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DOMINATION GAME PLAYED ON TREES AND SPANNING SUBGRAPHS

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Abstract

The domination game, played on a graph G, was introduced in [3]. Vertices are chosen, one at a time, by two players Dominator and Staller. Each chosen vertex must enlarge the set of vertices of G dominated to that point in the game. Both players use an optimal strategy–Dominator plays so as to end the game as quickly as possible, Staller plays in such a way that the game lasts as many steps as possible. The game domination number $\gamma_g(G)$ is the number of vertices chosen when Dominator starts the game and the Staller-start game domination number $\gamma_g'(G)$ when Staller starts the game.

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In this paper these two games are studied when played on trees and spanning subgraphs. A lower bound for the game domination number of a tree in terms of the order and maximum degree is proved and shown to be asymptotically tight. It is shown that for every k, there is a tree T with $(\gamma_g(T), \gamma_g'(T)) = (k, k+1)$ and conjectured that there is none with $(\gamma_g(T), \gamma_g'(T)) = (k, k-1)$. A relation between the game domination number of a graph and its spanning subgraphs is considered. It is proved that for any integer $\ell \geq 1$, there exists a graph G and its spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$. Moreover, there exist 3-connected graphs G having a spanning subgraph such that the game domination number of the spanning subgraph is arbitrarily smaller than that of G.

Keywords: domination game, game domination number, tree, spanning subgraph

AMS subject classification (2010): 05C57, 91A43, 05C69

1 Introduction

The domination game played on a graph G consists of two players, Dominator and Staller who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen by either player, at least one additional vertex is dominated. Dominator wishes to dominate the graph as fast as possible and Staller wishes to delay the process as much as possible. The *game domination number* $\gamma_g(G)$ is the number of vertices chosen when Dominator starts the game provided that both players play optimally. Similarly, the *Staller-start game domination number* $\gamma_g'(G)$ is defined for the game when Staller starts the game. The Dominator-start game and the Staller-start game will be briefly called *Game 1* and *Game 2*, respectively.

This game was introduced in 2010 ([3]) but was brought to the authors' attention back in 2003 by Henning [4]. Among other results, the authors of [3] proved a lower bound for the game domination number of the Cartesian product of graphs and established a connection with Vizing conjecure; for the latter see [2]. The Cartesian product was further investigated in [6] where the behavior of $\lim_{\ell\to\infty} \gamma_g(K_m \,\square\, P_\ell)/\ell$ was studied in detail.

In the rest of this section we give some notation, definitions, and recall results needed later. Then, In Section 2 we prove a general lower bound for the game domination number of a tree. In Section 3 we consider which pairs of integers (r, s) can be realized as $(\gamma_g(T), \gamma_g'(T))$, where T is a tree. It is shown that this is the case for all pairs but those of the form $(k, k-1)$. This increases the previously known pairs that can be realized by *connected* graphs. For the pairs $(k, k-1)$ we conjecture that they cannot be realized by trees. In the final section we study relations between the game domination number of a graph and its spanning subgraphs. Among other results we construct graphs G having spanning trees T with $\gamma_g(G) - \gamma_g(T)$ arbitrarily large. This is rather surprising because the domination number of a spanning tree (or a spanning subgraph) can never be smaller than the domination number of its supergraph.

Throughout the paper we will use the convention that $d_1, d_2, \ldots$ denotes the sequence of vertices chosen by Dominator and $s_1, s_2, \ldots$ the sequence chosen by Staller. We say that a pair (r, s) of integers is *realizable* if there exists a graph G such that $\gamma_g(G) = r$ and $\gamma_g'(G) = s$. Following [6], we make the following definitions. A *partially dominated graph* is a graph in which some vertices have already been dominated in some turns of the game already played. A vertex u of a partially dominated graph G is *saturated* if each vertex in $N[u]$ is dominated. The *residual graph* of G is the graph obtained from G by removing all saturated vertices and all edges joining dominated vertices. If G is a partially dominated graph then $\gamma_g(G)$ and $\gamma_g'(G)$ denote the optimal number of moves remaining in Game 1 and Game 2, respectively.

Contrary to the game chromatic number (see [1] for a survey on this related graph invariant), the game domination number of a graph G can be bounded in terms of the domination number $\gamma(G)$ of G:

Theorem 1.1 ([3]) *For any graph G, $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$.*

It was demonstrated in [3] that in general Theorem 1.1 cannot be improved. More precisely, for any integer $k \geq 1$ and any $0 \leq r \leq k - 1$, there exists a graph G with $\gamma(G) = k$ and $\gamma_g(G) = k + r$.

