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**INVERSE LIMITS IN THE
CATEGORY OF COMPACT
HAUSDORFF SPACES AND
UPPER SEMICONTINUOUS
FUNCTIONS**

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Inverse limits in the category of compact Hausdorff spaces and upper semicontinuous functions

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Abstract

We investigate inverse limits in the category \mathcal{CHU} of compact Hausdorff spaces with upper semicontinuous (usc) functions. We introduce the notion of weak inverse limits in this category and show that the inverse limits with upper semicontinuous set-valued bonding functions (as they were defined in [15]) together with the projections are not necessarily inverse limits in \mathcal{CHU} but they are always weak inverse limits in this category. This is a realization of our categorical approach to solving a problem stated by W. T. Ingram in [14].

Keywords: Upper semi-continuous functions, Inverse limits, Weak inverse limits

2000 Mathematics Subject Classification: primary 54C60; secondary 54B30

1 Introduction

W. T. Ingram in his book [14] states the following problem:

Problem 6.63. What can be said about inverse limits with set-valued functions if the underlying directed set is not a sequence of integers?

In this paper we present a categorical approach to solving the above problem.

Consider an inverse system $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ of compact Hausdorff spaces and continuous bonding functions. It is a well-known fact that the space

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) =$$

$$\{(x_\gamma)_{\gamma \in A} \in \prod_{\gamma \in A} X_\alpha \mid \text{for all } \alpha, \beta \in A, \alpha < \beta, x_\alpha = f_{\alpha\beta}(x_\beta)\}$$

together with the projection mappings $p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) \rightarrow X_\gamma$, $p_\gamma((x_\alpha)_{\alpha \in A}) = x_\gamma$, is in fact an inverse limit in the category \mathcal{CHC} of compact Hausdorff spaces with continuous functions.

In present paper we extend the category \mathcal{CHC} to the category \mathcal{CHU} of compact Hausdorff spaces with usc functions in such a way that \mathcal{CHC} is interpreted as a proper subcategory of \mathcal{CHU} . This can be done since every continuous function between compact Hausdorff spaces can be interpreted as a usc function.

As one of our main results we show that the inverse limits with upper semicontinuous set-valued bonding functions

$$\begin{aligned} & \varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) = \\ & \{(x_\gamma)_{\gamma \in A} \in \prod_{\gamma \in A} X_\alpha \mid \text{for all } \alpha, \beta \in A, \alpha < \beta, x_\alpha \in f_{\alpha\beta}(x_\beta)\} \end{aligned}$$

together with the projections

$$\begin{aligned} p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) & \rightarrow X_\gamma, \\ p_\gamma((x_\alpha)_{\alpha \in A}) & = \{x_\gamma\}, \end{aligned}$$

are not necessarily inverse limits in the category but they are always so called weak inverse limits in \mathcal{CHU} .

In the second section we give the basic definitions that are used in the paper.

In the third section we give a detailed description of the category \mathcal{CHU} of compact Hausdorff spaces with usc bonding functions.

In the fourth section we give results about inverse limits in the category \mathcal{CHU} .

In the last section we define objects in category \mathcal{CHU} that are called weak inverse limits in this category. We also show that for any inverse system $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ in \mathcal{CHU} , the corresponding inverse limit with upper semicontinuous set-valued bonding functions together with projections is always a weak inverse limit in category \mathcal{CHU} .

2 Definitions and notation

For any category \mathcal{K} the class of objects of \mathcal{K} will be denoted by $Ob(\mathcal{K})$, the class of morphisms of \mathcal{K} by $Mor(\mathcal{K})$, and the partial binary associative

operation (composition of morphisms) by \circ . For any $X \in \text{Ob}(\mathcal{K})$ the identity morphism on X will be denoted by $1_X : X \rightarrow X$.

