

**IMFM**

INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS  
JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

**Preprint series**

**Vol. 51 (2013), 1187**

ISSN 2232-2094

**INVERSE LIMITS WITH  
GENERALIZED MARKOV  
INTERVAL FUNCTIONS**

Iztok Banič      Tjaša Lunder

Ljubljana, May 17, 2013

# Inverse limits with generalized Markov interval functions

Iztok Banič and Tjaša Lunder  
Faculty of Natural Sciences and Mathematics,  
University of Maribor,  
Koroška 160, Maribor 2000, Slovenia  
E-mail: iztok.banic@uni-mb.si  
E-mail: tjasa.lunder@uni-mb.si

May 8, 2013

## Abstract

In [8] Markov interval maps were introduced and it was shown that any two inverse limits with Markov interval bonding maps with the same pattern were homeomorphic.

In this article we introduce generalized Markov interval functions, which are a generalization of Markov interval maps to set-valued functions, and show that any two generalized inverse limits with generalized Markov interval bonding functions with the same pattern are homeomorphic.

## 1 Introduction

In [8] Markov interval maps are defined as follows. Interval self-maps on  $I = [a_0, a_m]$  are Markov with respect to  $A = \{a_0, a_1, \dots, a_m\}$ , if

1.  $a_0 < a_1 < \dots < a_m$ ,
2.  $f(A) \subseteq A$ ,
3.  $f$  is injective on every component of  $I \setminus A$ .

---

2010 *Mathematics Subject Classification*: Primary 54C60; Secondary 54F15.

*Key words and phrases*: Inverse limits, Upper semicontinuous functions, Generalized Markov interval functions

Two interval self-maps,  $f$  and  $g$ , are Markov with the same pattern if  $f$  is Markov with respect to  $A = \{a_0, a_1, \dots, a_m\}$ ,  $g$  is Markov with respect to  $B = \{b_0, b_1, \dots, b_m\}$ , and  $f(a_j) = a_k$  if and only if  $g(b_j) = b_k$ .

The main theorem in [8] says that any two Markov interval maps with the same pattern have homeomorphic inverse limits:

**Theorem 1.1.** Let  $\{f_n\}_{n=0}^\infty$  be a sequence of surjective maps from  $I = [a_0, a_m]$  to  $I$ , which are all Markov interval maps with respect to  $A = \{a_0, a_1, \dots, a_m\}$  and let  $\{g_n\}_{n=0}^\infty$  be a sequence of surjective maps from  $J = [b_0, b_m]$  to  $J$ , which are all Markov interval maps with respect to  $B = \{b_0, b_1, \dots, b_m\}$ . If for each  $n$ ,  $f_n$  and  $g_n$  are Markov interval maps with the same pattern, then  $(I, f_n)$  is homeomorphic to  $(J, g_n)$ .

In this paper we introduce generalized Markov interval functions, which generalize Markov interval maps from [8] (in such a way that every Markov interval map is naturally interpreted as a generalized Markov interval function). In this generalization we allow a generalized Markov interval function to be non single-valued only on points in  $A$ , and include a condition that provides the injectivity of  $f$  on every component of  $I \setminus A$ . The definition of two generalized Markov interval functions with the same pattern will generalize the definition of two Markov interval maps with the same pattern (as it is defined in [8]). We prove the following theorem, which is a generalization of Theorem 1.1, as the main result of the paper:

**Theorem 1.2.** Let  $\{f_n\}_{n=0}^\infty$  be a sequence of u.s.c. functions from  $I = [a_0, a_m]$  to  $2^I$  with surjective graphs, which are all generalized Markov interval functions with respect to  $A = \{a_0, a_1, \dots, a_m\}$  and let  $\{g_n\}_{n=0}^\infty$  be a sequence of u.s.c. functions from  $J = [b_0, b_m]$  to  $2^J$  with surjective graphs, which are all generalized Markov interval functions with respect to  $B = \{b_0, b_1, \dots, b_m\}$ . If for each  $n$ ,  $f_n$  and  $g_n$  are generalized Markov interval functions with the same pattern, then  $(I, f_n)$  is homeomorphic to  $(J, g_n)$ .

Since techniques we used in the proof of Theorem 1.2 are quite different from the ones used in [8], our proof can serve as an alternative proof of Holte's result.

## 2 Definitions and notation

A map is a continuous function. In the case where  $X = Y = \mathbb{R}$ ,  $a \in \mathbb{R}$ , and  $f : X \rightarrow Y$  a map, we use  $\lim_{x \downarrow a} f(x)$  to denote the *right-hand limit* and  $\lim_{x \uparrow a} f(x)$

to denote the *left-hand limit* of a function  $f$  at the point  $a \in \mathbb{R}$ . A detailed introduction of such limits can be found in [21, p. 83–95].