The game domination number and the Staller-start game domination number are always close together as the next result asserts.

Theorem 1.2 ([3, 6]) *For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.*

By Theorem 1.2 only pairs of the form (r, r), $(r, r+1)$, and $(r, r-1)$ are realizable.

The following lemma, due to Kinnersley, West, and Zamani [6] in particular implies $\gamma_g'(G) \leq \gamma_g(G) + 1$, which is one half of Theorem 1.2. The other half was earlier proved in [3].

Lemma 1.3 (Continuation Principle) *Let G be a graph and $A, B \subseteq V(G)$. Let G_A and G_B be partially dominated graphs in which the sets A and B have already been dominated, respectively. If $B \subseteq A$, then $\gamma_g(G_A) \leq \gamma_g(G_B)$ and $\gamma_g'(G_A) \leq \gamma_g'(G_B)$.*

We wish to point out that Continuation Principle is a very useful tool for proving results about game domination number. In particular, suppose that at some stage of the game a vertex x is an optimal move for Dominator. Then, if there exists a vertex y such that the undominated part of $N[x]$ is contained in $N[y]$, then y is also an optimal selection for Dominator and we can thus assume (if necessary) that he plays y.

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2 A lower bound for trees

In this section we give a lower bound on the game domination number of trees and show that it is asymptotically sharp. Before we can state the main result, we need the following:

Lemma 2.1 *Let F be a partially dominated tree and suppose it is Staller's turn. Then Staller can make a move that dominates at most two new vertices.*

Proof. Let A be the set of saturated vertices of F and let B be the set of vertices of F that are dominated but not saturated. Let $C = V(F) - (A \cup B)$. Let F' be the subforest of F induced by $B \cup C$ but with edges induced by B removed (that is, F' is the residual graph). We may assume that $C \neq \emptyset$ since the game is not over yet. Then F' has a leaf x. If Staller plays x, she dominates at most two vertices in C. If $x \in B$, Staller dominates exactly one vertex in C. □

Note that the move guaranteed by Lemma 2.1 may not be an optimal move for Staller. For instance, the optimal first move of Staller when playing on P_5 is the middle vertex of P_5, thus dominating three new vertices. Also, we will see later that an optimal first move for Staller when playing Game 2 on the tree T_r from Figure 2 is $s_1 = w$ thus dominating $r + 1$ new vertices.

Theorem 2.2 *Let T be a tree on vertices $v_1, v_2, \ldots, v_n$, where $\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$. For $j \geq 1$, let $x_j = \sum_{i=1}^{j} \deg(v_i) + 3j$. Let r be the smallest integer such that $x_r \geq n$. Then $\gamma_g(T) \geq 2r - 1$ when $x_r - 2 \geq n$, and $\gamma_g(T) \geq 2r$ when $x_r \geq n$. In particular,*

$$\gamma_g(T) \geq \left\lceil \frac{2n}{\Delta(T) + 3} \right\rceil - 1\,.$$

Proof. By Lemma 2.1, Staller can move in such a way that at most two new vertices are dominated on each of her moves. Let us suppose that Dominator plays optimally when Staller plays to dominate at most two new vertices on each move. Let $d_1, s_1, d_2, s_2, \ldots, d_t, s_t$ be the resulting game, where we assume that s_t is the empty move if T is dominated after the move d_t. Let $f(d_i)$ (resp. $f(s_i)$) denote the number of newly dominated vertices when Dominator plays d_i (resp. when Staller plays s_i). Suppose the game ends on Staller's move. Then

$$n = \sum_{i=1}^{t} (f(d_i) + f(s_i)) \leq \sum_{i=1}^{t} ((\deg(v_i) + 1) + 2) = \sum_{i=1}^{t} \deg(v_i) + 3t = x_t\,.$$

By the choice of r we find that $t \geq r$. Since this strategy may not be an optimal one for Staller, it follows that $\gamma_g(T) \geq 2t \geq 2r$. Similar arguments gives $\gamma_g(T) \geq 2r - 1$ if the game ends on Dominator's move.