For a directed set A (A is nonempty and equipped with a reflexive and transitive binary relation \leq with the property that every pair of elements has an upper bound), a family of objects $\{X_\alpha \mid \alpha \in A\}$ of \mathcal{K} , and a family of morphisms $\{f_{\alpha\beta} : X_\beta \rightarrow X_\alpha \mid \alpha, \beta \in A, \alpha \leq \beta\}$ of \mathcal{K} , such that

1. for each $\alpha \in A$, $f_{\alpha\alpha} = 1_{X_\alpha}$,
2. for each $\alpha, \beta, \gamma \in A$, from $\alpha \leq \beta \leq \gamma$ it follows that $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$,

we call an inverse system (in \mathcal{K}) and denote it by

$$(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}).$$

We assume throughout the paper that A is cofinite, i.e. every $\alpha \in A$ has at most finitely many predecessors.

Next we define inverse limits in \mathcal{K} .

Definition 2.1. *An object $X \in \text{Ob}(\mathcal{K})$, together with morphisms $\{p_\alpha : X \rightarrow X_\alpha \mid \alpha \in A\}$ is an inverse limit of an inverse system $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ in the category \mathcal{K} , if*

1. for all $\alpha, \beta \in A$, from $\alpha \leq \beta$ it follows that the diagram

$$\begin{array}{ccc}
 X & & \\
 \downarrow p_\alpha & \searrow p_\beta & \\
 X_\alpha & \xleftarrow{f_{\alpha\beta}} & X_\beta
 \end{array} \tag{1}$$

commutes;

2. for any object $Y \in \mathcal{K}$ and any family of morphisms $\{\varphi_\alpha : Y \rightarrow X_\alpha \mid \alpha \in A\}$ it follows that if the diagram

$$\begin{array}{ccc}
 Y & & \\
 \downarrow \varphi_\alpha & \searrow \varphi_\beta & \\
 X_\alpha & \xleftarrow{f_{\alpha\beta}} & X_\beta
 \end{array} \tag{2}$$

commutes, then there is a unique morphism $\varphi : Y \rightarrow X$ such that for each $\alpha \in A$ the diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{\varphi} & X \\
 \downarrow \varphi_\alpha & & \swarrow p_\alpha \\
 X_\alpha & &
 \end{array}
 \tag{3}$$

commutes.

A *map* or *mapping* is a continuous function.

If X is a compact Hausdorff space, then 2^X denotes the set of all nonempty closed subsets of X .

The *graph* $\Gamma(f)$ of a function $f : X \rightarrow 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

A function $f : X \rightarrow 2^Y$ is *upper semi-continuous* function if for each $x \in X$ and for each open set $U \subseteq Y$ such that $f(x) \subseteq U$ there is an open set V in X such that

1. $x \in V$;
2. for all $v \in V$ it holds that $f(v) \subseteq U$.

The following is a well-known characterization of usc functions between Hausdorff compacta (see [15, p. 120, Theorem 2.1]).

Theorem 2.2. *Let X and Y be compact Hausdorff spaces and $f : X \rightarrow 2^Y$ a function. Then f is usc if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

At the end of this section we introduce the notion of inverse limits with usc set-valued bonding functions as it was introduced by Mahavier in [19] and Ingram and Mahavier in [15]. In the last section we use this notion as a motivation for defining inverse limits with usc set-valued bonding functions for arbitrary inverse systems.

An *inverse sequence* of compact Hausdorff spaces X_k with usc bonding functions f_k is a sequence $\{X_k, f_k\}_{k=1}^\infty$, where $f_k : X_{k+1} \rightarrow 2^{X_k}$ is usc for each k .

The *inverse limit with usc set-valued bonding functions* of an inverse sequence $\{X_k, f_k\}_{k=1}^\infty$ is defined to be the subspace of the product space

$\prod_{k=1}^{\infty} X_k$ of all $x = (x_1, x_2, x_3, \dots) \in \prod_{k=1}^{\infty} X_k$, such that $x_k \in f_k(x_{k+1})$ for each k . The inverse limit of $\{X_k, f_k\}_{k=1}^{\infty}$ is denoted by $\varprojlim \{X_k, f_k\}_{k=1}^{\infty}$.

Since the introduction of such inverse limits, there has been much interest in the subject and many papers appeared [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 16, 17, 18, 22, 23, 24, 25, 26].