Let  $X$  be a compact metric space, then  $2^X$  denotes the set of all nonempty closed subsets of  $X$ .

If  $f : X \rightarrow 2^Y$  is a function, then the *graph* of  $f$ ,  $\Gamma(f)$ , is defined as  $\Gamma(f) = \{(x, y) \in X \times Y \mid y \in f(x)\}$ .

A function  $f : X \rightarrow 2^Y$  has a *surjective graph*, if for each  $y \in Y$  there is an  $x \in X$ , such that  $y \in f(x)$ .

Let  $f : X \rightarrow 2^Y$  be a function. If for each open set  $V \subseteq Y$ , the set  $\{x \in X \mid f(x) \subseteq V\}$  is open in  $X$ , then  $f$  is an *upper semicontinuous* function (abbreviated u.s.c.) from  $X$  to  $2^Y$ .

The following is a well-known characterization of u.s.c. functions between metric compacta (for example, see [10, p. 120, Theorem 2.1]).

**Theorem 2.1.** Let  $X$  and  $Y$  be compact metric spaces and  $f : X \rightarrow 2^Y$  a function. Then  $f$  is u.s.c. if and only if its graph  $\Gamma(f)$  is closed in  $X \times Y$ .

Note that for any continuous function  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are compact metric spaces, the graph of  $f$  is a closed subset of  $X \times Y$ . Therefore the function  $F : X \rightarrow 2^Y$ , defined by  $F(x) = \{f(x)\}$ , is an u.s.c. function, since  $\Gamma(F) = \Gamma(f)$ . Also if  $F : X \rightarrow 2^Y$  is an u.s.c. function such that  $F(x) = \{y_x\}$  for each  $x \in X$ , then the function  $f : X \rightarrow Y$ , defined by  $f(x) = y_x$ , is continuous. Such functions  $F$  will be addressed as single-valued functions. In the paper we frequently deal with such u.s.c. functions. Understanding them as mappings will simplify the notation and make the proof more reader-friendly. That is why in this case we write  $y = F(x)$  instead of  $y \in F(x)$ .

Let  $A$  be a subset of  $X$  and let  $f : X \rightarrow 2^Y$  be a function. The *restriction* of  $f$  on the set  $A$ ,  $f|_A$ , is the function from  $A$  to  $2^Y$  such that  $f|_A(x) = f(x)$  for every  $x \in A$ .

A sequence  $\{X_k, f_k\}_{k=0}^\infty$  of compact metric spaces  $X_k$  and u.s.c. functions  $f_k : X_{k+1} \rightarrow 2^{X_k}$ , is an *inverse sequence with u.s.c. bonding functions*.

The *inverse limit* of an inverse sequence  $\{X_k, f_k\}_{k=0}^\infty$  with u.s.c. bonding functions is defined as the subspace of  $\prod_{k=0}^\infty X_k$  of all points  $(x_0, x_1, x_2, \dots)$ , such that  $x_k \in f_k(x_{k+1})$  for each  $k$ . The inverse limit of an inverse sequence  $\{X_k, f_k\}_{k=0}^\infty$  is denoted by  $(X_k, f_k)$ .

In this paper we deal only with the case when for each  $k$ ,  $X_k$  is a closed interval  $I$  and  $f_k : I \rightarrow 2^I$ . So, we denote the inverse limit simply by  $(I, f_k)$ .

The notion of inverse limits of inverse sequences with upper semi-continuous bonding functions (also known as generalized inverse limits) was introduced by

Mahavier in [15] and later by Ingram and Mahavier in [10]. Since then, inverse limits have appeared in many papers, such as [1, 2, 3, 4, 6, 9, 11, 12, 13, 17, 18, 19, 20, 22].

### 3 Proof of Theorem 1.2

In this section we introduce the notion of generalized Markov interval functions and prove Theorem 1.2.

**Definition 3.1.** Let  $a_0, a_m \in \mathbb{R}$ ,  $a_0 < a_m$ . We say that an u.s.c. function  $f$  from  $I = [a_0, a_m]$  to  $2^I$  is a *generalized Markov interval function with respect to  $A$* , where  $A = \{a_0, a_1, \dots, a_m\}$  is a subset of  $\mathbb{R}$ , if

1.  $a_0 < a_1 < \dots < a_m$ ,
2. the restriction of  $f$  on every component of  $I \setminus A$  is an injective single-valued function,
3. for each  $j = 0, 1, \dots, m$ , the image  $f(a_j)$  is an interval (possibly degenerated)  $[a_{r_1(j)}, a_{r_2(j)}]$  where  $a_{r_1(j)}, a_{r_2(j)} \in A$  ( $a_{r_1(j)} \leq a_{r_2(j)}$ ),
4. for each  $j = 0, 1, \dots, m - 1$ :  $\lim_{x \uparrow a_{j+1}} f(x), \lim_{x \downarrow a_j} f(x) \in A$ .