If Staller ends the game, then $n \le r(\Delta(T)+3) \le \frac{1}{2}\gamma_g(T)(\Delta(T)+3)$ and hence $\gamma_g(T) \ge \left\lceil \frac{2n}{\Delta(T)+3} \right\rceil$ since $\gamma_g(T)$ is integral. If the game ends on Dominator's move, then $\gamma_g(T) \ge 2r-1$ and hence

$$n \le r(\Delta(T)+3) \le \frac{\gamma_g(T)+1}{2}(\Delta(T)+3)\,.$$

This is equivalent to $2n \le (\gamma_g(T)+1)(\Delta(T)+3)$ which in turn implies that

$$\gamma_g(T) \ge \left\lceil \frac{2n}{\Delta(T)+3} - 1 \right\rceil = \left\lceil \frac{2n}{\Delta(T)+3} \right\rceil - 1$$

and we are done. □

To see that Theorem 2.2 is asymptotically optimal, consider the caterpillars $T(s,t)$ shown in Figure 1.

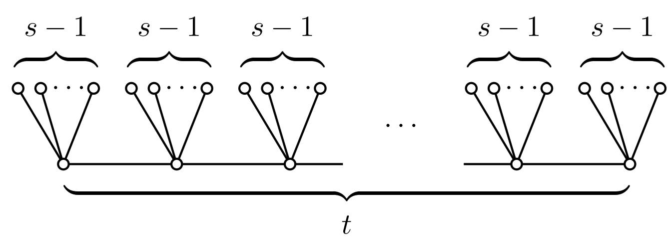

Figure 1: Caterpillar $T(s,t)$

Clearly, $T(s,t)$ has $n = st$ vertices. Let $s \ge t+1$, then it is easy to see that $\gamma_g(T(s,t)) = 2t-1$. Indeed, since $s-1 \ge t$, Staller can select a leaf after each of the first $t-1$ moves of Dominator. Hence after Dominator selects the t vertices of high degree, the game is over. By Theorem 2.2, $\gamma_g(T(s,t)) \ge \frac{2st}{s+4} - 1$, which for a fixed t, tends to $\frac{2n}{\Delta(T(s,t))+3} - 1 = \frac{2st}{s+4} - 1 \sim 2t-1$ as $s \to \infty$.

3 Pairs realizable by trees

In this section we are interested which of the possible realizable pairs (r,r), $(r,r+1)$, and $(r,r-1)$ can be realized on trees. It was observed in [3] that all pairs (k,k), $k \ge 1$, are realizable by trees. We now show that pairs $(k,k+1)$ are also realizable by trees. On the other hand, we prove that the pairs $(3,2)$ and $(4,3)$ cannot be realized by trees and conjecture that no pair $(k+1,k)$, $k \ge 1$, is realizable by a tree. (Clearly, no graph realizes the pair $(2,1)$.)

Theorem 3.1 *For any $k \ge 1$, there exits a tree T such that $\gamma_g(T) = k$ and $\gamma_g'(T) = k+1$.*

Proof. Stars $K_{1,n}$, $n \geq 2$, confirm the result for $k = 1$. For $k = 2$ consider the tree on five vertices obtained from $K_{1,3}$ by subdividing one edge. In the rest of the proof assume that $k \geq 3$. We distinguish three cases based on the parity of k mod 3.

Case 1: $(3r, 3r + 1)$.
Let $r \geq 1$ and consider the tree T_r of order $5r + 1$ from Figure 2.

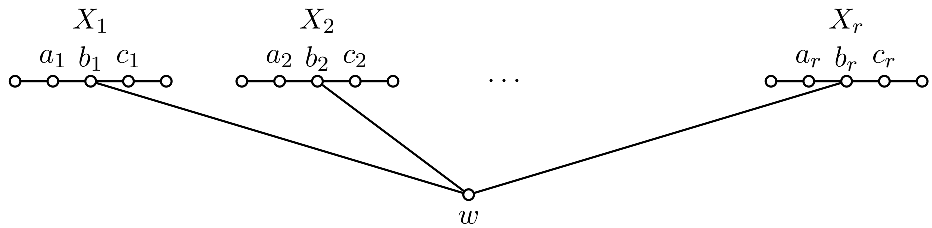

Figure 2: Tree T_r

We claim that $\gamma_g(T_r) = 3r$. Dominator can prevent Staller from forcing three moves in some X_i only if Staller does not follow Dominator on X_i but instead plays w. But this move by Staller uses w and eventually three moves will be made on each remaining X_j. (For example, play could be as follows: $d_1 = a_1$, $s_1 = b_1$, $d_2 = a_2$, $s_2 = b_2$, $d_3 = a_3$, $s_3 = w$, $d_4 = c_3$, $s_4 = b_4$, ... This game has $3r$ moves.) If, on the other hand, Staller continues to follow Dominator in all of X_i, then w is an illegal move. Hence $\gamma_g(T_r) = 3r$.