3 The category \mathcal{CHU}

The category \mathcal{CHU} of compact Hausdorff spaces and usc functions consists of the following objects and morphisms:

1. $Ob(\mathcal{CHU})$: compact Hausdorff spaces;
2. $Mor(\mathcal{CHU})$: the usc functions from X to Y is the set of morphisms from X to Y , denoted by $Mor(\mathcal{CHU})(X, Y)$.

We also define the partial binary operation \circ (the composition) as follows. For each $f \in Mor(\mathcal{CHU})(X, Y)$ and each $g \in Mor(\mathcal{CHU})(Y, Z)$ we define $g \circ f \in Mor(\mathcal{CHU})(X, Z)$ by

$$(g \circ f)(x) = g(f(x)) = \bigcup_{y \in f(x)} g(y)$$

for each $x \in X$.

Theorem 3.1. \mathcal{CHU} is a category.

Proof. First we show that \circ is well-defined. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be any morphisms. Let also $x \in X$ be arbitrary and let U be an open set in Z such that $(g \circ f)(x) \subseteq U$. Since g is usc and $f(x) \subseteq Y$, it holds that for each $y \in f(x)$ there is an open set W_y in Y such that

1. $y \in W_y$;
2. for all $w \in W_y$ it holds that $g(w) \subseteq U$.

Let $W = \bigcup_{y \in f(x)} W_y$. Since W is open in Y , $f(x) \subseteq W$, and since f is usc, there is an open set V in X such that

1. $x \in V$;
2. for all $v \in V$ it holds that $f(v) \subseteq W$.

Let $v \in V$ be arbitrary. Then

$$(g \circ f)(v) = g(f(v)) = \bigcup_{z \in f(v)} g(z) \subseteq U$$

since for each $z \in f(v)$, it holds that $g(z) \subseteq U$. Therefore \circ is well-defined.

It is obvious that the composition \circ of usc functions is an associative operation.

All that is left to show is that for each $X \in \text{Ob}(\mathcal{CHU})$ there is a morphism $1_X : X \rightarrow X$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any morphisms $f : Y \rightarrow X$ and $g : X \rightarrow Z$. We easily see that the identity map $1_X : X \rightarrow X$, defined by $1_X(x) = \{x\}$ for each $x \in X$, is the usc function satisfying the above conditions. \square

4 Inverse limits in \mathcal{CHU}

In this section we show that if $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is an inverse system of compact Hausdorff spaces and usc set-valued bonding functions, then

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$$

(see Definition 4.1) together with the projections is not necessarily an inverse limit in the category \mathcal{CHU} .

Motivated by [15, 19], we define in Definition 4.1 objects in \mathcal{CHU} , that are called inverse limits with usc set-valued bonding functions. Since such object were first introduced by Mahavier in [19] and Ingram and Mahavier in [15], where they call them the inverse limits with usc set-valued bonding functions, we continue to use the same name for them.

Definition 4.1. *Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ be any inverse system in \mathcal{CHU} . We call the object*

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) = \{x \in \prod_{\alpha \in A} X_\alpha \mid \text{for all } \alpha < \beta, x_\alpha \in f_{\alpha\beta}(x_\beta)\}$$

an inverse limit with usc set-valued bonding functions.

In the following theorem we prove that $\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is really an object of \mathcal{CHU} .

Theorem 4.2. *Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ be any inverse system in \mathcal{CHU} . Then the inverse limit with usc set-valued bonding functions*

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$$

is a compact Hausdorff space.

Proof. For each $\gamma \in A$, X_γ is a compact Hausdorff space, therefore the product $\prod_{\gamma \in A} X_\gamma$ is a compact Hausdorff space. Since $\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is a subspace of the Hausdorff space, it is also a Hausdorff space. We show that $\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is a closed subset of the compact space $\prod_{\gamma \in A} X_\gamma$ to show that it is compact.