Obviously,  $f$  can be single-valued on some points  $a_j$  in  $A$ . In this case  $r_1(j) = r_2(j)$  for some  $0 \leq j \leq m$  and  $f(a_j) = \{a_{r_1(j)}\}$ . Additionally, taking into account property 4. above, we see that:

1. if  $0 < j < m$ , then  $\lim_{x \uparrow a_j} f(x) = \lim_{x \downarrow a_j} f(x) = a_{r_1(j)} = a_{r_2(j)}$ ,
2. if  $j = 0$ ,  $\lim_{x \downarrow a_j} f(x) = a_{r_1(j)} = a_{r_2(j)}$ ,
3. if  $j = m$ ,  $\lim_{x \uparrow a_j} f(x) = a_{r_1(j)} = a_{r_2(j)}$ .

**Definition 3.2.** Let  $A = \{a_0, a_1, \dots, a_m\}$  and  $B = \{b_0, b_1, \dots, b_m\}$ , where  $a_0 < a_1 < \dots < a_m$  and  $b_0 < b_1 < \dots < b_m$ . Then we say that  $(a, b) \in A \times B$  is a *pair of similar points (with respect to  $A$  and  $B$ )*, if  $a = a_i$  and  $b = b_i$  for some  $i = 0, 1, \dots, m$ .

In the following definition we define when two generalized Markov interval functions follow the same pattern.

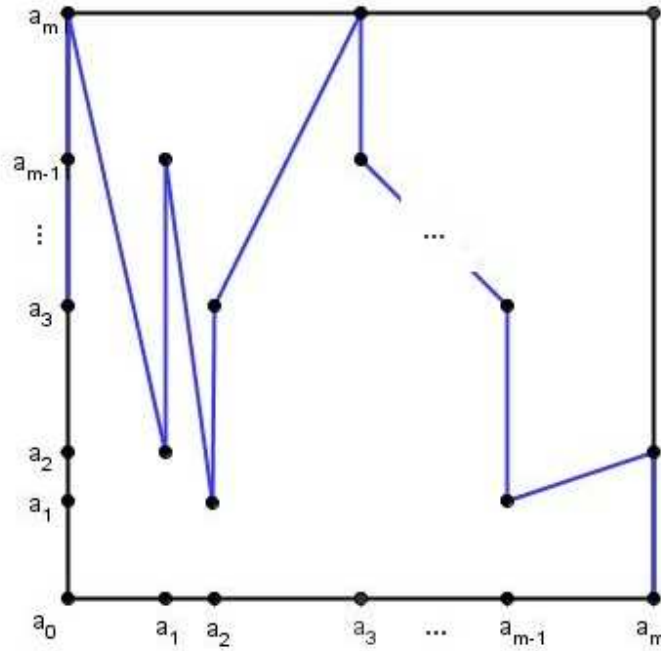


Figure 1: A generalized Markov function

**Definition 3.3.** Let  $f : I = [a_0, a_m] \rightarrow 2^I$  be a generalized Markov interval function with respect to  $A = \{a_0, a_1, \dots, a_m\}$  and let  $g : J = [b_0, b_m] \rightarrow 2^J$  be a generalized Markov interval function with respect to  $B = \{b_0, b_1, \dots, b_m\}$ .

We say that  $f$  and  $g$  are *generalized Markov interval function with the same pattern* if *i*) and *ii*) hold true:

- i*) for every  $j = 0, 1, \dots, m$ :  $f(a_j) = [a_{r_1(j)}, a_{r_2(j)}]$  if and only if  $g(b_j) = [b_{r_1(j)}, b_{r_2(j)}]$ ,
- ii*) for every  $j = 0, 1, \dots, m - 1$ :  $(\lim_{x \uparrow a_{j+1}} f(x), \lim_{y \uparrow b_{j+1}} g(y))$  and  $(\lim_{x \downarrow a_j} f(x), \lim_{y \downarrow b_j} g(y))$  are pairs of similar points.

Finally we prove Theorem 1.2.

*Proof.* Since we have different functions  $f_k, g_k$ , we introduce functions  $r_1^k, r_2^k : \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, m\}$  serving as  $r_1, r_2$  from Definition 3.1, i.e. such that  $f_k(a_j) = [a_{r_1^k(j)}, a_{r_2^k(j)}]$  for each  $j = 0, 1, \dots, m$  and each  $k = 0, 1, 2, \dots$ . According to Definition 3.3 the same functions  $r_1^k, r_2^k$  are also used for  $g_k$ , i.e.  $g_k(b_j) = [b_{r_1^k(j)}, b_{r_2^k(j)}]$ .