Consider now Game 2. Let Staller play $s_1 = w$ and then follow Dominator on X_i as soon as Dominator plays on X_i. In this way, on every X_i, $1 \leq i \leq r$, three vertices will be played in the game, hence $\gamma_g'(T_r) \geq 3r + 1$. By Theorem 1.2, $\gamma_g'(T_r) = 3r + 1$.

Case 2: $(3r + 1, 3r + 2)$.
For $r \geq 1$ let T_r' be the graph of order $5r + 3$ obtained from T_r (the tree from Figure 2) by attaching a path of length 2 to w with new vertices y and z, where z is a pendant vertex (and y adjacent to w and z). Suppose Dominator plays a_1 in Game 1. If Staller now plays $s_1 = w$ and Dominator plays $d_2 = c_1$, then Staller guarantees 3 from each of $X_2, X_3, \ldots, X_r$. Thus if Staller plays $s_1 = w$, then $3r + 1$ moves will be made. On the other hand, Staller could play $s_1 = b_1$ which forces three vertices to be played on X_1. Dominator would respond with $d_2 = a_2$. Continuing in this manner (with Staller responding with $s_i = b_i$ to $d_i = a_i$) until two moves have been made on each of $X_1, X_2, \ldots, X_r$, then Dominator would play $d_{r+1} = y$. In this way, $3r + 1$ moves are made.

If Staller follows Dominator on $X_1, X_2, \ldots, X_i$, but when $d_{i+1} = a_{i+1}$ Staller suddenly plays $s_{i+1} = w$, then Dominator plays $d_{i+2} = c_{i+1}$. This uses $3i + 2 + 2 + 3(r - (i + 1)) = 3r + 1$ moves. Hence $\gamma_g(T_r') = 3r + 1$.

We play Game 2 next. Let $s_1 = w$. Then each X_i will have three played vertices since Staller follows Dominator on each X_i and hence $\gamma_g'(T_r') \geq 3r + 2$. By

Theorem 1.2, $\gamma_g'(T_r') = 3r + 2$.

Case 3: $(3r + 2, 3r + 3)$.
In this case consider the tree T_r'' $(r \geq 1)$ obtained from the tree T_r (of Figure 2) by attaching a path of length 2 to b_1, say $P = b_1, p, q$, and a path of length 2 to b_r, say $Q = b_r, u, v$. Note that if $r = 1$, then P and Q are attached at the same vertex $b_1 = b_r$.

Proof that $\gamma_g(G) = 3r + 2$ is similar to the proof of Case 1 in that Dominator either forces Staller to follow in each X_i or saves one move in some X_i in exchange for Staller playing w. Two more moves are required in order to dominate q and v. (Note that the same argument works also for $r = 1$.) An optimal strategy in Game 2 is again $s_1 = w$, now each X_i will have three played vertices and there will be two more played vertices because of q and v. Hence $\gamma_g'(T_r'') \geq 3r + 3$ and we conclude that $\gamma_g'(T_r'') = 3r + 3$. □

For the $(k, k-1)$ case we pose:

Conjecture 3.2 *No pair $(k, k-1)$, $k \geq 3$, can be realized by a tree.*

In the rest of the section we prove the first two cases of the conjecture:

Theorem 3.3 *No tree realizes the pair $(3, 2)$ or the pair $(4, 3)$.*

Proof. Suppose that a tree T realizes $(3, 2)$. It is easy to see that $\gamma_g'(T) = 2$ implies that T is either a star $K_{1,n}$ for $n \geq 2$ or a P_4. In both cases $\gamma_g(T) \leq 2$, thus $(3, 2)$ is not realizable on trees.

Suppose T is a tree that realizes $(4, 3)$, and let $d_1 = x$ be an optimal first move for Dominator. The residual graph T' has at most 3 components, each of which is a partially dominated subtree of T. Note that if one of these partially dominated components F has $\gamma_g'(F) = 1$, then F has exactly one undominated vertex.

Suppose first that T' has three partially dominated components T_1, T_2, T_3 with T_i rooted at the dominated vertex v_i. If at least one of these subtrees, say T_1 has more than one undominated vertices, then Staller can force at least two moves in T_1. Because the other two subtrees each require at least one move, it follows that $\gamma_g(T) \geq 5$, a contradiction. Hence, each of T_1, T_2, T_3 has exactly one undominated vertex, and T is the tree of order 7 formed by identifying a leaf from three copies of P_3. However, this tree has Staller-start game domination number 4, again contradicting our initial assumption.

