induction assumption, σ restricted to Γ_{n-1} has a single equivalence class, the same holds for the restriction of σ to Γ_{n-2}. The edges between these two subgraphs are in relation Θ. One of these edges is the edge between $000\ldots0$ and $100\ldots0$. It is in relation τ with the edge between $000\ldots0$ and $010\ldots0$ that lies in Γ_{n-1}. Hence σ has only one equivalence class. □

4 Enumeration results

In this section we present several enumeration results on Fibonacci cubes, most of them being obtained by the method of generating functions. The results include the degree sequence, the cube polynomial, and the eccentricity sequence of Fibonacci cubes. These results cover numerous partial results obtained earlier in different papers. It nevertheless seems appealing to first present some specific formulas. In [33] it is proved that for $n \geq 1$,

$$|E(\Gamma_n)| = \frac{nF_{n+1} + 2(n+1)F_n}{5},$$

while in [22] the number of edges of Γ_n is expressed as

$$|E(\Gamma_n)| = F_{n+1} + \sum_{i=1}^{n-2} F_i F_{n+1-i}$$

and the number of 4-cycles of Γ_n as

$$-\frac{3n}{25}F_{n+1} + \left(\frac{n^2}{10} + \frac{3n}{50} - \frac{1}{25}\right) F_n \, .$$

4.1 Degree sequence

A fundamental structural property of a given (family of) graph(s) is the number of vertices of a given degree. Q_n is an n-regular graph, hence nothing has to be said for hypercubes. On the other hand, the situation with Fibonacci cubes is much more interesting. To determine the degree sequence of Γ_n, let $a_{n,k}$ and $b_{n,k}$ be the number of vertices of A_n and B_n, respectively, of degree k, where $n \geq 1$ and $0 \leq k \leq n$. Consider a vertex $x \in A_n$ of degree k. Then it is of degree $k-1$ in the subgraph Γ_{n-2} of Γ_n induced by A_n. Since x lies in exactly one of the corresponding sets A_{n-2} and B_{n-2}, we get

$$a_{n,k} = a_{n-2,k-1} + b_{n-2,k-1} \, .$$

Similarly, a vertex $y \in B_n$ either has a neighbor in A_n (if it starts with 00) or has no neighbor in A_n. In the first case, it is a vertex of the corresponding set B_{n-1}, in the second case, a vertex of A_{n-1}. Therefore,

$$b_{n,k} = b_{n-1,k-1} + a_{n-1,k} \, .$$

Hence the degree sequences in the subgraphs induced by A_n and B_n satisfy the system of linear recurrences and initial conditions

$$\begin{aligned}
a_{n,k} &= a_{n-2,k-1} + b_{n-2,k-1} \quad (n \geq 2,\ k \geq 1),\\
b_{n,k} &= b_{n-1,k-1} + a_{n-1,k} \quad (n \geq 1,\ k \geq 1),\\
a_{0,k} &= a_{n,0} = 0 \quad (n \geq 0,\ k \geq 0), \quad a_{1,1} = 1, \quad a_{1,k} = 0 \quad (k \geq 2),\\
b_{0,0} &= 1, \quad b_{0,k} = b_{n,0} = 0 \quad (n \geq 1,\ k \geq 1).
\end{aligned}$$

Their generating functions $a(x,y) = \sum_{n,k\geq 0} a_{n,k}x^n y^k$ and $b(x,y) = \sum_{n,k\geq 0} b_{n,k}x^n y^k$ therefore satisfy the system

$$\begin{aligned}
a(x,y) - xy &= x^2y\, a(x,y) + x^2y\, b(x,y),\\
b(x,y) - 1 &= xy\, b(x,y) + x\, a(x,y),
\end{aligned}$$

whose solution is

$$\begin{aligned}
a(x,y) &= \frac{xy(1+x-xy)}{(1-xy)(1-x^2y) - x^3y},\\
b(x,y) &= \frac{1}{(1-xy)(1-x^2y) - x^3y}.
\end{aligned}$$

After standard, but technical, computation we arrive at the following result.

Theorem 4.1 ([25]) *Let $n \geq k \geq 0$. Then the number of vertices of Γ_n having degree k is equal to*

$$\sum_{i=0}^{k} \binom{n-2i}{k-i}\binom{i+1}{n-k-i+1}.$$

In Theorem 4.1, the summation can be restricted to the interval between $\lceil (n-k)/2 \rceil$ and $\min(k, n-k)$ as the other terms are equal to zero.