For each  $j = 0, 1, \dots, m - 1$  we define the subinterval  $I_j = [a_j, a_{j+1}] \subseteq I = [a_0, a_m]$ , and the subinterval  $J_j = [b_j, b_{j+1}] \subseteq J = [b_0, b_m]$ . We also define a

piecewise linear mapping  $h : I \rightarrow J$  such that  $h(a_j) = b_j$  for all  $j = 0, 1, \dots, m$  by

$$h(x) = \begin{cases} \frac{b_1 - b_0}{a_1 - a_0}(x - a_0) + b_0; & \text{if } x \in I_0, \\ \frac{b_2 - b_1}{a_2 - a_1}(x - a_1) + b_1; & \text{if } x \in I_1, \\ \vdots & \\ \frac{b_m - b_{m-1}}{a_m - a_{m-1}}(x - a_{m-1}) + b_{m-1}; & \text{if } x \in I_{m-1}. \end{cases}$$

The mapping  $h : I \rightarrow J$  is obviously continuous, monotone and surjective, therefore it is a homeomorphism.

Let  $x = (x_0, x_1, x_2, \dots)$  be any element of  $(I, f_n)$ . We show first that there is a uniquely determined point  $y = (y_0, y_1, y_2, \dots)$  in  $(J, g_n)$ , where  $y_0 = h(x_0)$ , such that for all  $i = 0, 1, 2, \dots$ ,  $I(i)$  and  $II(i)$  hold true. Here for each  $i$ ,  $I(i)$  and  $II(i)$  are defined as the following statements:

$I(i) \dots x_i \in \text{Int}(I_j)$  if and only if  $y_i \in \text{Int}(J_j)$ , for each  $j = 0, 1, \dots, m - 1$ ,

$II(i) \dots x_i = a_j$  if and only if  $y_i = b_j$ , for each  $j = 0, 1, \dots, m$ .

To determine the point  $y$  we construct inductively the coordinates  $y_i$  of  $y$  as follows.

First we construct  $y_0$  as  $y_0 = h(x_0)$ . It follows from the definition of  $h$  that  $I(0)$  and  $II(0)$  hold true.

Suppose we have already constructed  $y_0, y_1, y_2, \dots, y_k$  such that  $I(i)$  and  $II(i)$  hold true for each  $i = 0, 1, \dots, k$ , and  $y_{i-1} \in g_{i-1}(y_i)$  hold true for each  $i = 1, 2, \dots, k$ .

Now we construct  $y_{k+1}$  such that  $I(k + 1)$ ,  $II(k + 1)$ , and  $y_k \in g_k(y_{k+1})$ . We consider the following two possible cases.

1.  $x_{k+1} = a_j$  for some  $j = 0, 1, \dots, m$ . In this case we define  $y_{k+1} = b_j$ . Obviously,  $I(k + 1)$  and  $II(k + 1)$  hold true. Next we show that  $y_k \in g_k(y_{k+1})$ . Since  $x_k \in f_k(x_{k+1}) = f_k(a_j) = [a_{r_1^k(j)}, a_{r_2^k(j)}]$  for some  $a_{r_1^k(j)}, a_{r_2^k(j)} \in A$ , and since  $g_k$  and  $f_k$  have the same pattern, it follows that  $g_k(y_{k+1}) = g_k(b_j) = [b_{r_1^k(j)}, b_{r_2^k(j)}]$ .

If  $a_{r_1^k(j)} \neq a_{r_2^k(j)}$ , then fix an integer  $\ell_0$  such that  $x_k \in I_{\ell_0} \subseteq [a_{r_1^k(j)}, a_{r_2^k(j)}]$ . Then  $y_k \in J_{\ell_0} \subseteq [b_{r_1^k(j)}, b_{r_2^k(j)}] = g_k(y_{k+1})$ . If  $a_{r_1^k(j)} = a_{r_2^k(j)}$ , then  $x_k = a_{r_1^k(j)}$ . It follows from the induction assumption  $II(k)$  that  $y_k = b_{r_1^k(j)}$  and therefore  $y_k \in [b_{r_1^k(j)}, b_{r_2^k(j)}] = g_k(y_{k+1})$ .