Let for all $\alpha, \beta \in A$, $\alpha < \beta$,

$$G_{\alpha\beta} = \Gamma(f_{\alpha\beta}) \times \prod_{\gamma \in A \setminus \{\alpha, \beta\}} X_\gamma = \{x \in \prod_{\gamma \in A} X_\gamma \mid x_\alpha \in f_{\alpha\beta}(x_\beta)\}.$$

Since the graph $\Gamma(f_{\alpha\beta})$ of $f_{\alpha\beta}$ is by Theorem 2.2 a closed subset of $X_\beta \times X_\alpha$, $G_{\alpha\beta}$ is also a closed subset of $\prod_{\gamma \in A} X_\gamma$. It is obvious that

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) = \bigcap_{\alpha, \beta \in A, \alpha < \beta} G_{\alpha\beta}$$

and hence $\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is a closed subset of $\prod_{\gamma \in A} X_\gamma$. \square

In the following example we construct an inverse limit with usc set-valued bonding functions that is not an inverse limit in \mathcal{CHU} regardless of the choice of morphisms $\{p_\alpha : X \rightarrow X_\alpha \mid \alpha \in A\}$.

Example 4.3. Let $A = \mathbb{N}$, $X_k = [0, 1]$, and let $f_{k(k+1)} = f$ for each $k \in \mathbb{N}$, where $f : [0, 1] \rightarrow 2^{[0,1]}$ is the function on $[0, 1]$ defined by its graph

$$\Gamma(f) = \{(t, t) \in [0, 1] \times [0, 1] \mid t \in [0, 1]\} \cup (\{1\} \times [0, 1]).$$

Also let $X = \varprojlim(\mathbb{N}, \{[0, 1]\}_{k \in \mathbb{N}}, \{f_{k\ell}\}_{k, \ell \in \mathbb{N}})$ and let $\{p_i : X \rightarrow X_i \mid i \in \mathbb{N}\}$ be any set of morphisms in \mathcal{CHU} , such that the diagrams (1) always commute. We show that X with $\{p_i : X \rightarrow X_i \mid i \in \mathbb{N}\}$ is not an inverse limit of $(\mathbb{N}, \{[0, 1]\}_{k \in \mathbb{N}}, \{f_{k\ell}\}_{k, \ell \in \mathbb{N}})$ in \mathcal{CHU} . Let $Y = [0, 1]$ be an object in \mathcal{CHU} and let $\{\varphi_k : Y \rightarrow X_k \mid k \in \mathbb{N}\}$ be the family of morphisms where $\varphi_k(t) = [0, 1]$ for each k and each $t \in Y$. The diagram (2) always commutes. We distinguish the following two cases.

1. If there is a positive integer i_0 , such that $1 \notin p_{i_0}(x)$ for each $x \in X$, then suppose that Φ is any morphism $Y \rightarrow X$. Then $\varphi_{i_0}(t) = [0, 1]$ but $1 \notin p_{i_0}(\Phi(t))$ for any $t \in Y$. Therefore the diagram (3) does not commute for $\alpha = i_0$.
2. If for each positive integer i there is $x^i \in X$ such that $1 \in p_i(x^i)$, then let $s \in X$ be an accumulation point of the sequence $\{x^i\}_{i=1}^\infty$. We show

first that $p_i(s) = [0, 1]$ for each i . Let k be any positive integer. Then for each $\ell > k$, it follows from

$$[0, 1] \supseteq p_k(x^\ell) = f_{k\ell}(p_\ell(x^\ell)) \supseteq f_{k\ell}(1) \supseteq [0, 1]$$

that $p_k(x^\ell) = [0, 1]$. Let $\{n_i\}_{i=1}^\infty$ be any increasing sequence of positive integers such that

- $n_i > k$ for each i ;
- $\lim_{i \rightarrow \infty} x^{n_i} = s$.

It follows from $p_k(x^{n_i}) = [0, 1]$ that $\{x^{n_i}\} \times [0, 1] \subseteq \Gamma(p_k)$ for each i . This means that for each $t \in [0, 1]$, the point $(x^{n_i}, t) \in \Gamma(p_k)$. Therefore $\varinjlim (x^{n_i}, t) = (s, t) \in \Gamma(p_k)$ for each t , since $\Gamma(p_k)$ is a closed subset of $X \times [0, 1]$. It follows that $\{s\} \times [0, 1] \subseteq \Gamma(p_k)$ and hence $p_k(s) = [0, 1]$.