Now suppose that T' is the disjoint union of T_1 and T_2. Note that in this case x cannot be a support vertex in the original tree T. Indeed, if x is adjacent to a leaf y, then when Game 2 is played on T Staller can play first at y which is easily shown to force at least four moves. Thus, $\deg(x) = 2$. If $\gamma_g'(T_1) = 1 = \gamma_g'(T_2)$, then $T = P_5$ and $\gamma_g(T) = 3$, a contradiction.

Note that the Staller-start game domination number of any of these two partially dominated trees cannot exceed 2. We may thus assume without loss of generality that $2 = \gamma_g'(T_1) \geq \gamma_g'(T_2)$. Suppose that $\gamma_g'(T_2) = 2$. Staller can then play at vertex x when Game 2 is played on T. After Dominator's first move at least one of T_1 or T_2 is part of the residual graph, and Staller can then force at least two more moves again contradicting the assumption that $\gamma_g'(T) = 3$. Therefore, T_2 is the path of order 2 with one of its vertices dominated.

If T_1 is a star with v_1 as its center or as one of its leaves, then $\gamma(T) = 2$ and hence $4 = \gamma_g(T) \leq 2 \cdot 2 - 1$, an obvious contradiction. Therefore, $\gamma_g'(T_1) = 2$, but T_1 is not a star. A short analysis shows that T_1 must be one of the partially dominated trees in Figure 3. Each of these candidates for T_1 together with $T_2 = P_2$ yields a tree T with either $\gamma_g(T) \neq 4$ or $\gamma_g'(T) \neq 3$, again contradicting our assumption about T. This implies that the residual graph T' has exactly one component.

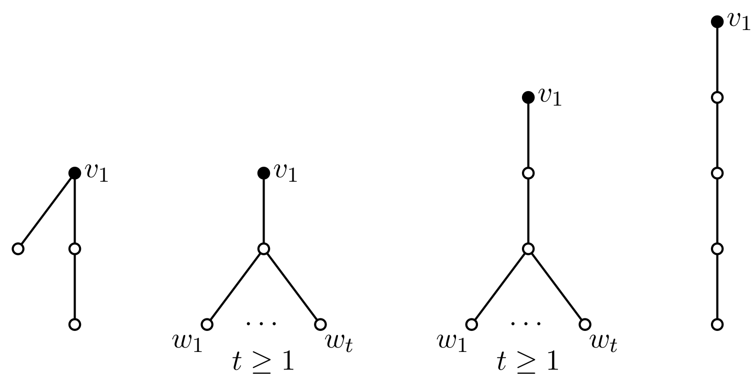

Figure 3: Possible partially dominated trees

By Continuation Principle (Lemma 1.3) it must be the case that x is a support vertex in T because we assumed that x was an optimal move by Dominator. Let $A = \{v_1\}$ and $B = \emptyset$. Since $\gamma_g'(T_1) = 3$ we apply Lemma 1.3 to get

$$3 = \gamma_g'(T) = \gamma_g'(T_B) \geq \gamma_g'(T_A) \geq 1 + \gamma_g'(T_1) = 4\,.$$

This establishes the theorem. □

4 Game on spanning subgraphs

We now turn our attention to relations between the game domination number of a graph and its spanning subgraphs, in particular spanning trees.

Note that since any graph is a spanning subgraph of the complete graph of the same order, the ratio $\gamma_g(H)/\gamma_g(G)$ where H is a spanning subgraph of G can be arbitrarily large. On the other hand the following result shows that this ratio is bounded below by one half.

Proposition 4.1 *Let G be a graph and let H be a spanning subgraph of G. Then*

$$\gamma_g(H) \geq \frac{\gamma_g(G)+1}{2}\,.$$

In particular, if T is a spanning tree of connected G, then $\gamma_g(T) \geq (\gamma_g(G)+1)/2$.

Proof. Clearly, $\gamma(H) \geq \gamma(G)$. By Theorem 1.1, $\gamma_g(H) \geq \gamma(H)$ and $\gamma_g(G) \leq 2\gamma(G)-1$. Then $\gamma_g(H) \geq \gamma(H) \geq \gamma(G) \geq (\gamma_g(G)+1)/2$. □

To see that a spanning subgraph can indeed have game domination number much smaller that its supergraph, consider the graph G_t consisting of t blocks isomorphic to the house graph and its spanning subgraph H_t, see Figure 4. Let x be the vertex where the houses of G_t are amalgamated. Note that Dominator needs at least two moves to dominate each of the blocks of G_t. Hence his strategy is to play $d_1 = x$ and then finish dominating one block at each move. On the other hand, if not all blocks are already dominated, Staller can play the vertex of degree 2 adjacent to x of such a block B in order to force one more move on B. So in half of the blocks two vertices will be played (not counting the move on x) which in turn implies that $\gamma_g(G_t)$ is about $3t/2$. On the other hand, playing Game 1 on H_t, the optimal first move for Dominator is $d_1 = x$. After that Staller and Dominator will in turn dominate each of the triangles, hence $\gamma_g(H_t) = t+1$.