2.  $x_{k+1} \in \text{Int}(I_j)$  for some  $j = 0, 1, \dots, m - 1$ . In this case, since  $f_k|_{\text{Int}(I_j)}$  is single-valued,

$$x_k = f_k(x_{k+1}) = f_k|_{\text{Int}(I_j)}(x_{k+1}) \in f_k(\text{Int}I_j) = (a_{\ell_1}, a_{\ell_2}),$$

for some  $a_{\ell_1}, a_{\ell_2} \in A$  (where  $\{a_{\ell_1}, a_{\ell_2}\} = \{\lim_{x \downarrow a_j} f(x), \lim_{x \uparrow a_{j+1}} f(x)\}$ ). Therefore  $y_k \in (b_{\ell_1}, b_{\ell_2}) = g_k(\text{Int} J_j)$  since  $f_k$  and  $g_k$  follow the same pattern. We choose  $y_{k+1} \in \text{Int}(J_j)$  such that  $y_k = g_k|_{\text{Int}(J_j)}(y_{k+1})$ . Such a point  $y_{k+1}$  exists and is uniquely determined since  $g_k|_{\text{Int}(J_j)} : \text{Int}(J_j) \rightarrow (b_{\ell_1}, b_{\ell_2})$  is bijective.

Next we show, that if we fix  $y_0 = h(x_0)$ , there is exactly one point  $y = (y_0, y_1, y_2, \dots)$  in  $(J, g_n)$ , such that for each nonnegative integer  $i$ ,  $I(i)$  and  $II(i)$  hold true. Suppose that  $y = (y_0, y_1, y_2, \dots)$  and  $y' = (y_0, y'_1, y'_2, \dots) \in (J, g_n)$  are two such points. We show using the induction on  $i$  that  $y_i = y'_i$  for any  $i$ , hence it follows that  $y = y'$ . Suppose that for each  $k = 0, 1, 2, \dots, i - 1$ ,  $y_k = y'_k$ . We prove that  $y_i = y'_i$ . We examine the following two cases.

1. For some  $j = 0, 1, 2, \dots, m - 1$ ,  $x_i \in \text{Int}(I_j)$ . Then  $y_i, y'_i$  are both in  $\text{Int}(J_j)$  by  $I(i)$ .  
 Since  $y_{i-1} = y'_{i-1}$ , it follows that  $g_{i-1}|_{\text{Int}(J_j)}(y_i) = g_{i-1}(y_i) = y_{i-1} = y'_{i-1} = g_{i-1}(y'_i) = g_{i-1}|_{\text{Int}(J_j)}(y'_i)$ . Since  $g_{i-1}|_{\text{Int}(J_j)}$  is one-to-one, it follows that  $y_i = y'_i$ .
2. For each  $j = 0, 1, 2, \dots, m - 1$ ,  $x_i \notin \text{Int}(I_j)$ . This means that  $x_i = a_j$  for some  $j = 0, 1, \dots, m$ . In this case  $y_i = b_j = y'_i$  by  $II(i)$ .

Next we define a function  $H : (I, f_n) \rightarrow (J, g_n)$  and prove that it is a homeomorphism.

For each  $(x_0, x_1, x_2, \dots) \in (I, f_n)$  we define  $H(x_0, x_1, x_2, \dots)$  to be the unique point  $(y_0, y_1, y_2, \dots)$  in  $(J, g_n)$  such that  $y_0 = h(x_0)$  and for each  $i = 0, 1, 2, \dots$ ,  $I(i)$  and  $II(i)$  hold true.

We have already seen that  $H$  is well defined. Next we show that  $H$  is continuous, by proving that for any sequence  $\{x^i\}_{i=0}^\infty$  in  $(I, f_n)$  converging to  $x \in (I, f_n)$ , the sequence  $\{y^i\}_{i=0}^\infty$ , where  $y^i = H(x^i)$  for each  $i$ , is convergent and its limit equals  $H(x)$ .

$(I, f_n)$  and  $(J, g_n)$  are both compact metric spaces since  $I$  and  $J$  are compact (for details see [10]).

Let  $\{x^i\}_{i=0}^\infty$  be a convergent sequence of elements in  $(I, f_n)$ , where  $x^i = (x_0^i, x_1^i, x_2^i, \dots)$  for all  $i = 0, 1, \dots$ . Let  $x = (x_0, x_1, x_2, \dots)$  be the limit of this sequence. This means that  $x_j$  is the limit of the sequence  $\{x_j^i\}_{i=0}^\infty$  for each  $j$ .