Next, let $\Phi, \Psi : Y \rightarrow X$ be the morphisms in \mathcal{CHU} , defined by

$$\Phi(t) = X,$$

$$\Psi(t) = \{s\}$$

for each $t \in Y$. It follows from

$$p_k(\Phi(t)) = p_k(X) = [0, 1] = \varphi_k(t)$$

and

$$p_k(\Psi(t)) = p_k(\{s\}) = [0, 1] = \varphi_k(t)$$

that the diagram (3) commutes for both $\varphi = \Phi$ and $\varphi = \Psi$. Therefore there is no unique morphism φ such that all diagrams (3) commute.

Note that in the second part of Example 4.3, $\Psi(t) \subseteq \Phi(t) = (\prod_{k=1}^\infty \varphi_k(t)) \cap X$ holds true for each $t \in Y$. The following lemma shows that such an inclusion is not accidental. It will be used in the proof of Theorem 5.5.

Lemma 4.4. *Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ be any inverse system in \mathcal{CHU} and let $X = \varprojlim (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$. Suppose that for an object Y of \mathcal{CHU} and a family of morphisms $\{\varphi_\alpha : Y \rightarrow X_\alpha \mid \alpha \in A\}$ the diagram (2) commutes for any α and β , $\alpha < \beta$. Then $\varphi : Y \rightarrow X$, defined by $\varphi(y) = (\prod_{\gamma \in A} \varphi_\gamma(y)) \cap X$ for each $y \in Y$, is a morphism in \mathcal{CHU} such that for each $\alpha \in A$ the diagram (3) commutes. Even more, for any morphism $\Psi : Y \rightarrow X$ such that $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$ for each $\alpha \in A$ and for each $y \in Y$, $\Psi(y) \subseteq \varphi(y)$ holds true for all $y \in Y$.*

Proof. We show that φ satisfies all the conditions in the following steps.

1. The set $\prod_{\gamma \in A} \varphi_\gamma(y)$ is a closed subset of $\prod_{\alpha \in A} X_\alpha$, therefore $\varphi(y)$ is a closed subset of X for any $y \in Y$.
2. Next we show that for any $y \in Y$, the set $\varphi(y)$ is nonempty. Let $y \in Y$ be arbitrarily chosen. Next, let for each positive integer n , $A_n \subseteq A$ be the set of all elements $\alpha \in A$ that have exactly $n - 1$ predecessors. For any $\alpha \in A_1$ we arbitrarily choose $t_\alpha \in \varphi_\alpha(y)$. For any $\beta \in A_2$ there is an $\alpha \in A_1$ such that $\alpha < \beta$. For any such α and β it follows from $t_\alpha \in \varphi_\alpha(y) \subseteq f_{\alpha\beta}(\varphi_\beta(y))$ that there is $t_\beta \in \varphi_\beta(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$. We choose and fix such t_β for each $\beta \in A_2$. Suppose that we have already constructed $t_\alpha \in \varphi_\alpha(y)$ for all $\alpha \in A_n$. Then for any $\beta \in A_{n+1}$ there is an $\alpha \in A_n$ such that $\alpha < \beta$. For any such α and β it follows from $t_\alpha \in \varphi_\alpha(y) \subseteq f_{\alpha\beta}(\varphi_\beta(y))$ that there is $t_\beta \in \varphi_\beta(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$. We choose and fix such t_β for each $\beta \in A_{n+1}$.
Then obviously $x = (t_\alpha)_{\alpha \in A} \in \varphi(y)$ and therefore $\varphi(y)$ is nonempty.