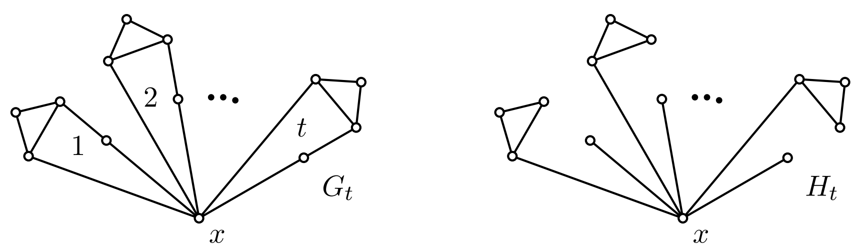

Figure 4: Graph G_t and its spanning subgraph H_t

The example of Figure 4 might indicate that spanning subgraphs can have smaller game domination number than their supergraphs provided none is 2-connected. However:

Theorem 4.2 *For any $m \geq 3$ there exists a 3-connected graph G_m and its 2-connected spanning subgraph H_m such that $\gamma_g(G_m) \geq 2m-2$ and $\gamma_g(H_m) = m$.*

Proof. We form a graph G_m of order $n = m(m+2)$ as follows. Let $X_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m}\} \cup \{x_i, y_i\}$ for each $1 \leq i \leq m$, and then set

$$V(G_m) = \bigcup_{i=1}^{m} X_i\,.$$

The edges are the following. We let $\{x_1, y_1, x_2, y_2, \ldots, x_m, y_m\}$ induce a complete graph of order $2m$. For each p, $1 \le p \le m$ we let X_i induce a complete graph of order $m+2$. In addition, for each $1 \le i \le m-1$ and each $i \le j \le m-1$ we add the edge $a_{i,j}a_{j+1,i}$. See Figure 5 for G_4.

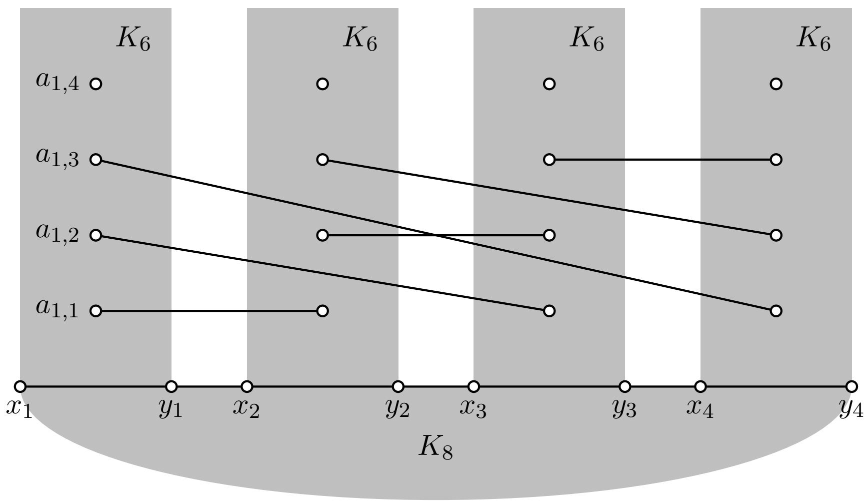

Figure 5: Graph G_4

Suppose first that $d_1 = x_1$. Then Staller plays in X_1, say $s_1 = a_{1,1}$. Then, in each of the next rounds, Continuation Principle implies that Dominator must play in some X_i that has not been played in before and on a vertex of X_i that has a neighbor outside X_i. It will always be possible for Staller to follow Dominator and also play in X_i in each of her first $m-2$ moves. Hence by this time, $2m-4$ moves were made. At this stage, there are two undominated vertices in different X_i's with no common neighbor. Hence two more moves are needed to end the game which thus ends in no less than $2m-2$ moves.

Assume next that $d_1 = a_{1,1}$. Then Staller plays $s_1 = x_1$ and we are in the first case. Note that playing $d_1 = x_1$ or $d_1 = a_{1,1}$ covers all the cases due to symmetry. Hence $\gamma_g(G_m) \ge 2m-2$.