Several results closely related to Theorem 4.1 were obtained earlier. In the first paper on Fibonacci cubes it was observed [14, Lemma 6] that the degrees of Γ_n lie between $\lfloor (n+2)/3 \rfloor$ and n. In [10] a recursive formula (depending on the recursive structure of Γ_n and the value of the integer that represents the given binary vertex) for computing the vertex degrees is given. The approach was further developed in [36] in order to investigate the domination number of Γ_n. The related main result ([36, Theorem 2.6]) gives an explicit description of the vertices of degrees between $n-3$ and n. In fact, Theorem 4.1 implies the following:

Corollary 4.2 ([25]) *Let $m \geq 0$ and let $n \geq 2m+2$. Then the number of vertices of Γ_n of degree $n-m$ is equal to*

$$\begin{cases} 1; & m = 0, \\ 2; & m = 1, \\ n+1; & m = 2, \\ 3n-8; & m = 3, \\ n^2/2 + 3n/2 - 21; & m = 4, \\ 2n^2 - 16n + 10; & m = 5\,. \end{cases}$$

More generally, $f_{n,n-m}$ is a polynomial in n of degree $\lfloor m/2 \rfloor$. Its leading coefficient is $\frac{1}{(m/2)!}$ when m is even, and $\frac{\lceil m/2 \rceil + 1}{\lfloor m/2 \rfloor !}$ when m is odd.

For a result parallel to Corollary 4.2 for vertices of Γ_n of small degrees see [25].

The *weight* of a vertex $b_1 b_2 \ldots b_n \in V(\Gamma_n)$ is $\Sigma_{i=1}^{n} b_i$. There is also an expression for the number of vertices of Γ_n of a given weight:

Theorem 4.3 ([25]) *Let k, n, w be integers with $k, w \leq n$. Then the number of vertices of Γ_n having degree k and weight w is equal to*

$$\binom{w+1}{n-w-k+1}\binom{n-2w}{k-w}.$$

4.2 Cube polynomial

In [14, 22, 34] several expressions involving the number of vertices, the number of edges, and the number of induced 4-cycles of Γ_n were obtained. A more general approach is to consider the *cube polynomial*

$$C(G, x) = \sum_{n \geq 0} c_n(G) x^n$$

of a graph G, where $c_n(G)$ denotes the number of induced subgraphs of G isomorphic to Q_n. This concept was introduced in [2] and in particular encompasses the number of vertices $c_0(G) = |V(G)|$, the number of edges $c_1(G) = |E(G)|$, and the number of induced 4-cycles $c_2(G)$ of G.

Using the fundamental decomposition of Γ_n it is not hard to deduce that the generating function of the sequence $\{C(\Gamma_n, x)\}_{n=0}^{\infty}$ is

$$\sum_{n\geq 0} C(\Gamma_n, x)y^n = \frac{1 + y(1 + x)}{1 - y - y^2(1 + x)},$$

from which the cube polynomial of Γ_n can be deduced:

Theorem 4.4 ([24]) *For any $n \geq 0$, $C(\Gamma_n, x)$ is of degree $\lfloor \frac{n+1}{2} \rfloor$ and*

$$C(\Gamma_n, x) = \sum_{a=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-a+1}{a} (1 + x)^a .$$

Theorem 4.4 then in turn implies that the number of induced Q_k, $k \geq 0$, in Γ_n is equal to

$$\sum_{i=k}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1}{i} \binom{i}{k} .$$

Roots of $C(\Gamma_n, x)$ can also be explicitly computed [24]. As a consequence the sequences of coefficients of $C(\Gamma_n)$ are log-concave and unimodal.

Mollard also obtained the number of maximal (with respect to inclusion) induced k-cubes in Fibonacci cubes:

Theorem 4.5 ([31]) *For any $k \geq 1$, the number of* maximal *induced Q_k in Γ_n is equal to*

$$\binom{k+1}{n+1-2k} .$$

4.3 Distance invariants

Hsu [14] proved that the diameter of Γ_n is n, Munarini and Zagaglia Salvi [34] followed by proving that the radius of Γ_n is $\lceil n/2 \rceil$. They also determined the center of Γ_n. Very recently, Castro and Mollard proved the following appealing result:

Theorem 4.6 ([5]) *Let $n \geq k \geq 1$. Then the number of vertices of Γ_n with eccentricity k is equal to*

$$\binom{k}{n-k} + \binom{k-1}{n-k} .$$

5 Fibonacci cubes as median graphs

We have already observed that Γ_n is an induced subgraph of Q_n. But much more is true:

