

IMFM

INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS
JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

Preprint series

Vol. 50 (2012), 1168

ISSN 2232-2094

**SHARP UPPER AND
LOWER BOUNDS ON THE
NUMBER OF SPANNING
TREES IN CARTESIAN
PRODUCT GRAPHS**

Jernej Azarija

Ljubljana, January 3, 2012

Sharp upper and lower bounds on the number of spanning trees in Cartesian product graphs

Jernej Azarija *

January 1, 2012

Abstract

Let G_1 and G_2 be simple graphs and let $n_1 = |V(G_1)|$, $m_1 = |E(G_1)|$, $n_2 = |V(G_2)|$ and $m_2 = |E(G_2)|$. In this paper we derive sharp upper and lower bounds for the number of spanning trees τ in the Cartesian product $G_1 \square G_2$ of G_1 and G_2 . We show that:

$$\tau(G_1 \square G_2) \geq \frac{2^{(n_1-1)(n_2-1)}}{n_1 n_2} (\tau(G_1) n_1)^{\frac{n_2+1}{2}} (\tau(G_2) n_2)^{\frac{n_1+1}{2}}$$

and

$$\tau(G_1 \square G_2) \leq \tau(G_1) \tau(G_2) \left[\frac{2m_1}{n_1 - 1} + \frac{2m_2}{n_2 - 1} \right]^{(n_1-1)(n_2-1)}.$$

We also characterize the graphs for which equality holds. As a by-product we derive a formula for the number of spanning trees in $K_{n_1} \square K_{n_2}$ which turns out to be $n_1^{n_1-2} n_2^{n_2-2} (n_1 + n_2)^{(n_1-1)(n_2-1)}$.

1 Introduction

An important invariant in graph theory is $\tau(G)$, the number of spanning trees of a graph G . The first result related to $\tau(G)$ dates back to 1847 and is attributed to Kirchhoff. In his celebrated theorem he has shown that the number of spanning trees of a graph G is closely related to the cofactor of a special matrix (the *Laplacian matrix*) that can be obtained after subtracting the *degree matrix* (a diagonal matrix with vertex degrees on the diagonals) from the respective *adjacency matrix*. If by $Q(G)$ we denote the Laplacian matrix of a graph G of order n with eigenvalues $0 = \lambda_1 \leq \dots \leq \lambda_n$, then a corollary of Kirchoff theorem can be stated as:

$$\tau(G) = \frac{1}{n} \lambda_2 \cdots \lambda_n. \quad (1)$$

For example, as the eigenvalue n of $Q(K_n)$ has multiplicity $(n - 1)$, it follows that

$$\tau(K_n) = n^{n-2}. \quad (2)$$

*Department of Mathematics, University of Ljubljana, Jadranska 21, 1000 Ljubljana, Slovenia,
e-mail: jernej.azarija@gmail.com

Equation (2) is also referred to as *Cayley formula* as a tribute to its discoverer Arthur Cayley [8]. For a survey of known results related to the Laplacian spectrum of graphs we refer the reader to [13].

Since the result of Cayley, many interesting identities for the number of spanning trees for various classes of graphs have been derived. For example, it was shown by Baron et al. [3] that the number of spanning trees of the square of a cycle C_n^2 equals to $n^2 f_n$ where f_n is the n 'th *Fibonacci number*. Bogdanowicz [4] later showed that the number of spanning trees of the n -fan F_{n+1} equals to f_{2n} . A similar result relating the number of spanning trees of the wheel graph to *Lucas numbers* is also known [10]. Counting the number of spanning trees is not only an area that is rich with surprising identities but also holds a fundamental role in other scientific areas such as physics [14, 9, 7] and networking theory [5, 6]. As a network invariant τ is a crucial measure of network reliability [1] and also finds applications in the study of various electrical networks [2]. Since *graph products* (as defined in [11]) form a basis for many networks it is natural to study the function τ in relation with various graph products.