The spanning subgraph H_m of G_m is obtained by removing all the edges $a_{i,j}a_{j+1,i}$. By Continuation Principle we may without loss of generality assume that $d_1 = x_1$ when Game 1 is played on H_m. But then on each successive move of either of the players the newly dominated vertices are a subset of the X_i on which they play. Hence $\gamma_g(H_m) = m$. □

If $\gamma_g(G)$ attains one of the two possible extremal values, $\gamma(G)$ or $2\gamma(G)-1$, we can say more.

Proposition 4.3 *Let G be a graph with $\gamma_g(G) = \gamma(G)$ and let H be a spanning subgraph of G. Then $\gamma_g(H) \ge \gamma_g(G)$.*

Proof. $\gamma_g(H) \geq \gamma(H) \geq \gamma(G) = \gamma_g(G)$. □

In particular, every spanning tree T of a connected graph G with $\gamma_g(G) = \gamma(G)$ has $\gamma_g(T) \geq \gamma_g(G)$.

Proposition 4.4 *Let G be a graph with $\gamma_g(G) = 2\gamma(G) - 1$ and let H be a spanning subgraph of G with $\gamma(H) = \gamma(G)$. Then $\gamma_g(H) \leq \gamma_g(G)$.*

Proof. $\gamma_g(H) \leq 2\gamma(H) - 1 = 2\gamma(G) - 1 = \gamma_g(G)$. □

Since every graph G has a spanning forest F such that $\gamma(G) = \gamma(F)$, see [5, Exercise 10.14], we infer:

Corollary 4.5 *Let G be a graph with $\gamma_g(G) = 2\gamma(G) - 1$. Then G contains a spanning forest F (spanning tree if G is connected) such that $\gamma_g(F) \leq \gamma_g(G)$.*

In the rest of this section we focus on spanning trees. First we show that a graph can have the property that all of its spanning trees have game domination number much larger than that of the supergraph. Let $n \geq 3$, $m = 2r$, and let S be the star with center x and leaves $v_1, v_2, \ldots, v_m$. Let G_m be the graph (of order $nm + 1$) constructed by identifying a vertex of a complete graph of order n with v_i, for each i, $1 \leq i \leq m$; see Figure 6.

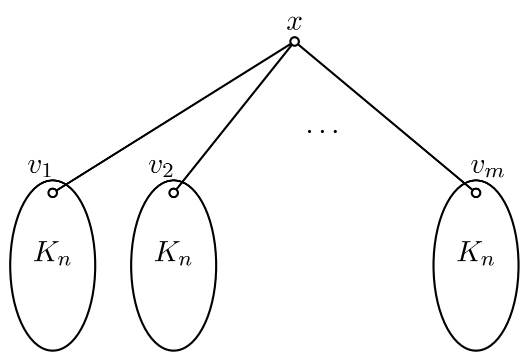

Figure 6: Graph G_m

We first note that $\gamma_g(G_m) = m + 1$. Let T be any spanning tree of G_m. T has at least one leaf ℓ_i in the subtree T_i of T rooted at v_i when the edge xv_i is removed from T (choose $\ell_i \neq v_i$). When Game 1 is played on T, Staller can choose at least half of these leaves $(\ell_1, \ell_2, \ldots, \ell_m)$ or let Dominator choose them. Thus in at least half of $T_1, T_2, \ldots, T_m$, two vertices will be chosen. Therefore $\gamma_g(T) \geq m + m/2 = 3m/2$, hence we conclude that

$$\gamma_g(T) - \gamma_g(G_m) \geq \frac{3}{2}m - m - 1 = r - 1\,.$$

Next we give an example of a spanning tree with game domination number smaller than the one of its supergraph. Consider the graph G and its spanning tree T from Figure 7.

Figure 7: Graph G and its spanning tree T

For each of the following pairs of vertices (x, y) from G, if Dominator plays x then Staller can play y and then the game domination number of the resulting residual graph G' will be 2: $(1,6); (2,3); (3,2); (4,8); (8,4); (7,3); (6,1); (5,1)$. Therefore, $\gamma_g(G) \geq 4$. Consider now the spanning tree T and let Dominator play 2 on T. For each of the following vertices a, the residual graph T' when Staller plays a is listed in Figure 8. For instance, the left case is when Staller plays $a = 5$ in which case the residual graph is induced by vertices $6, 7, 8$ and the vertex 6 of the residual graph is already dominated as indicated by the filled vertex.