Theorem 5.1 ([22]) *For any $n \geq 0$, Γ_n is a median graph.*

Proof. The result is clearly true for $n \leq 2$. Let $n \geq 3$ and let $b = b_1 \ldots b_n$, $b' = b'_1 \ldots b'_n$, and $b'' = b''_1 \ldots b''_n$ be vertices of Γ_n. Then if a median c of b, b', b'' exists, it must be obtained by the majority rule: c_i must be equal to the element that appears at least twice among b_i, b'_i, and b''_i. Suppose that $c \notin \mathcal{F}_n$, that is, for some i, $c_i = c_{i+1} = 1$. Then at least two of b_i, b'_i, b''_i must be equal to 1, and at least two of $b_{i+1}, b'_{i+1}, b''_{i+1}$ must be equal to 1 as well. But then at least one of b, b, b'' is not a Fibonacci string by the pigeonhole principle. Hence Γ_n is an induced subgraph of Q_n such that to any triple of vertices its median also lies in Γ_n. By a classical theorem of Mulder [32] we conclude that Γ_n is a median graph. □

Another way to deduce Theorem 5.1 is to recall the main result of [28] which asserts that resonance graphs of catacondensed even ring systems are median graphs. As fibonaccenes belong to the family of catacondensed even ring systems, Fibonacci cubes are indeed median graphs.

To present a characterization of Fibonacci cubes among median graphs, we need the following two concepts. For a partial cube G, the vertices of its *τ-graph* G^τ are the Θ-classes of G, different Θ-classes E and F are adjacent whenever $E \neq F$ and there exist edges $e \in E$ and $f \in F$ with $e\tau f$. A Θ-class F of a partial cube G is called *peripheral* if at least one of the (two) connected components of $G - F$ has $|F|$ vertices. Vesel characterized Fibonacci cubes as follows:

Theorem 5.2 ([41]) *Let G be a median graph. Then G is isomorphic to Γ_n if and only if any Θ-class of G is peripheral and $G^\tau = P_n$.*

We add here that τ-graphs are universal in the sense that for every graph G there exists a median graph M such that $G = M^\tau$ [23].

Parallel to the decomposition of Fibonacci strings $\mathcal{F}_n$ into the sets A_n and B_n, there is a decomposition of $\mathcal{F}_n$ into the sets

$$A'_n = \{b_1 b_2 \ldots b_n \in \mathcal{F}_n \mid b_n = 1\} \quad \text{and} \quad B'_n = \{b_1 b_2 \ldots b_n \in \mathcal{F}_n \mid b_n = 0\}$$

of Fibonacci strings that end with 1 and 0, respectively. B'_n induces a subgraph of Γ_n isomorphic to Γ_{n-1}, just like it does B_n. Moreover, this subgraph is convex because the edges between it and the subgraph induced by A'_n (isomorphic to Γ_{n-2}) form a Θ-class of Γ_n. Taranenko and Vesel [40] proved that the subgraphs of Γ_n induced by B_n and by B'_n are the only convex subgraphs of Γ_n isomorphic to Γ_{n-1}. Combining this result (it is used to prove that the conditions of the next theorem are sufficient) with the fundamental decomposition leads to the following characterization of Fibonacci cubes, where $W_{uv} = \{w \mid d(u,w) < d(v,w)\}$, F_{uv} is the set of edges between W_{uv} and W_{vu}, and U_{uv} is the set of those vertices of W_{uv} that have a neighbor in W_{vu}.

Theorem 5.3 ([40]) *Let G be a connected, bipartite graph and let $uv \in E(G)$ be an edge of G with $\deg(u) = n$ and $\deg(v) = n-1$. Then G is isomorphic to Γ_n if and only if the following hold:*

(i) U_{uv} *is convex in* W_{uv}*,*
(ii) F_{uv} *is a matching that defines an isomorphism between* U_{uv} *and* W_{vu}*,*
(iii) W_{uv} *is isomorphic to* Γ_{n-1} *and* W_{vu} *is isomorphic to* Γ_{n-2}*,*
(iv) $U_{vu} = W_{vu}$*.*

Taranenko and Vesel used Theorem 5.3 to design a recognition algorithm for Fibonacci cubes that runs in $O(|E(G)| \log |V(G)|)$ time.

To conclude the section we add that the natural embedding of Γ_n into Q_n, considered as a median embedding, leads to further connections between Fibonacci cubes and Fibonacci numbers, see [26] how the Fibonacci triangle is reflected in Fibonacci cubes.