In this paper we study the number of spanning trees in the Cartesian product of graphs. For simple graphs G_1 and G_2 , the Cartesian product of $G_1 \square G_2$ is defined as the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in G_2 , or $u' = v'$ and u is adjacent to v in G_1 .

In what follows G_1 and G_2 will denote simple graphs of order n_1 and n_2 such that $m_1 = |E(G_1)|$ and $m_2 = |E(G_2)|$. Moreover, we will denote by $\lambda_1, \dots, \lambda_{n_1}$ and μ_1, \dots, μ_{n_2} the eigenvalues of $Q(G_1)$ and $Q(G_2)$ respectively. Using this notation, we can state the well known (see [13] for a survey of results related to the Laplacian spectrum) fact relating the eigenvalues of G_1 and G_2 to the eigenvalues of $G_1 \square G_2$ which are:

$$\lambda_i + \mu_j \quad \text{for } i = 1, \dots, n_1 \quad \text{and} \quad j = 1, \dots, n_2.$$

Applying the later equality to identity (1) and using the fact that $\lambda_1 = \mu_1 = 0$ one obtains the following formula for the number of spanning trees for the Cartesian product of G_1 and G_2 :

$$\tau(G_1 \square G_2) = \tau(G_1)\tau(G_2) \prod_{i=2}^{n_1} \prod_{j=2}^{n_2} (\lambda_i + \mu_j). \tag{3}$$

2 Upper and lower bounds for $\tau(G_1 \square G_2)$

We are going to simplify equation (3) as to obtain upper and lower bounds for $\tau(G_1 \square G_2)$. Furthermore we will characterize the graphs for which equality holds and derive a formula for the number of spanning trees of the *Rook's graph* $K_{n_1} \square K_{n_2}$.

Theorem 1. $\tau(G_1 \square G_2) \geq \frac{2^{(n_1-1)(n_2-1)}}{n_1 n_2} (\tau(G_1) n_1)^{\frac{n_2+1}{2}} (\tau(G_2) n_2)^{\frac{n_1+1}{2}}$ where equality holds if and only if G_1 or G_2 is not connected or $n_1 = n_2$ and $G_1 \simeq G_2 \simeq K_{n_1}$.

Proof. Consider the expression:

$$\prod_{i=2}^{n_1} \prod_{j=2}^{n_2} (\lambda_i + \mu_j).$$

By the inequality of arithmetic and geometric means $\lambda_i + \mu_j \geq 2\sqrt{\lambda_i\mu_j}$ for every i, j , it therefore follows that:

$$\prod_{i=2}^{n_1} \prod_{j=2}^{n_2} (\lambda_i + \mu_j) \geq \prod_{i=2}^{n_1} \prod_{j=2}^{n_2} 2\sqrt{\lambda_i\mu_j} = 2^{(n_1-1)(n_2-1)} \prod_{i=2}^{n_1} \prod_{j=2}^{n_2} \sqrt{\lambda_i\mu_j}.$$

The last expression can also be written as:

$$2^{(n_1-1)(n_2-1)} \prod_{i=2}^{n_1} \sqrt{\lambda_i^{n_2-1}} \prod_{j=2}^{n_2} \sqrt{\mu_j^{n_1-1}}.$$

We now multiply and divide the last expression by $\sqrt{n_1^{n_2-1}n_2^{n_2-1}}$ and obtain:

$$2^{(n_1-1)(n_2-1)} \sqrt{n_1^{n_2-1}n_2^{n_2-1}} \frac{\prod_{i=2}^{n_1} \sqrt{\lambda_i^{n_2-1}}}{\sqrt{n_1^{n_2-1}}} \frac{\prod_{j=2}^{n_2} \sqrt{\mu_j^{n_1-1}}}{\sqrt{n_2^{n_1-1}}}$$

which, according to (1), equals

$$2^{(n_1-1)(n_2-1)} (\tau(G_1)n_1)^{\frac{n_2-1}{2}} (\tau(G_2)n_2)^{\frac{n_1-1}{2}}.$$

The stated inequality now follows after combining the derived result with equation (3).