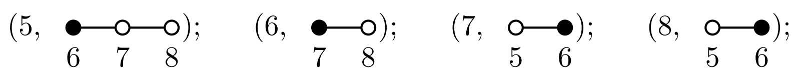

Figure 8: Staller's possible moves

In each case we find that the residual graph has game domination number 1 and therefore,

$$\gamma_g(T) \leq 3 < \gamma_g(G)\,.$$

This rather surprising fact demonstrates the intrinsic difficulty and unusual behavior of the game domination number. But even more can be shown:

Theorem 4.6 *For any integer $\ell \geq 1$, there exists a graph G and its spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$.*

Proof. We introduce the family of graphs G_k and their spanning trees T_k in the following way. Let k be a positive integer, and for each i between 1 and k, $x_i^1, x_i^2, x_i^3, x_i^4, x_i^5$ are non-adjacent vertices in T_k, and $Q_i : y_i^1 y_i^2 y_i^3 y_i^4 y_i^5$ is a path isomorphic to P_5 in T_k. Finally x and y are two vertices, such that x is adjacent to $x_i^1, x_i^2, x_i^3, x_i^4$ and x_i^5 for all $i \in \{1, \ldots, k\}$, while y is adjacent to y_i^1 for all $i \in \{1, \ldots, k\}$, and x and y are also adjacent. The resulting graph T_k is a tree on $10k + 2$ vertices. We obtain G_k by adding edges between x_i^j and y_i^j for $1 \leq i \leq k$,

$1 \leq j \leq 5$. See Figure 9 for G_4, from which T_4 is obtained by removing all vertical edges except xy.

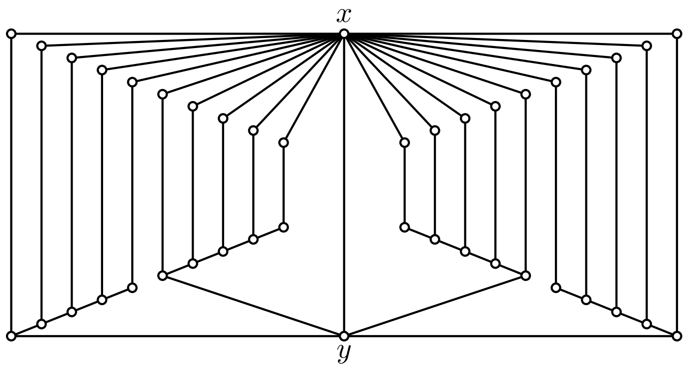

Figure 9: Graph G_4

To complete the proof it suffices to show that for any integer $k \geq 1$,

$$\gamma_g(G_k) \geq \frac{5}{2}k - 1 \quad \text{and} \quad \gamma_g(T_k) \leq 2k + 3\,.$$

Let us first verify the second inequality, concerning trees T_k. To prove it we need to show that Dominator has a strategy by which at most $2k+3$ moves will be played during Game 1. His strategy is as follows. In his first two moves, he ensures that x and y are chosen. He plays x in his first move, and y in his second move, unless already Staller played y (we will consider this case later). Now, $s_1 \neq y$ implies that s_1 is in some Q_i, without loss of generality let this be Q_1. Hence in Dominator's third move, since y_i^1 is dominated for each i, he can dominate all vertices of Q_1. One by one, Staller will have to pick a new Q_i to play in, which Dominator will entirely dominate in his next move. Altogether, in each Q_i (with a possible exception of one Q_i, where Staller can force three vertices to be played), there will be only two vertices played, which yields $2k+3$ as the total number of moves in this game. On the other hand, if $s_1 = y$, then $d_2 = y_1^3$ ensures that in Q_1 only two vertices will be played. Moreover, Dominator can follow Staller in each of the remaining Q_is to force only two moves in each. Hence the game will finish in $2k+2$ moves.

To prove the first inequality we need to show that Staller has a strategy to enforce at least $\frac{5}{2}k-1$ moves played during Game 1 in G_k. Her strategy in each of the first k moves of the game is to play an x_i^4 for which no vertex from $Q_i \cup \{x_i^1, x_i^2, x_i^3, x_i^5\}$ has been played before her move. This is clearly possible as Dominator made at most $\lceil \frac{k}{2} \rceil$ of these moves. Using this strategy she ensures that in each of these $\lfloor \frac{k}{2} \rfloor$ Q_is at least two more moves will be needed (since at least y_i^2, y_i^3 and y_i^5 are left undominated). The remaining $\lceil \frac{k}{2} \rceil$ paths Q_i require two moves each. Hence altogether, there will be at least $2k + \lfloor \frac{k}{2} \rfloor$ moves played during Game 1, which implies $\gamma_g(G_k) \geq \frac{5}{2}k - 1$.
□

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