3. We show that φ is a usc function. Let $y \in Y$ be arbitrary point and let

$$U = (U_{\gamma_1} \times U_{\gamma_2} \times U_{\gamma_3} \times \cdots \times U_{\gamma_n}) \times \prod_{\gamma \in A \setminus \{\gamma_1, \gamma_2, \dots, \gamma_n\}} X_\gamma$$

be an open set in X such that $\varphi(y) \subseteq U$, where for each $i = 1, 2, 3, \dots, n$, U_{γ_i} is an open set in X_{γ_i} . It follows from the definitions of φ and U that $\varphi_{\gamma_i}(y) \subseteq U_{\gamma_i}$ for each $i = 1, 2, 3, \dots, n$. Since each φ_{γ_i} is usc, there is an open set V_i in Y such that

- (a) $y \in V_i$;
- (b) for each $x \in V_i$, it holds that $\varphi_{\gamma_i}(x) \subseteq U_{\gamma_i}$

for each i . We define $V = \bigcap_{i=1}^n V_i$. Then V is an open set in Y for which

- (a) $y \in V$;
- (b) for each $x \in V$, it holds that $\varphi(x) = \prod_{\gamma \in A} \varphi_\gamma(x) \subseteq U$

holds true. Therefore φ is a usc function and so it is a morphism from Y to X .

4. Next we show that the diagram (3) commutes, i.e. for any $\alpha \in A$ and any $y \in Y$, $\varphi_\alpha(y) = (p_\alpha \circ \varphi)(y)$ holds true. Choose any $\alpha \in A$ and any

$y \in Y$. Obviously

$$p_\alpha(\varphi(y)) = p_\alpha\left(\left(\prod_{\gamma \in A} \varphi_\gamma(y)\right) \cap X\right) \subseteq p_\alpha\left(\prod_{\gamma \in A} \varphi_\gamma(y)\right) = \varphi_\alpha(y).$$

Next we show that $\varphi_\alpha(y) \subseteq p_\alpha(\varphi(y))$. Let $z \in \varphi_\alpha(y)$ be arbitrarily chosen. We show that $z \in p_\alpha(\varphi(y))$ by showing that there is a point $x \in \varphi(y)$ such that $z \in p_\alpha(x)$. Let k be the positive integer such that $\alpha \in A_k$. For each $\gamma \in A_k \setminus \{\alpha\}$ let $t_\gamma \in \varphi_\gamma(y)$ be arbitrary and let $t_\alpha = z$. For each $\gamma \in A_{k-1}$ we choose $t_\gamma \in \varphi_\gamma(y)$ such that if $\alpha \in A_{k-1}$, $\beta \in A_k$, and $\alpha < \beta$, then $t_\alpha \in f_{\alpha\beta}(t_\beta)$. This can be done since $f_{\alpha\beta}(\varphi_\beta(y)) = \varphi_\alpha(y)$ and therefore $f_{\alpha\beta}(t_\beta) \subseteq \varphi_\alpha(y)$.

Continuing in the same fashion we choose for each $i = 1, 2, 3, \dots, k-1$ and each $\gamma \in A_i$ an element $t_\gamma \in \varphi_\gamma(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$ for each $\alpha \in A_i$, $\beta \in A_{i+1}$, $\alpha < \beta$.

Next, for each $\beta \in A_{k+1}$ and for each $\alpha \in A_k$ such that $\beta > \alpha$, since $t_\alpha \in \varphi_\alpha(y) = f_{\alpha\beta}(\varphi_\beta(y))$, there is $t_\beta \in \varphi_\beta(y)$, such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$.

We continue inductively in the same fashion and choose for each $i = k+1, k+2, k+3, \dots$ and each $\beta \in A_{i+1}$ an element $t_\beta \in \varphi_\beta(y)$ such that $t_\alpha \in f_{\alpha\beta}(t_\beta)$ for each $\alpha \in A_i$, such that $\alpha < \beta$.

Let $x \in \prod_{\gamma \in A} X_\gamma$ be such an element that $p_\gamma(x) = \{t_\gamma\}$ for each $\gamma \in A$. It follows from the construction of x that $x \in \varphi(y)$ and $z \in p_\alpha(x)$.

5. Suppose that $\psi : Y \rightarrow X$ is a morphism in \mathcal{CHU} such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\psi(y)) = \varphi_\alpha(y)$. Let $y \in Y$ be arbitrary and let $z \in \psi(y)$. Obviously $z \in X$ since ψ is a morphism from Y to X . It follows from $p_\alpha(z) \subseteq p_\alpha(\psi(y)) = \varphi_\alpha(y)$ (for each α) that $z \in \prod_{\gamma \in A} \varphi_\gamma(y)$. Therefore $z \in \varphi(y)$ and hence $\psi(y) \subseteq \varphi(y)$.