We now examine the cases in which equality holds. If G_1 or G_2 is not connected, then equality clearly holds as $\tau(G_1 \square G_2) = 0$. Therefore, let us assume G_1 and G_2 are connected. As we derived our inequality using the inequality of arithmetic geometric mean it follows that equality holds if and only if $\lambda_i = \mu_j$ for every $i = 2, \dots, n_1$ and $j = 2, \dots, n_2$. The later holds if and only if

$$\lambda_1 = \dots = \lambda_{n_1} = \mu_1 = \dots = \mu_{n_2},$$

which means that $Q(G_1)$ and $Q(G_2)$ have eigenvalues of multiplicity $n_1 - 1$ and $n_2 - 1$, respectively. As the only graph of order k whose Laplacian matrix has an eigenvalue of multiplicity $k - 1$ is K_k , it follows, that $n_1 = n_2$ and thus $G_1 \simeq K_{n_1} \simeq G_2$. \square

In the proof of Theorem 1 we applied the inequality of arithmetic geometric mean to each summand of (3) individually. Observe that the same inequality can be applied to the factors of equation (3). In the next theorem we use this observation and the fact that $\sum_{i=2}^{n_1} \lambda_i = 2|E(G_1)| = 2m_1$ and $\sum_{i=2}^{n_2} \mu_i = 2|E(G_2)| = 2m_2$ in order to derive an upper bound for $\tau(G_1 \square G_2)$.

Theorem 2. $\tau(G_1 \square G_2) \leq \tau(G_1)\tau(G_2) \left[\frac{2m_1}{n_1-1} + \frac{2m_2}{n_2-1} \right]^{(n_1-1)(n_2-1)}$, where equality holds if and only if G_1 or G_2 is not connected or $G_1 \simeq K_{n_1}$ and $G_2 \simeq K_{n_2}$.

Proof. As observed, we can bound equation (3) by applying the inequality of geometric arithmetic mean on its factors. We then obtain:

$$\tau(G_1 \square G_2) = \tau(G_1)\tau(G_2) \prod_{i=2}^{n_1} \prod_{j=2}^{n_2} (\lambda_i + \mu_j) \leq \tau(G_1)\tau(G_2) \left[\frac{\sum_{i=2}^{n_1} \sum_{j=2}^{n_2} (\lambda_i + \mu_j)}{(n_1 - 1)(n_2 - 1)} \right]^{(n_1-1)(n_2-1)},$$

which we further simplify to:

$$\tau(G_1)\tau(G_2) \left[\frac{(n_2 - 1) \sum_{i=2}^{n_1} \lambda_i + (n_1 - 1) \sum_{j=2}^{n_2} \mu_j}{(n_1 - 1)(n_2 - 1)} \right]^{(n_1-1)(n_2-1)}.$$

Applying the identity for the summation of the eigenvalues of the Laplacian matrix we obtain:

$$\tau(G_1)\tau(G_2) \left[\frac{2m_1}{n_1 - 1} + \frac{2m_2}{n_2 - 1} \right]^{(n_1-1)(n_2-1)},$$

which is what we wanted to show.

Observe now, that if G_1 or G_2 is not connected, equality in the stated bound clearly holds. Thus, let us assume G_1 and G_2 are connected. Equality will then hold if and only if

$$\lambda_i + \mu_j = \lambda_{i'} + \mu_{j'} \quad \text{for } i, i' = 1, \dots, n_1 \quad \text{and} \quad j, j' = 1, \dots, n_2.$$

The later holding if and only if

$$\lambda_1 = \dots = \lambda_{n_1} \quad \text{and} \quad \mu_1 = \dots = \mu_{n_2},$$

which means $G_1 \simeq K_{n_1}$ and $G_2 \simeq K_{n_2}$ as these are the only graphs of order n_1 and n_2 having eigenvalues of multiplicity $n_1 - 1$ and $n_2 - 1$, respectively. \square

The statements of Theorems 1 and 2 simplify substantially if G_1 and G_2 are trees. In this case we can write the implications of Theorem 1 and Theorem 2 as the following corollary.