6 Fibonacci dimension and invariants of Fibonacci cubes

The *Fibonacci dimension*, $\mathrm{fdim}(G)$, of a graph G, is the smallest integer n such that G admits an isometric embedding into Γ_n. This concept was introduced in [3]. Let in addition $\mathrm{idim}(G)$ be the *isometric dimension* of G, that is, the smallest n such that G isometrically embeds into Q_n. These graph dimensions are related through the following basic facts:

Proposition 6.1 *Let G be a connected graph. Then $\mathrm{fdim}(G)$ is finite if and only if $\mathrm{idim}(G)$ is finite if and only if G is a partial cube. Moreover,*

$$\mathrm{idim}(G) \le \mathrm{fdim}(G) \le 2\,\mathrm{idim}(G) - 1\,.$$

To derive the upper bound of Proposition 6.1, consider G isometrically embedded into Q_n and insert 0 between every consecutive coordinates of embedded vertices. This yields Fibonacci strings of length $2n-1$ with the same distance function as on the orginal strings. The bound can be improved to

$$\mathrm{fdim} \le \mathrm{idim}(G) + \mathrm{ldim}(G) - 1\,,$$

where $\mathrm{ldim}(G)$ (the *lattice dimension* of G) denotes the smallest integer n such that G admits an isometric embedding into $\mathbb{Z}^n$, or, equivalently, into the Cartesian product of n paths. Clearly, $\mathrm{ldim}(G) \le \mathrm{idim}(G)$. For additional bounds and exact results on the Fibonacci dimension see [3], let us extract here that $\mathrm{ldim}(G) \le \lceil \mathrm{fdim}(G)/2 \rceil$ holds for any partial cube G.

Algorithmic aspects of the Fibonacci dimension were also studied in [3]. Since the problem of computing the isometric dimension is polynomial ($\mathrm{idim}(G)$ is equal to the number of Θ-classes of G), and the same holds for the lattice dimension [11], the following result comes with a surprise:

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Theorem 6.2 *It is NP-complete to decide whether* $\mathrm{idim}(G) = \mathrm{fdim}(G)$ *for a given graph* G*.*

On a positive side, Vesel [42] developed a linear algorithm for the computation of the Fibonacci dimension of the resonance graphs of catacondensed benzenoid graphs. (Recall from the introduction that Fibonacci cubes are precisely the resonance graphs of a special class of catacondensed benzenoid graphs.)

In the rest of the section we turn our attention to graph invariants that were studied on Fibonacci cubes.

Observability Since Γ_n is bipartite, its chromatic number and chromatic index are known. Of more interest is another coloring invariant defined as follows. The *observability* of a graph G is the minimum number of colors needed in a proper edge-coloring of G such that distinct vertices receive different sets of colors on their incident edges. Dedó, Torri, and Zagaglia Salvi [7] proved that the observability of Γ_n is n, that is, it is equal to its chromatic index. To construct appropriate edge-colorings they recursively applied the fundamental decomposition a suitable number of times.

Decycling number and fault-tolerance The *decycling number* $\nabla(G)$ of a graph G is the size of a smallest set $X \subseteq V(G)$ such that $G - X$ has no cycle.

Ellis-Monaghan, Pike, and Zou [10] studied the decycling number of Fibonacci cubes and determined it exactly up to $n = 9$. For instance, $\nabla(\Gamma_8) = 19$ and $\nabla(\Gamma_9) = 33$. Hence the first open case is Γ_{10} for which the general bounds obtained imply $53 \leq \nabla(\Gamma_{10}) \leq 55$. They also conjectured that $\nabla(\Gamma_n) = \nabla(\Gamma_{n-1}) + \nabla(\Gamma_{n-2}) + \mu(n)$, where $\mu(n)$ is a non-decreasing function of n.

In the decycling problem one searches for the smallest number of vertices of Γ_n that destroy all cycles. A reverse problem is to search for the smallest number of vertices of a graph that destroy all Fibonacci subcubes. This problem was attacked by Gregor in [13]. More precisely, he was interested in the smallest number of vertices that need to be removed from Q_n such that all induced subgraphs isomorphic to Γ_m are destroyed. Denoting this number with $\psi(n,m)$, Gregor among other results determined $\psi(n,m)$ for all n and $m \leq 3$. For instance, $\psi(n,3) = \lfloor 2^n/3 \rfloor$, $n \geq 3$. Several general bounds are also established, for instance, $\psi(n,m) \geq 2\psi(n-4,m-4)$ holds for any $n \geq m \geq 4$.