Let  $s = (s_0, s_1, s_2, \dots) \in (J, g_n)$  be any accumulation point of the sequence  $\{y^i\}_{i=0}^\infty$ . Let  $k_i$  be a strictly increasing sequence of nonnegative integers such that  $\lim_{i \rightarrow \infty} y^{k_i} = s$ .



$$\begin{array}{ccc}
 (x_0^{k_0}, x_1^{k_0}, x_2^{k_0}, \dots) \in (I, f_n) & \xrightarrow{H} & (y_0^{k_0}, y_1^{k_0}, y_2^{k_0}, \dots) \in (J, g_n) \\
 (x_0^{k_1}, x_1^{k_1}, x_2^{k_1}, \dots) \in (I, f_n) & \xrightarrow{H} & (y_0^{k_1}, y_1^{k_1}, y_2^{k_1}, \dots) \in (J, g_n) \\
 (x_0^{k_2}, x_1^{k_2}, x_2^{k_2}, \dots) \in (I, f_n) & \xrightarrow{H} & (y_0^{k_2}, y_1^{k_2}, y_2^{k_2}, \dots) \in (J, g_n) \\
 \downarrow i \rightarrow \infty & & \downarrow i \rightarrow \infty \\
 & & (s_0, s_1, s_2, \dots) \in (J, g_n) \\
 (x_0, x_1, x_2, \dots) \in (I, f_n) & \xrightarrow{H} & (y_0, y_1, y_2, \dots) \in (J, g_n)
 \end{array}$$

Let  $y = (y_0, y_1, \dots) = H(x)$ . We prove that  $s = y$ .

One can easily see, that

$$s_0 = \lim_{i \rightarrow \infty} y_0^{k_i} = \lim_{i \rightarrow \infty} h(x_0^{k_i}) = h(\lim_{i \rightarrow \infty} x_0^{k_i}) = h(x_0) = y_0.$$

Suppose we have already shown that  $y_k = s_k$  for each  $k = 0, 1, 2, \dots, \ell - 1$ . We show that  $y_\ell = s_\ell$  by distinguishing the following cases.

1.  $s_\ell \in \text{Int}(J_j)$  for some  $0 \leq j \leq m - 1$ . The point  $s_\ell$  is the limit of the sequence  $\{y_\ell^{k_i}\}_{i=0}^\infty$ . This means that there exists a nonnegative integer  $i_0$  such that  $y_\ell^{k_i} \in \text{Int}(J_j)$ , for all  $i \geq i_0$ . Therefore  $x_\ell^{k_i} \in \text{Int}(I_j)$ , for all  $i \geq i_0$  by  $I(\ell)$ . Since  $x_\ell = \lim_{i \rightarrow \infty} x_\ell^{k_i}$ , it follows that  $x_\ell \in I_j$ . We consider the following two subcases.

- (a) If  $x_\ell \in \text{Int}(I_j)$ , then  $y_\ell \in \text{Int}(J_j)$  by  $I(\ell)$ . Then

$$g_{\ell-1}|_{\text{Int}(J_j)}(y_\ell) = y_{\ell-1} = s_{\ell-1} = g_{\ell-1}|_{\text{Int}(J_j)}(s_\ell)$$

and since  $g_{\ell-1}|_{\text{Int}(J_j)}$  is single-valued and one-to-one,  $y_\ell = s_\ell$  follows.

- (b) If  $x_\ell \in A$ , then  $x_\ell = a_j$  or  $x_\ell = a_{j+1}$  (recall that  $x_\ell \in I_j$ ). Without loss of generality, assume that  $x_\ell = a_j$ . It follows from

- i.  $\lim_{i \rightarrow \infty} x_\ell^{k_i} = a_j$ , and  $a_j$  is the left-hand endpoint of  $I_j$ ,
- ii. for all  $i \geq i_0$ ,  $x_\ell^{k_i} \in \text{Int}(I_j)$  and  $f_{\ell-1}$  is single-valued on  $\text{Int}(I_j)$ ,

that

$$x_{\ell-1} = \lim_{i \rightarrow \infty} x_{\ell-1}^{k_i} \stackrel{\text{ii.}}{=} \lim_{i \rightarrow \infty} f_{\ell-1}(x_\ell^{k_i}) \stackrel{\text{i.}}{=} \lim_{t \downarrow a_j} f_{\ell-1}(t) = a_r$$

where  $a_r \in [a_{r_1^{\ell-1}(j)}, a_{r_2^{\ell-1}(j)}] = f_{\ell-1}(a_j)$  (recall that  $f_{\ell-1}$  is a generalized Markov function with respect to  $A$ ). Therefore,  $x_{\ell-1} = a_r$  and by definition of  $H$ , it follows that  $y_{\ell-1} = b_r$ . We also know that  $(\lim_{t \downarrow a_j} f_{\ell-1}(t), \lim_{t \downarrow b_j} g_{\ell-1}(t))$  is a pair of similar points since  $f_{\ell-1}$  and  $g_{\ell-1}$  follow the same pattern and therefore  $\lim_{t \downarrow b_j} g_{\ell-1}(t) = b_r$ .