Corollary 1. *If G_1 and G_2 are trees of order $n_1 \geq 3$ and $n_2 \geq 3$ respectively, then*

$$2^{(n_1-1)(n_2-1)} n_1^{\frac{n_2-1}{2}} n_2^{\frac{n_1-1}{2}} < \tau(G_1 \square G_2) < 2^{2(n_1-1)(n_2-1)}.$$

As we saw in Theorem 2, the derived bound for $\tau(G_1 \square G_2)$ is tight whenever $G_1 \simeq K_{n_1}$ and $G_2 \simeq K_{n_2}$. This, in combination with equation (2), readily gives an exact formula for the number of spanning trees of $K_{n_1} \square K_{n_2}$:

Corollary 2. $\tau(K_{n_1} \square K_{n_2}) = n_1^{n_1-2} n_2^{n_2-2} (n_1 + n_2)^{(n_1-1)(n_2-1)}$.

Observe, that the same argument as used in Theorems 1 and 2 could be applied to the other standard graph products provided that a similar characterisation of their Laplacian spectrum is known. At present no result of this type was known to the author, hence we leave it as future work to investigate upper and lower bounds for the other graph products.

3 Acknowledgements

The author is thankful to Sandi Klavžar for constructive discussions related to the problem.

References

- [1] T. Atajan and H. Inaba, *Network reliability analysis by counting the number of spanning trees*, *ISCIT 2004, IEEE International Symposium on Communication and Information technology 2004* **1** 601–604.
- [2] R. B. Bapat and S. Gupta, Resistance distance in wheels and fans, *Indian J. Pure Appl. Math.* **41** (2010) 1–13.
- [3] G. Baron, F. Boesch, H. Prodinger, R. Tichy and J. Wang, The Number of Spanning Trees in the Square of Cycle, *The Fibonacci Quart.* **23** (1985) 258–264.
- [4] Z. Bogdanowicz, Formulas for the Number of Spanning Trees in a Fan, *Applied Mathematical Sciences* **16** (2008) 781–786.
- [5] F. T. Boesch, On unreliability polynomials and graph connectivity in reliable network synthesis, *J. Graph Theory* **10** (1986) 339–352.
- [6] F. T. Boesch, A. Satyanarayana and Charles L. Suffel, A survey of some network reliability analysis and synthesis results, *Networks* **54** (2009) 99–107.
- [7] R. Burton and R. Pemantle, *Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances*, *Ann. Prob.* **21** (1993) 1329–1371.
- [8] G. A. Cayley, *A theorem on trees*, *Quart. J. Math* **23** (1889) 276–378.
- [9] J. Felker and R. Lyons, *High-precision entropy values for spanning trees in lattices*, *J. Phys. A.* **36** (2003) 8361–8365.
- [10] M. H. S. Haghghi and K. Bibak, *Recursive Relations for the Number of Spanning Trees*, *Applied Mathematical Sciences*, **46** (2009) 2263–2269.
- [11] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs 2nd Edition*, CRC press (2011).
- [12] G. G. Kirchhoff, *Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird*, *Ann. Phy. Chem.* **72** (1847) 497–508.
- [13] B. Mohar, *The laplacian spectrum of graphs*, in: Y. Alavi, G. Chartrand, O. Oellermann and A. Schwenk (Eds.), *Graph Theory, Combinatorics, and Applications*, Wiley, New York, NY, USA, 1991, pp. 871–898
- [14] R. Shrock and F. Y. Wu, *Spanning trees on graphs and lattices in d dimensions*, *J. Phys. A.* **33** (2000) 3881–3902.