Domination number The domination number of Fibonacci cubes was first investigated by Pike and Zou [36]. The fundamental decomposition clearly implies that $\gamma(\Gamma_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-2})$, while for the upper bound Pike and Zou proved:

Theorem 6.3 ([36]) *For any* $n \geq 4$*,*

$$\gamma(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 3}{n - 2} \right\rceil .$$

By computer search, Pike and Zou determined exact values of $\gamma(\Gamma_n)$ for $n \leq 8$. In [4] it was demonstrated that $\gamma(\Gamma_9) \leq 17$ and conjectured that $\gamma(\Gamma_9) = 17$ holds. Using linear programming approach and computer, Ilić and Milošević [18] confirmed this conjecture and furthermore established that $\gamma(\Gamma_{10}) = 25$.

Two additional invariants related to domination were investigated on Fibonacci cubes, the 2-packing number in [4, 18], and the independent domination number in [18].

7 Related classes of graphs

In this section we list numerous variants and generalizations of Fibonacci cubes that were proposed in the literature.

7.1 Lucas cubes

Lucas cubes form a class of graphs closely related to Fibonacci cubes. The Lucas cube Λ_n, $n \geq 1$, is the subgraph of the n-cube induced by Fibonacci strings $b_1 \ldots b_n$ such that not both b_1 and b_n are equal to 1. In other words, Λ_n is obtained from Γ_n by removing all vertices that begin and start with 1. The name of these cubes is justified with the fact that for any $n \geq 1$, $|V(\Lambda_n)| = L_n$, where L_n are the *Lucas numbers* defined with $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Moreover, $|E(\Lambda_n)| = nF_{n-1}$, see [33].

The definition of the Lucas cubes can be considered as a symmetrization of the definition of the Fibonacci cubes which in turn leads to more symmetric graphs. To be more precise, for $n \geq 3$, the automorphism group of Λ_n is isomorphic to the dihedral group D_{2n}, while the automorphism group of Γ_n is $\mathbb{Z}_2$, see [4]. Because of the close similarity between Fibonacci cubes and Lucas cubes, they are frequently studied together. For numerous results on Lucas cubes see [4, 7, 13, 22, 24, 25, 26, 33, 49].

7.2 Extended Fibonacci cubes

Motivated by the fact that only one third of all Fibonacci cubes are hamiltonian, Wu [45] introduced extended Fibonacci cubes Γ_n^i with $n \geq i \geq 0$ as follows. $V(\Gamma_{n+2}^i) = 0V(\Gamma_{n+1}^i) \cup 10V(\Gamma_n^i)$, with initial conditions $V(\Gamma_i^i) = \mathcal{B}_i$ and $V(\Gamma_{i+1}^{i+1}) = \mathcal{B}_{i+1}$. Note that $\Gamma_i^i = Q_i$, $\Gamma_{i+1}^i = Q_{i+1}$, and $\Gamma_n^0 = \Gamma_n$. Whitehead and Zagaglia Salvi [43, Theorem 2.1] extended these observations by proving that $\Gamma_n^i = \Gamma_{n-1}^{i-1} \,\square\, K_2$ which in turn implies that $\Gamma_n^i = \Gamma_{n-i} \,\square\, Q_i$. Hence, the study of Extended Fibonacci cubes can often be reduced to the study of Fibonacci cubes via the Cartesian product decomposition. For instance, it is well known that if G and H have hamiltonian paths, then $G \,\square\, H$ has a hamiltonian cycle unless both G and H are bipartite of odd order, see [20, Exercise 3.7]. Since Fibonacci cubes and hypercubes have hamiltonian paths (hypercubes also have hamiltonian cycles), this result immediately implies that Extended Fibonacci cubes have hamiltonian cycles. Similarly, Extended Fibonacci cubes are median graphs, see [22], because the Cartesian product of two median graphs is median.

7.3 Enhanced Fibonacci cubes

Qian and Wu [37] introduced enhanced Fibonacci cubes as graphs that contain Fibonacci cubes as subgraphs, maintain their important properties and posses additional ones, like having hamiltonian cycles. The first four *Enhanced Fibonacci cubes* EFC_n, $n \leq 4$, are isomorphic to the first four Fibonacci cubes, respectively; for $n \geq 5$, EFC_n is the subgraph of Q_n induced by the vertex set $00V(EFC_{n-2}) \cup 10V(EFC_{n-2}) \cup 0100V(EFC_{n-4}) \cup 0101V(EFC_{n-4})$.