Using the fact that  $g_{\ell-1}$  is one-to-one on  $\text{Int}(J_j)$  and that  $s_\ell \in \text{Int}(J_j)$ , we conclude that

$$s_{\ell-1} = g_{\ell-1}(s_\ell) \neq \lim_{t \downarrow b_j} g_{\ell-1}(t) = b_r = y_{\ell-1}.$$

Therefore  $y_{\ell-1} \neq s_{\ell-1}$  which contradicts the inductive assumption.

2.  $s_\ell = b_j$  for some  $0 \leq j \leq m$ .

If there exists a nonnegative integer  $i_1$  such that  $y_\ell^{k_i} = b_j$ , for all  $i \geq i_1$ , then by  $II(\ell)$ ,  $x_\ell^{k_i} = a_j$  holds true for all  $i \geq i_1$ . This means that  $x_\ell = a_j$ , since it is the limit of the sequence  $\{x_\ell^{k_i}\}_{i=0}^\infty$ . Therefore,  $y_\ell = b_j = s_\ell$ .

If such an integer  $i_1$  does not exist, then we consider the following two possible cases:

(a)  $0 < j < m$ . We chose a strictly increasing sequence of positive integers  $n_i$ , such that

- $\{y_\ell^{k_{n_i}}\}_{i=0}^\infty$  is a subsequence of  $\{y_\ell^{k_i}\}_{i=0}^\infty$ ,
- $y_\ell^{k_{n_i}} \neq b_j$  for all  $i$ ,
- $y_\ell^{k_{n_i}} \in \text{Int}(J_{j-1})$  for all  $i$  or  $y_\ell^{k_{n_i}} \in \text{Int}(J_j)$  for all  $i$ .

Assume without any loss of generality that  $y_\ell^{k_{n_i}} \in \text{Int}(J_j)$  for all  $i$ . Recall that  $\lim_{i \rightarrow \infty} y_\ell^{k_{n_i}} = s_\ell = b_j$  and that by  $I(\ell)$ ,  $x_\ell^{k_{n_i}} \in \text{Int}(I_j)$  for all  $i$ . This means that  $x_\ell \in I_j$  and we distinguish the following possibilities:

i. If  $x_\ell \in A$ , then either  $x_\ell = a_j$  or  $x_\ell = a_{j+1}$ . One can see, using similar arguments as in 1.(b), that

$$s_{\ell-1} = \lim_{i \rightarrow \infty} y_{\ell-1}^{k_{n_i}} = \lim_{i \rightarrow \infty} g_{\ell-1}(y_\ell^{k_{n_i}}) = \lim_{t \downarrow b_j} g_{\ell-1}(t) = b_r$$

where  $b_r \in [b_{r_1^{\ell-1}(j)}, b_{r_2^{\ell-1}(j)}] = g_{\ell-1}(b_j)$ . By inductive assumption  $y_{\ell-1} = s_{\ell-1} = b_r$ . By definition of  $H$ , it follows that  $x_{\ell-1} = a_r$ . Since  $f_{\ell-1}$  and  $g_{\ell-1}$  follow the same pattern, it follows that  $x_{\ell-1} = a_r = \lim_{t \downarrow a_j} f_{\ell-1}(t) \neq \lim_{t \uparrow a_{j+1}} f_{\ell-1}(t)$  since  $f_{\ell-1}$  is one-to-one on  $\text{Int}(I_j)$ . Therefore,  $x_\ell$  cannot equal to  $a_{j+1}$ , hence  $x_\ell = a_j$ . By definition of  $H$ , it follows that  $y_\ell = b_j = s_\ell$ .

ii. If  $x_\ell \in \text{Int}(I_j)$ , then

$$x_{\ell-1} = \lim_{i \rightarrow \infty} x_{\ell-1}^{k_{n_i}} = \lim_{i \rightarrow \infty} f_{\ell-1}(x_\ell^{k_{n_i}}) \neq \lim_{t \downarrow a_j} f_{\ell-1}(t)$$

since  $f_{\ell-1}$  is single-valued and injective on  $\text{Int}(I_j)$ . Using the same arguments as in i. we conclude that also in this case

$s_{\ell-1} = b_r$  where  $b_r \in [b_{r_1^{\ell-1}(j)}, b_{r_2^{\ell-1}(j)}] = g_{\ell-1}(b_j)$ . Since  $f_{\ell-1}$  and  $g_{\ell-1}$  follow the same pattern,  $\lim_{t \downarrow a_j} f_{\ell-1}(t) = a_r$  follows from  $\lim_{t \downarrow b_j} g_{\ell-1}(t) = b_r$ . Therefore  $x_{\ell-1} \neq a_r$ . It follows that  $y_{\ell-1} \neq b_r = s_{\ell-1}$ , which contradicts our inductive assumption.