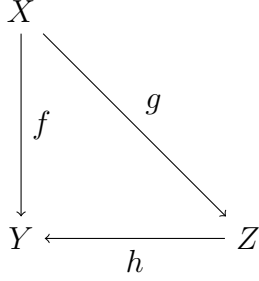
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5 Weak inverse limits in \mathcal{CHU}

In this section we introduce the notion of weak inverse limits in \mathcal{CHU} and show that $\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ (together with the projections) is always a weak inverse limit in \mathcal{CHU} .

In Definition 5.1 we define a weak commutation of a diagram in the category \mathcal{CHU} .

Definition 5.1. *Let $X, Y, Z \in \text{Ob}(\mathcal{CHU})$ and let $f : X \rightarrow Y$, $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ be any morphisms in \mathcal{CHU} . The diagram*

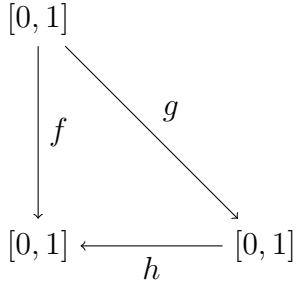


weakly commutes, if for any $x \in X$, $f(x) \subseteq (h \circ g)(x)$.

Example 5.2. Let $f : [0, 1] \rightarrow 2^{[0,1]}$, $g : [0, 1] \rightarrow 2^{[0,1]}$ be identity functions on $[0, 1]$ and let $h : [0, 1] \rightarrow 2^{[0,1]}$ be defined by

$$h(x) = [0, 1]$$

for all $x \in [0, 1]$. Then the diagram



weakly commutes but does not commute.

In the following definition we generalize the notion of inverse limits in \mathcal{CHU} .

Definition 5.3. An object $X \in \text{Ob}(\mathcal{CHU})$, together with morphisms $\{p_\alpha : X \rightarrow X_\alpha \mid \alpha \in A\}$, is a weak inverse limit of an inverse system

$$(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$$

in \mathcal{CHU} , if

1. for all $\alpha, \beta \in A$, from $\alpha \leq \beta$ it follows that the diagram (1) weakly commutes;
2. for any object $Y \in \mathcal{CHU}$ and any family of morphisms $\{\varphi_\alpha : Y \rightarrow X_\alpha \mid \alpha \in A\}$ it follows that if the diagram (2) commutes, then for any morphism $\Psi : Y \rightarrow X$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq (\prod_{\gamma \in A} \varphi_\gamma(y)) \cap X$ holds true for all $y \in Y$.

Note that each inverse limit in \mathcal{CHU} is always a weak inverse limit in \mathcal{CHU} .

Example 5.4. Let $X = \varprojlim(\mathbb{N}, \{[0, 1]\}_{k \in \mathbb{N}}, \{f_{k\ell}\}_{k, \ell \in \mathbb{N}})$ be the inverse limit with usc set-valued bonding functions that we defined in Example 4.3. Then X , together with the projection mappings, is obviously not an inverse limit but it is a weak inverse limit in \mathcal{CHU} .

We show in the following theorem that the inverse limits with upper semicontinuous set-valued bonding functions together with projections are always weak inverse limits in \mathcal{CHU} .

Theorem 5.5. Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ be any inverse system in \mathcal{CHU} . Then the inverse limit with usc set-valued bonding functions

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}),$$

together with projections

$$p_\gamma : \varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) \rightarrow X_\gamma,$$

$$p_\gamma((x_\alpha)_{\alpha \in A}) = \{x_\gamma\},$$

is a weak inverse limit of the inverse system $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ in \mathcal{CHU} .

Proof. Let $X = \varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$. First we prove that the diagram (1) weakly commutes. Choose any $x \in X$ and let $\alpha < \beta$. Then $p_\alpha(x) = \{x_\alpha\} \subseteq f_{\alpha\beta}(\{x_\beta\}) = (f_{\alpha\beta} \circ p_\beta)(x)$.