7.4 Fibonacci (p, r)-cubes

Egiazarian and Astola [9] introduced a wide generalization of Fibonacci cubes as follows. Let $p, r \leq n$, then a *Fibonacci (p, r)-string* of length n is a binary string of length n in which there are at most r consecutive 1s and at least p 0s between two substrings composed of (at most r) consecutive 1s. Then the *Fibonacci (p, r)-cube* $\Gamma_n^{(p,r)}$ is the subgraph of Q_n induced by the Fibonacci (p, r)-strings (of length n). Note that $\Gamma_n^{(1,n)} = Q_n$ and $\Gamma_n^{(1,1)} = \Gamma_n$. Moreover, graphs $\Gamma_n^{(p,1)}$ were previously introduced by Wu and Yang [46] as the *n-dimensional postal networks with series* $p + 1$.

Recall that Fibonacci cubes are precisely the resonance graphs of fibonaccenes. Ou, Zhang, and Yao [35] extended this result by obtaining the complete list of all Fibonacci (p, r)-cubes that can be represented as the resonance graph of some plane bipartite graph.

7.5 Fibonacci hypercubes

Consider the Fibonacci strings $\mathcal{F}_n$ of length n as vectors in $\mathbb{R}^n$ and let P be their convex hull. Then Rispoli and Cosares [38] defined the *Fibonacci hypercube* FQ_n as the graph with vertex set $\mathcal{F}_n$ in which two vertices are adjacent if they form an edge in the polytope P. In the rest we briefly mention results obtained by Rispoli and Cosares. First, and utmost important, the adjacencies in FQ_n can be described with coordinates as follows. Let $b = b_1 \dots b_n$ and $b' = b'_1 \dots b'_n$ be vertices of FQ_n, and let $D(b, b') = \{i \mid b_i \neq b'_i\}$. Then b is adjacent to b' if and only if $D(b, b')$ consists of consecutive elements of $\{1, \dots, n\}$. Note that this in particular implies that Γ_n is a proper spanning subgraph of FQ_n. The number of edges of FQ_n satisfies $E(FQ_n) = E(FQ_{n-1}) + E(FQ_{n-2}) + F_{n+2} - 1$, $n \geq 3$, and the degree of a given vertex of a FQ_n can be expressed as the function of the occurrence of the string 010. FQ_n is n-connected, has diameter $\lceil n/2 \rceil$, and contains a hamiltonian cycle.

7.6 Generalized Fibonacci cubes

Another wide generalization of Fibonacci cubes was recently introduced in [17]. Suppose f is an arbitrary binary string and $n \geq 1$. Then the *generalized Fibonacci cube*, $Q_n(f)$, is the graph obtained from Q_n by removing all vertices that contain f as a substring. In this notation, $\Gamma_n = Q_n(11)$. Much earlier, in 1993, Hsu and Chung [15] introduced, under the same name, the graphs $Q_n(1^s)$, that is, the graphs obtained from Q_n by removing all vertices that contain s consecutive 1s. The graphs $Q_n(1^s)$ were further studied in [30, 48].

As already said, the Fibonacci cube $Q_n(11)$ is an isometric (well, even median) subgraph of Q_n. To characterize strings f and integers n, for which $Q_n(f)$ is an isometric subgraph of Q_n seems a difficult question. In [17] the problem is solved for all strings f of length at most five and for all strings consisting of at most three blocks. Moreover, several embeddable and non-embeddable infinite series are given.

8 Concluding remarks

We close with some open problems and ideas for further investigation.

1. Can Fibonacci cubes be recognized in linear (with respect to the number of edges) time?

2. A progress on any of the studied invariants (decycling number, domination number, 2-packing number, ...) would be welcome.

3. Let Γ_n be a Fibonacci cube of odd order. For which vertices v of Γ_n, the graph $\Gamma_n - v$ contains a hamiltonian cycle?

4. It might be interesting to consider infinite Fibonacci cubes. Of course, the graph X defined on all infinite Fibonacci strings is disconnected, since X contains vertices that differ in infinite many coordinates. Therefore, a way to define Γ_∞ would be to define it as the connected component of X that contains the vertex that contains only 0s.

5. Study generalized Fibonacci cubes. Which generalized Fibonacci cubes are hamiltonian? Determine more strings f and integers n, for which $Q_n(f)$ is isometric or non-isometric in Q_n. Ideally, classify embeddable strings.

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