(b)  $j = 0$  or  $j = m$ . Assume without any loss of generality that  $j = 0$ . We chose a strictly increasing sequence of positive integers  $n_i$ , such that

- $\{y_\ell^{k_{n_i}}\}_{i=0}^\infty$  is a subsequence of  $\{y_\ell^{k_i}\}_{i=0}^\infty$ ,
- $y_\ell^{k_{n_i}} \neq b_j$  for all  $i$ ,
- $y_\ell^{k_{n_i}} \in \text{Int}(J_0)$  for all  $i$ .

The rest of the proof is similar to the proof of (a), replacing  $j$  with 0.

We can define  $H^{-1} : (J, g_n) \rightarrow (I, f_n)$  in the same fashion as we did with  $H$ . Every element  $(y_0, y_1, \dots)$  of  $(J, g_n)$  has the unique image  $(x_0, x_1, \dots)$  in  $(I, f_n)$ , such that  $x_0 = h^{-1}(y_0)$  and for each  $i = 0, 1, 2, \dots$ ,  $I(i)$  and  $II(i)$  hold true. Therefore  $H$  is a homeomorphism.  $\square$

We conclude the paper with the following corollary that easily follows from Theorem 1.2.

**Corollary 3.4.** Let  $f : I = [a_0, a_m] \rightarrow 2^I$  be a generalized Markov interval function with respect to  $A = \{a_0, a_1, \dots, a_m\}$  with a surjective graph and  $g : J = [b_0, b_m] \rightarrow 2^J$  be a generalized Markov interval function with respect to  $B = \{b_0, b_1, \dots, b_m\}$  with a surjective graph. If  $f$  and  $g$  are generalized Markov interval functions with the same pattern, then  $(I, f)$  is homeomorphic to  $(J, g)$ .  $\square$

### Acknowledgements

This work was supported in part by the Slovenian Research Agency, under Grant P1-0285.

We thank Uroš Milutinović for constructive suggestions.

### References

[1] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Limits of inverse limits*, Topology Appl. 157 (2010), 439–450.

- [2] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Paths through inverse limits*, Topology Appl. 158 (2011), 1099–1112.
- [3] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Towards the complete classification of generalized tent maps inverse limits*, Topology Appl. 160 (2013), 63–73.
- [4] W. J. Charatonik and R. P. Roe, *Inverse limits of continua having trivial shape*, Houston J. Math. 38 (2012), 1307–1312.
- [5] W. J. Charatonik and R. P. Roe, *Mappings between inverse limits of continua with multivalued bonding functions*, Topology Appl. 159 (2012), 233–235.
- [6] A. N. Cornelius, *Weak crossovers and inverse limits of set-valued functions*, preprint (2009).
- [7] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, London, Sydney, Toronto, 1966.
- [8] S. Holte, *Inverse limits of Markov interval maps*, Topology Appl. 123, 2002, 421–427.
- [9] A. Illanes, *A circle is not the generalized inverse limit of a subset of  $[0, 1]^2$* , Proc. Amer. Math. Soc. 139 (2011), 2987–2993.
- [10] W. T. Ingram and W. S. Mahavier, *Inverse limits of upper semicontinuous set valued functions*, Houston J. Math. 32 (2006), 119–130.
- [11] W. T. Ingram, *Inverse limits of upper semicontinuous functions that are unions of mappings*, Topology Proc. 34 (2009), 17–26.
- [12] W. T. Ingram, *Inverse limits with upper semicontinuous bonding functions: problems and some partial solutions*, Topology Proc. 36 (2010), 353–373.
- [13] J. A. Kennedy and S. Greenwood, *Generic generalized inverse limits*, Houston J. Math. 38 (2012), 1369–1384.
- [14] K. Kuratowski, *Topology, vol. 2*, Academic Press, New York, London, Warsaw, 1968.
- [15] W. S. Mahavier, *Inverse limits with subsets of  $[0, 1] \times [0, 1]$* , Topology Appl. 141 (2004), 225–231.

- [16] S. B. Nadler, *Continuum theory. An introduction*, Marcel Dekker, Inc., New York, 1992.
- [17] V. Nall, *Inverse limits with set valued functions*, Houston J. Math. 37 (2011), 1323–1332.
- [18] V. Nall, *Connected inverse limits with set valued functions*, Topology Proc. 40 (2012), 167–177.
- [19] V. Nall, *Finite graphs that are inverse limits with a set valued function on  $[0, 1]$* , Topology Appl. 158 (2011), 1226–1233.
- [20] A. Palaez, *Generalized inverse limits*, Houston J. Math. 32 (2006), 1107–1119.
- [21] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, Tokyo et al., 1976.
- [22] S. Varagona, *Inverse limits with upper semi-continuous bonding functions and indecomposability*, Houston J. Math. 37 (2011), 1017–1034.