Next, suppose that for an object $Y \in \mathcal{CHU}$ and a family of morphisms $\{\varphi_\alpha : Y \rightarrow X_\alpha \mid \alpha \in A\}$ the diagram (2) commutes. By Lemma 4.4, for any morphism $\Psi : Y \rightarrow X$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq (\prod_{\gamma \in A} \varphi_\gamma(y)) \cap X$ holds true for all $y \in Y$. \square

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References

- [1] I. Banič, *On dimension of inverse limits with upper semicontinuous set-valued bonding functions*, Topology Appl. 154 (2007), 2771–2778.
- [2] I. Banič, *Inverse limits as limits with respect to the Hausdorff metric*, Bull. Austral. Math. Soc. 75 (2007), 17–22.
- [3] I. Banič, *Continua with kernels*, Houston J. Math. 34 (2008), 145–163.
- [4] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Limits of inverse limits*, Topology Appl. 157 (2010), 439–450.
- [5] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Paths through inverse limits*, Topology Appl. 158 (2011), 1099–1112.
- [6] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Towards the complete classification of generalized tent maps inverse limits*, <http://dx.doi.org/10.1016/j.topol.2012.09.017>.
- [7] I. Banič, M. Črepnjak, M. Merhar, U. Milutinović and T. Sovič *Ważewski’s universal dendrite as an inverse limit with one set-valued bonding function*, preprint (2012).
- [8] W. J. Charatonik and R. P. Roe, *Inverse limits of continua having trivial shape*, to appear in Houston J. Math.
- [9] W. J. Charatonik and R. P. Roe, *Mappings between inverse limits of continua with multivalued bonding functions*, preprint
- [10] A. N. Corneliu, *Weak crossovers and inverse limits of set-valued functions*, preprint (2009),
- [11] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, London, Sydney, Toronto, 1966.
- [12] A. Illanes, *A circle is not the generalized inverse limit of a subset of $[0, 1]^2$* , Proc. Amer. Math. Soc. 139 (2011), 2987–2993.
- [13] A. Illanes, S. B. Nadler, *Hyperspaces. Fundamentals and recent advances*, Marcel Dekker, Inc., New York, 1999.
- [14] W. T. Ingram, *An Introduction to Inverse Limits with Set-valued Functions*, Springer, New York et al., 2012.

- [15] W. T. Ingram, W. S. Mahavier, *Inverse limits of upper semi-continuous set valued functions*, Houston J. Math. 32 (2006), 119–130.
- [16] W. T. Ingram, *Inverse limits of upper semicontinuous functions that are unions of mappings*, Topology Proc. 34 (2009), 17–26.
- [17] W. T. Ingram, *Inverse limits with upper semicontinuous bonding functions: problems and some partial solutions*, Topology Proc. 36 (2010), 353–373.
- [18] J. A. Kennedy and S. Greenwood, *Pseudoarcs and generalized inverse limits*, preprint (2010).
- [19] W. S. Mahavier, *Inverse limits with subsets of $[0, 1] \times [0, 1]$* , Topology Appl. 141 (2004), 225–231.
- [20] S. Mardešić and J. Segal, *Shape theory*, North-Holland, Amsterdam, 1982.
- [21] S. B. Nadler, *Continuum theory. An introduction*, Marcel Dekker, Inc., New York, 1992.
- [22] V. Nall, *Inverse limits with set valued functions*, Houston J. Math. 37 (2011), 1323–1332.
- [23] V. Nall, *Connected inverse limits with set valued functions*, Topology Proc. 40 (2012), 167–177.
- [24] V. Nall, *Finite graphs that are inverse limits with a set valued function on $[0, 1]$* , Topology Appl. 158 (2011), 1226–1233.
- [25] A. Palaez, *Generalized inverse limits*, Houston J. Math. 32 (2006), 1107–1119.
- [26] S. Varagona, *Inverse limits with upper semi-continuous bonding functions and indecomposability*, Houston J. Math. 37 (2011), 1017–1034.

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